Additional Exercises for Chapter 7

In these exercises we will use more precise data than that found in Chapter 7. These are taken from the websites jpl.nasa sponsored by the Jet Propulsion Laboratory (JPL) and the National Aeronautics and Space Administration (NASA). For example, 1 AU = 149,597,890 kilometers. Use of 1 foot = 0.3048 meters (this is exact), gives 1 mile = 1.609344 kilometers. Hence 1 AU = 92,955,819 miles. Also, the sidereal year of 365.2564 days is a better estimate for the time of the completion of one orbit than the 365.2422 day long tropical year. (See "For the Instructor" for Chapter 4.) Finally, we will use the estimate $G = 6.6726 \times 10^{-11}$ in M.K.S. for the gravitational constant $G$. When an elliptical orbit of a planet, satellite, comet, or asteroid is considered, $a$ will be its semimajor axis, $b$ its semiminor axis, $e$ its linear eccentricity, $\varepsilon$ its astronomical eccentricity, $T$ its period, and $\kappa$ its Kepler constant. Recall that $e = a\varepsilon$ and $a^2 = b^2 + e^2$.

On Newton’s Verification of Kepler’s Second Law

We begin with a reformulation of Exercises 2 to 5 that illustrates Newton’s proof of Kepler’s second law more realistically. Consider the orbit of the Earth around the Sun $S$. Recall that the semimajor axis of the Earth’s orbit is $a = 1$ AU and that its astronomical eccentricity is $\varepsilon = 0.0167$. Because $a = 1$ AU, the linear eccentricity is $e = 0.0167$ AU. Let $P$ be the perihelion position of the Earth and let $Q$ be its position $t_1 = $ two months = 61 days later. Note that $SP = a - e = 0.9833$ AU. Let $\Delta t = 1$ day and let $P_1$ be the position of the Earth one day after it is at $P$. Figure 7.24a illustrates the matter. Observe, in contrast, that Figure 7.24 of the Exercises and the discussion that precedes this figure made the assumption that the orbit of the Earth is an off centered circle.

![Figure 7.24a](image)

2a. As in the original Exercise 2, since $\Delta t = 1$ day and $t_1 = 61$ days, the number of triangles inscribed in the sector $SPQ$ is 61.

3a. The area of the Earth’s elliptical orbit is orbit is $ab\pi = a\sqrt{a^2 - e^2}\pi = 1(0.99986)\pi = 3.14115$ AU$^2$. Kepler’s constant $\kappa = \frac{3.14115}{365.2564} = 0.00860$ AU$^2$ per day is the same as before. The use of $\kappa$ shows that the areas of the sectors $SPP_1$ and $SPQ$ are 0.0086 AU$^2$ and 61(0.0086) = 0.5246 AU$^2$ respectively.

4a. The region $SPP_1$ is approximately equal to a circular sector of radius $a - e = 0.9833$ AU. In fact the method of successive approximation of Sections 4.7 - 4.9 applied with $t = 1$ day
can be used to confirm that 0.9833 AU is the correct value (up to four decimal places) for the distance between the Sun to the Earth at \( P \) and hence throughout its motion from \( P \) to \( P_1 \). Assuming equality and using Exercise 3a, we get \( \frac{1}{2}(a - e)^2 \theta = 0.0086 \) and hence \( \theta = 0.0178 \) radians. As in the original Exercise 4, the area \( \Delta A \) of the triangle \( \Delta SPP_1 \) is \( \frac{1}{2}(a - e)^2 \sin \theta = 0.0086 \) AU².

5a. Combining the conclusions of Exercises 2a, 4a and 5a, we get \( n(\Delta A) = 61(0.0086) = 0.5246 = A_1 \) in AU² accurate up to four decimal places. The round off procedure explains why \( n(\Delta A) = 61(0.0086) = 0.5247 = A_1 \) in the original Exercise 5.

**About the Velocity of Planets, Comets, and Asteroids**

27. By Kepler’s second law, a planet (or comet or asteroid) attains the maximum speed \( v_{\text{max}} \) in its orbit at perihelion. Why? Consider a very short stretch of the orbit from \( P \) to \( P' \) that has the perihelion position at its midpoint and let \( \Delta t \) be the time the planet requires to trace it out. Approximate the short stretch of the orbit from \( P \) to \( P' \) by an arc of a circle of radius \( a - e \) centered at the center of the Sun \( S \). Let \( \Delta s \) be the length of this arc.

i. Use the formula for the area of a circular sector to show that the area traced out by the segment \( SP \) during the time \( \Delta t \) is approximately equal to \( \frac{1}{2}(a - e)\Delta s \).

ii. Show that \( \frac{\Delta s}{\Delta t} \approx \frac{2ab\pi}{(a-e)T} = \frac{2b\pi}{(1-e)T} = \frac{2a\pi\sqrt{1-e^2}}{(1-e)T} \).

iii. Why can you conclude that \( v_{\text{max}} = \frac{2\pi a}{T}\sqrt{1+\varepsilon} = \sqrt{\frac{GM}{a}}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \)?

28. Again by Kepler’s second law, a planet (or comet or asteroid) attains the minimum speed \( v_{\text{min}} \) in its orbit at aphelion. Proceed as in Exercise 27 to show that \( v_{\text{min}} = \frac{2\pi a}{T}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} = \sqrt{\frac{GM}{a}}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \).

Using the approximations 1 AU = 92,956,000 miles (see Section 4.7) and 1 year = 31,558,000 seconds (in current context, a year consists of 365.2564 days), we get

\[
1 \text{ AU/year} = \frac{92,956,000 \text{ miles}}{1 \text{ year}} \times \frac{1 \text{ year}}{31,558,000 \text{ seconds}} = 2.9456 \text{ miles/second}.
\]
29. Show that the maximum and minimum speeds of the Earth in its orbit are equal to 18.82 and 18.20 miles per second respectively.

30. Compute the maximum and minimum orbital speeds of Mercury and Halley in miles per second. (For Mercury, use the data provided in Section 4.7. For Halley, use \(a = 17.94\) AU, \(\varepsilon = 0.97\), and \(T = 76\) years.)

31. The figure below considers a planet in position \(P\) at the ”top” of its orbit and again at a time \(\Delta t\) later at \(Q\). The point \(C\) is the center of the ellipse. For a small \(\Delta t\), \(PQ\) is approximately equal to \(PQ'\) and the area of the elliptical sector \(SPQ\) is approximately equal to the area of the triangle \(SPQ'\). Compute Kepler’s constant in two ways to show that the velocity of the planet at \(P\) is equal to \(v_P = \frac{2\pi a}{T}\).

32. Refer to Figure 7.10 and let \(\Delta t\) be the time it takes for the planet to travel from \(P\) to \(Q\). Consider the following sequence of steps for a computation of the velocity \(v_P\) of the planet at any position \(P\) in its orbit:

\[
\frac{1}{2}(SP \times QT) \approx \frac{ab\pi}{T} \cdot SP \times QT \approx \frac{2ab\pi}{T} \cdot \frac{QP}{QT}, \quad \frac{QP}{QT} \approx \frac{2ab\pi}{T} \cdot \frac{1}{SP} \cdot \frac{QP}{QT}.
\]

i. Take limits and put \(\lim_{Q \to P} \frac{QP}{QT} = c\), to get

\[
v_P = \lim_{\Delta t \to 0} \frac{QP}{\Delta t} = \frac{2ab\pi}{T} \cdot \frac{1}{SP} \cdot \lim_{Q \to P} \frac{QP}{QT} = \frac{2abc\pi}{T} \cdot \frac{1}{SP}.
\]

In particular, the velocity of a planet at any point \(P\) in its orbit is inversely proportional to its distance \(SP\) from the Sun. The constant of proportionality is \(\frac{2abc\pi}{T}\).

ii. Is this assertion consistent with the conclusions of Exercises 31 and 27. If not, what has gone wrong?

**Comment:** See the Additional Exercises for Chapter 8 for the formula for the velocity of a planet that is valid throughout an orbit.

**Maximal Velocities of Objects in the Solar System**

Go to the site ”Computer Model of Elliptical Orbits Generated by Kepler’s Equations” attached to the website for Chapter 4. Experiment with the simulations of the orbits of the inner, outer
planets, and the comets and asteroids for which the relevant data is provided. It turns out, perhaps surprisingly, that there is a "speed limit" on the velocity of the planets, comets, or asteroid in the solar system.

33. Consider any object in orbit around the Sun.

   i. Show that $v_{\text{max}} = \frac{2\pi \sqrt{1 + \varepsilon}}{\sqrt{a \sqrt{1 - \varepsilon}}} = \frac{2\pi \sqrt{1 + \varepsilon}}{\sqrt{a - \varepsilon}}$ AUs/year. [Hint: What does Kepler’s third law tell us about the relationship between $a$ and $T$ in these units? Use Exercise 27 iii.]

   Recall that $a - e = a(1 - \varepsilon)$, the distance at perihelion, is the minimum distance of the object from the Sun. Because $1 \leq \sqrt{1 + \varepsilon} \leq \sqrt{2}$, $v_{\text{max}}$ depends primarily on this minimum distance. The smaller the minimum distance, the larger the corresponding $v_{\text{max}}$.

   ii. Assuming that the object is not on collision course with the Sun, $a - e$ needs to be greater than the radius of the Sun. Use the fact that this radius is 432,000 miles (see Chapter 1.5), to show that the maximum speed of any orbiting object is less than 384 miles per second.

34. Consider the term $a - e = a(1 - \varepsilon)$ for the planets, comets, and asteroids that are featured on the site "Computer Model ... ". The parameters $a$ and $\varepsilon$ are provided. For the planets, just click on their pictorial representations. By a combination of inspection and computation, confirm that the comet Hyakutake has the greatest maximum speed of any orbiting object on the site. The $a$ and $\varepsilon$ for this comet are respectively $a = 25.74$ AUs and $\varepsilon = 0.9998$.

   (The latter is much more accurate than the figure $\varepsilon = 0.99$ provided on the site.) Use these data to compute the maximum speed of Hyakutake. How much below the "speed limit" is its maximum speed?

About the Moon

The website

http://sse.jpl.nasa.gov/planets/profile.cfm?Object=Moon

provides the following information about our Moon’s orbit. The average distance of the Moon from the Earth is 384,401 kilometers (or 238,856 miles), the eccentricity of the orbit is $\varepsilon = 0.05490049$, its period is $T = 27.321661$ days, and the average orbital velocity of the Moon is 1.023 kilometers (or 0.6357 miles) per second. In the exercises below take $a = 238,900$ miles, $\varepsilon = 0.0549$, and $T = 27.32$ days.

35. You are looking out at the rising Moon on a cloudless night. You observe it change position relative to the horizon. Does the Earth’s rotation or the Moon’s own motion have the greater effect on this change of position?
36. Estimate the maximum and minimum distances of the Moon from the Earth in miles. Use the formulas $v_{\text{max}} = \frac{2\pi a}{T} \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ and $v_{\text{min}} = \frac{2\pi a}{T} \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$ to estimate the maximum and minimum orbital velocities of the Moon in miles per second.

37. Consider Figures 7.20 and 7.21 of the text. The fact that the distance between the Earth and the Moon varies, means that the distance between the center $C$ of the Earth and the barycenter $B$ of the Earth-Moon system varies also. Use the results of Exercise 36 and refer to the solution of Exercise 16 to estimate the variation in the distance between $C$ and $B$.

**Earth Satellites**

This group of exercises focuses on artificial satellites of the Earth. As was already observed in Chapter 1 (see Note 9), the Earth is a sphere that is slightly flattened at the poles. So the distance from the Earth’s Center to its surface varies slightly. The average radius at the equator is 6,378 kilometers and the radius at the poles is 6,357 kilometers. For the exercises that follow, take the overall average of 6,371 kilometers. Precise measurements of the sizes and periods of the orbits of scientific satellites (such as LAGEOS) have shown that Kepler’s constant $K_E = \frac{a^3}{T^2}$ for bodies orbiting the Earth is $1.0097 \times 10^{13}$ in M.K.S. Inserting the estimate $G = 6.6726 \times 10^{-11}$ into the formula $K_E = \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$ in Section 7.6, this provides the estimate

$$M = \frac{4\pi^2}{G} \cdot \frac{T^2}{a^3} = 5.9742 \times 10^{24} \text{ kilograms}$$

for the mass of the Earth.

38. The launching of the satellite Sputnik 1 by the Soviet Union on October 4, 1957 inaugurated the Space Age. This exercise will study it using data that is more accurate and complete than that supplied by Exercise 22. In Sputnik’s elliptical orbit, its distance from the Earth’s surface ranged from 230 kilometers to 942 kilometers. It circled the Earth once every 96 minutes and had a mass of 83.6 kilograms. It remained in orbit until early in 1958 when it burned up in the Earth’s atmosphere.

i. Assume that Sputnik 1 orbited over the Earth’s equator and compute its orbital data $a$, $\varepsilon$, and $\kappa$ in M.K.S. Calculate the maximum and minimum orbital velocities of Sputnik in kilometers per second.

ii. Compute Sputnik’s weight in newtons, taking $g = 9.80 \text{ meters/second}^2$. Provide both an upper and lower estimate for the Earth’s gravitational force (again in newtons) on the orbiting Sputnik.

iii. Your estimates in (ii) should both be less than Sputnik’s weight. Now turn to Exercise 25 and its solution. Notice that the force required to keep the twirled object on its circular path is much greater than its weight! How do you explain this apparent contradiction?
39. The satellite Explorer 1, launched from Cape Kennedy (then known as Cape Canaveral) in Florida on January 31, 1958, was our American response to Sputnik 1. Its orbit took it from a distance of 360 kilometers to a distance of 2,534 kilometers above the Earth’s surface. Its period was 114.9 minutes. About 60% of its mass of 13.92 kilograms consisted of instruments. This included a cosmic-ray detection package, temperature sensors, a micrometeorite impact microphone, and micrometeorite erosion gauges. The data collected by these instruments were transmitted back to Earth. Repeat parts (i) and (ii) of Exercise 38 for Explorer 1.

40. An artificial satellite that has been spectacularly successful is the Hubble Space Telescope. It was put into orbit in 1990 by the space shuttle. The size of a large schoolbus, it has a mass of 11,110 kilograms. Its orbit is circular at a distance of 612 kilometers above the Earth’s surface and the plane of the orbit makes an angle of 28.5° with the plane of the Earth’s equator. Hubble orbits the Earth once every 97 minutes. It has taken an array of spectacular images of objects in the solar system and beyond. Explore the site http://hubble.nasa.gov/ for a look.

41. The International Space Station ...

42. Suppose that a moving object \( P \) is propelled by a single force that is centripetal in the direction of a fixed point \( S \). So by one of Newton’s conclusions, the segment \( SP \) sweeps out equal areas in equal times. Suppose the orbit is an ellipse and that the segment \( SP \) also sweeps out equal angles in equal times. Show that the orbit must be a circle. [Hint: Make use of Figure 4.38 in the Exercises Section of Chapter 4.]

43. A communications satellite is placed into an orbit that keeps it over a designated fixed point on the Earth’s equator. This is referred to as a “geosynchronous orbit.” In such an orbit it orbits the Earth exactly once every 24 hours. Show that the orbit of such a satellite must be a circle and that its radius must be about 42,000 kilometers.

44. The world’s Global Positioning System (GPS) is an interesting application of satellite technology. The system consists of 24 satellites. The 24 satellites of the original system were launched between 1989 and 1994. (As older satellites are being replaced by newer ones, there are often more than 24 in orbit.) The 24 satellites are carefully spaced in the following configuration. They circle the Earth in six different orbital planes. These planes are equally spaced and each makes an angle of about 55° with the plane of the Earth’s equator. In each of the six planes...
there are four satellites in orbit. All orbits are circles with the Earth at the center. The satellites are equally spaced in each plane and complete one orbit in 12 hours. The 24 satellites are staggered relative to each other in such a way that they are spread out uniformly as they speed high over the surface of the Earth. The path of every satellite of the system is carefully monitored by ground stations. The figure below illustrates what has been described. Why do all the satellites move on the same sphere with center the center of the Earth? What is the radius of this sphere?

This system allows users who carry a GPS receiver - whether they are sailing somewhere in the Atlantic or exploring the jungles of the Amazon - to obtain precise coordinates of their positions. Here is how. From any point on Earth there are at any time between five and eight (and almost always at least six) satellites within range. The radio signals that each of the satellites emits, allows the receiver to determine the position of the satellite and its distance from the receiver. Suppose for instance that at a given time four satellites are within range of the receiver $R$. Suppose they are in positions $P_1, P_2, P_3,$ and $P_4$ in their orbits and that their distances from the receiver are $d_1, d_2, d_3,$ and $d_4$. So the receiver lies simultaneously on four spheres. Since the intersection of two spheres is a circle (sit back and picture this), it follows that $R$ lies at the intersection of two circles. So there are only two different possibilities for the position of $R$. Because one of these can usually be ruled out (GPS receivers have mathematical methods of eliminating impossible locations), the location of $R$ is determined. See the site

http://www.colorado.edu/geography/gcraft/notes/gps/gps_f.html

for more information.

45. The N.Y. Times article "SpaceElevator" (go to the site attached to Chapter 7) engages in a futuristic speculation about how to get to space more economically (than via the currently used rockets and shuttles). According to one of the assertions of the article a satellite in
geosynchronous orbit circles the Earth about 22,300 miles, or 35,880 kilometers, above its surface. Is this consistent with the conclusion of Exercise 44?

**Space Probes**

A wealth of information about our solar system, especially some of the distant planets, has been gathered by space probes such as Voyager, Galileo, and Cassini. See http://www.jpl.nasa.gov/missions/ for an overview of current NASA missions. This section will use Newton’s equations to develop information about the trajectories of the Voyager probes in the solar system.

Suppose a space probe $V$ has left the realm of the inner planets and is heading towards the farther reaches of the solar system. The probe $V$ is far from the Sun and any planet. The fact that its mass is relatively small means that the gravitational forces acting on $V$ are negligible. By Newton’s second law, the probe now travels in a straight line with constant velocity. Let’s say that after some time its path brings it into the ”gravitational neighborhood” of an outer planet. At this point two motions will occur simultaneously. On the one hand, the center of mass of the ”planet-probe system” (more accurately, of the system consisting of the planet, its satellites, and the probe) will be in elliptical orbit around the Sun in the same way that this happens for the Earth-Moon system. Given the relatively small mass of the probe (and that of any satellite the planet might have), the center of mass of this system is in essence the center of mass of the planet. On the other hand, relative to this center of mass the gravitational force of the planet will act as a centripetal force on the probe. By one of Newton’s conclusions, the segment from the probe to the planet now sweeps out equal areas in equal times. As the probe draws nearer, the planet’s gravitational force begins to bend the probe’s path. It turns out that the probe is deflected along a hyperbolic arc around the planet with the center of the planet at a focal point of the hyperbola. The cameras and sensors on $V$ can now begin to study the planet. If the probe’s thrusters are fired to slow it down, it can be brought into orbit around the planet.

46. On April 11, 2000, the New York Times (in the Science Times Section) reported that the craft NEAR Shoemaker, guided by gentle shoves from small thruster rockets, was taking up a circular orbit of 99.8 kilometers from the center of the asteroid Eros in order to conduct a study of the asteroid. The 772 kilogram craft had been named NEAR (Near Earth Asteroid Rendezvouz) Shoemaker in honor of Eugene Shoemaker, a pioneer in the study of asteroids and comets. Eros is one of the larger asteroids, measuring about 34 kilometers long and 13 kilometers thick. It is a potato-shaped rock covered with craters with a large gouge in the center. Instead of turning on its long axis it rolls end over end. Eros’s gravitational field is so slight that the space craft must keep its speed down to 4.8 kilometers per hour in order to stay in orbit. Can you estimate the mass of Eros? Can you estimate its density?

The next example shows that natural satellites can provide such information as well.
On October 12, 1999, the New York Times (in the Science Times Section) reported that a team of astronomers using a telescope on Mauna Kea, Hawaii, had discovered a satellite orbiting the asteroid Eugenia (in the main asteroid belt between Mars and Jupiter). The moon is about 13 kilometers across, and orbits Eugenia at a distance of 1130 kilometers once every 4.7 days. Use this information to estimate the mass of Eugenia.

When no thrusters are fired, the probe $V$ will continue on its hyperbolic trajectory, fly past the planet and push on. Such "fly-bys" produce a gravitational "sling shot effect" often designed to propel the probe towards the next object to be studied. NASA used this strategy to send the Voyager probes on "planet hopping" fact finding missions of the solar system. See http://voyager.jpl.nasa.gov/index.html for the homepage of the probes Voyager 1 and 2.

Suppose a probe $V$ is on a hyperbolic fly-by of a planet. Let $M$ be the mass of the planet and let $P$ be the planet’s center. Let

\[ K_P = \frac{GM}{4\pi^2} \]

be Kepler’s constant for satellites orbiting this planet (in M.K.S. for example). See Section 7.6. Let $\kappa$ be the Kepler constant of the probe’s trajectory.

Place an $x$-$y$ coordinate system into the plane determine by the planet and the probe’s path in such a way that that the equation of the hyperbola has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where both $a$ and $b$
are positive. The solution of Exercise 26 of Chapter 4 tells us how this equation is derived and that the \( x \)-coordinates of the focal points of this hyperbola are \(-e\) and \(e\) with \(e = \sqrt{a^2 + b^2}\). The graph of the equation is provided below. Review the study of the hyperbola that concludes Section 5.1. Recall, in particular, that the lines \(y = \frac{b}{a}x\) and \(y = -\frac{b}{a}x\) determined by the "box" in the figure are the asymptotes of the hyperbola. Assume that the right branch of the hyperbola represents the trajectory of the probe. So the center \(P\) of the planet is the focus \((e, 0)\). The number \(a\) is the semimajor axis and \(\varepsilon = \frac{e}{a}\) is the astronomical eccentricity of the hyperbola. The semimajor axis and the eccentricity determine the hyperbola, for if \(a\) and \(\varepsilon\) are given, then one can solve for \(b\).

Notice that the situation is analogous to that of the ellipse (as discussed in Section 4.5).

48. Show that the latus rectum of the hyperbola is \(L = \frac{2b^2}{a}\). Combine the formulas \(F = \frac{8c^2m}{L_{\text{tip}}}\) and \(F = GMm/r^2\) to show that

\[
\kappa = \sqrt{GMa \cdot b^2/2}.
\]

49. Let \(Q = (x, y)\) be any point on the asymptote \(y = \frac{b}{a}x\) and let \(d\) be the distance from this point to the focus \(P = (e, 0)\). Express \(d^2\) as a function of \(x\) and show that \(d^2\) is a minimum for \(Q = (\frac{a}{\varepsilon}, \frac{b}{\varepsilon})\). Conclude that the length of the perpendicular from the asymptote to the focus \(P = (e, 0)\) is equal to \(b\). Show that the velocity \(v\) of the probe when it is still some distance from the planet \(P\) satisfies

\[
v \approx \frac{2\kappa}{b} = \sqrt{\frac{GM}{a}}.
\]

[Hint: Regard the probe \(V\) to be traveling along the asymptote for a short time \(\Delta t\) and compute the area traced out by the segment \(VP\).]

50. Why does the probe reach its maximum velocity \(v_{\text{max}}\) at the instant it is closest to the planet \(P\)? Show that

\[
v_{\text{max}} = \frac{2\kappa}{a(\varepsilon - 1)} = \sqrt{\frac{GM}{a}} \sqrt{\frac{\varepsilon + 1}{\varepsilon - 1}}.
\]

[Hint: Approximate the probe’s path near the point of closest approach by a small circular arc of radius \(e - a\) and proceed as in Exercise 27.]

Let’s return to our space probe \(V\) on approach to a planet of mass \(M\). Suppose that \(V\) is far enough away from the planet, so that relative to the planet, it travels in essentially a straight line at constant velocity. A sequence of three position measurements taken over some interval of time \(t\) confirms that the probe has traced out a certain straight path \(D\) at a constant speed \(v\). So the length of \(D\) is equal to \(d = vt\). In time, \(V\) will be close enough, the planet’s gravity will act as centripetal force on \(V\), and the hyperbolic bend in the path will begin. If the position of the segment \(D\) is understood relative to the center \(P\) of the planet, then the hyperbolic path of the probe around \(P\) can be predicted.

Here is a way to see how. On a sheet of paper draw \(D\) with a straight edge. Make \(D\) one centimeter long, so that your map of the trajectory will be scaled at \(d = 1\) cm. Put in the center
of the planet. Because an asymptote closely approximates the hyperbola away from its focus, we will regard the segment $D$ to lie on an asymptote of a hyperbola. Extend the segment $D$ to a straight line $L$ in the general direction of $P$. Carefully draw in the perpendicular segment from $P$ to $L$. By Exercise 48, the length of this perpendicular segment is equal to the parameter $b$ of the hyperbola. By the same exercise, Kepler’s constant of the probe can be approximated by $\kappa = \frac{1}{2} bv$. By Exercise 47, the semimajor axis $a$ for the hyperbola is $a = \frac{GMb^2}{4\pi^2}$. Now compute $e$ by making use of $e^2 = a^2 + b^2$. Knowing $e$ allows us choose a point $O$ on $L$ at a distance $e$ from $P$ that will serve as the origin of an $x$-$y$ coordinate system. (To see why, refer to the sketch of the hyperbola on the previous page.) The $x$-axis is determined by the line through $O$ and $P$. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ pins down the hyperbolic trajectory of the probe. Put in the $y$ coordinates corresponding to $x = e$, $x = 2a$, $x = 3a$, $\cdots$, and you will be able to draw in the probe’s hyperbolic path.

Jupiter: Satellites, Galileo, and the Voyagers

Jupiter is the largest planet of the solar system. It has the shape of a flattened sphere. It bulges out at the equator, because it spins rapidly about its axis (about $2\frac{1}{2}$ revolutions per day). Its equatorial diameter is 143,000 kilometers. Its mass is $1.8987 \times 10^{27}$ kilograms. The four satellites that Galileo discovered now have the names Io, Europa, Ganymede, and Callisto. The modern versions of their average distances from Jupiter are 421,600, 670,900, 1,070,000, and 1,883,000 kilometers, respectively. Their diameters (in the same order) are 3,642, 3,130, 5,268, and 4,806 kilometers. Their orbital periods are respectively, 1.769, 3.551, 7.155, and 16.689 Earth days. The astronomical eccentricities of their very nearly circular orbits are respectively, 0.004, 0.009, 0.002, and 0.007. Since the time of Galileo, more than two dozen additional satellites of Jupiter have been discovered. These are all small, however. Himalia with a diameter of 170 kilometers is the largest. The semimajor axis of Jupiter’s orbit around the Sun is 778,412,020 kilometers. Refer to the websites

http://www.jpl.nasa.gov/solar_system/planets/jupiter_index.html

http://solarsystem.nasa.gov/planets/profile.cfm?Object=Jupiter

for much more information about Jupiter and its satellites.

51. Study the information that the N.Y. Times article ”GalileoProbe” provides about Jupiter and its moons.

52. Use the modern data provided above to assess the accuracy of Newton’s data for the moons of Jupiter (as supplied in Exercise 10) and to verify Kepler’s third law. What estimate for the mass of Jupiter does Newton’s version of Kepler’s third law provide?
We now turn to the space probes Voyager 1 and Voyager 2. They are identical craft with a mass at launch of 815 kg. They have in essence the weight and size of a small car. The current approximate weight of Voyager 1 is 733 kg and that of Voyager 2 is 735 kg. The difference is the total amount of fuel available for the spacecrafts’ thrusters. A set of small thrusters provides the Voyagers with the capability for attitude control and trajectory correction. Each of these tiny assemblies generates a thrust of only about 0.834 newtons. By comparison, if you have a stack of 10 pennies in your palm, they will push down on your hand with a force of about 0.270 newtons. The above information (and much more) about the Voyagers can be found on the Voyager home page

http://voyager.jpl.nasa.gov/index.html

See also the FAQ page on this site.

53. Take the average value of 775 kilograms for the mass of a Voyager craft. Compute its weight. Use $1.989 \times 10^{30}$ kg for the mass of the Sun and estimate the gravitational force of the Sun on Voyager when it was as far away from the Sun as Jupiter’s average distance from the Sun.

54. Estimate the gravitational pull of Jupiter on one of the Voyagers when it’s distance from Jupiter was 20 million kilometers, 10 million kilometers, 2 million kilometers, and 750,000 kilometers. Approximately how far from Jupiter does the Voyagers’ “gravitational neighborhood” begin?

55. Compute the orbit constant $K_J = \frac{GM}{4\pi^2}$ for Jupiter in M.K.S.

For the data about the trajectories of both Voyager probes refer to


We will now focus on Voyager 1. This craft was in hyperbolic flyby of Jupiter in February and March of 1979. Its closest approach occurred on March 5, 1979. The parameters for the hyperbolic trajectory of the fly-by are $a = 1,092,356$ kilometers and $\varepsilon = 1.318976$. [For technical reasons, the semimajor axes of hyperbolic trajectories on this site are listed as negative numbers.]

The solutions of some of the exercises below require the formulas developed in the section ”Space Probes.”

56. Compute the following in M.K.S. for the hyperbolic fly-by of Jupiter by Voyager 1. The parameters $e$ and $b$ and Kepler’s constant $\kappa$. How close did Voyager get to Jupiter’s center of mass? To its surface?

57. Let 1 centimeter correspond to 1 million kilometers and choose an $x$-$y$ coordinate system so that the path of Voyager 1 corresponds to the right half of the graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. 

12
i. Take a blank sheet of paper and put the longer side horizontally and have a sharp pencil ready. (A compass would add to the precision of your effort.) Draw a $2a \times 2b$ rectangular box and place the origin at the point where the diagonals intersect. Draw in the asymptotes $y = \pm \frac{b}{a}x$, the $x$-axis, and place Jupiter at the point $J = (e, 0)$.

ii. Put in the $y$ coordinates corresponding to $x = e$ (these are the endpoints of the latus rectum), $x = 2a$, $x = 4a$, and $x = 6a$. Sketch Voyager 1’s hyperbolic path.

58. Estimate the velocity that Voyager 1 had just before it approached Jupiter’s “gravitational neighborhood.”

59. Estimate Voyager 1’s maximum velocity. Is your answer consistent with the assertion (on the Voyager website) that “Voyager used the enormous gravity field of Jupiter to be hurled on to Saturn, experiencing a Sun-relative speed increase of roughly 35,700 mph.”?

The space probe Voyager 2 had its closest approach to Jupiter on July 9, 1979. The following information is consistent with the Voyagers orbital data. When it was about 20 million kilometers from Jupiter (as it approached Jupiter’s ”gravitational neighborhood”) it moved (relative to Jupiter) along a straight line at a speed of about 7,600 meters per second (or 27,360 kilometers per hour). If Voyager 2 would have continued on this straight path, its closest approach to Jupiter’s center of mass would have been about 1,920,000 kilometers.

60. Use the information provided about Voyager 2’s fly-by of Jupiter to provide estimates for
   i. The orbital parameter $b$.
   ii. The Kepler constant $\kappa$.
   iii. The orbital parameters $a$ and $e$ and Voyager 2’s maximum speed relative to Jupiter.

61. Using a scale of $1 \text{ cm} = 1,000,000 \text{ kilometers}$, sketch Voyager 2’s hyperbolic fly-by of Jupiter by following the directives at the end of the Exercise Section ”Space Probes.”

Voyager 1 had its closest approach to Saturn on November 12, 1980. It then left the orbital plane of the planets, but it is still going strong and sending back information. It was 90 AUs from the Sun on November 5, 2003 and is moving out of the solar system with a speed of 3.6 AU per year. It is the most distant man made object in the universe. Voyager 2 reached its closest approach of Saturn on August 25, 1981. It then continued to Uranus (the closest approach occurred on January 24, 1986) and Neptune (with closest approach on on August 25, 1989). Voyager 2 is currently escaping the solar system at a speed of about 3.3 AU per year.

**Saturn: Satellites, Cassini, and the Voyagers**

Refer to the websites

http://www.jpl.nasa.gov/solar_system/planets/saturn_index.html
http://solarsystem.nasa.gov/planets/profile.cfm?Object=Saturn

for basic information about Saturn and its satellites.

62. Estimate the mass of Saturn by making use of the orbital data provided for some of its satellites. Compare this with the mass listed on the sites.

The websites supplied earlier provide detailed information about the Voyagers hyperbolic fly-bys of Saturn. The diagram below depicts Voyager 1’s flyby of Saturn. It was scanned from page 359 of the volume


The diagram shows the hyperbolic flyby as seen from ”above” the orbital plane of Saturn’s rings and satellites. The inner disc in the diagram represents Saturn. The white part shows the half of Saturn that is lit by the Sun and the dark part the half that is dark. Moving outward from Saturn, the diagram depicts Saturn’s rings as well as the shadow that Saturn casts in their direction. Saturn’s satellites are placed at the points of Voyager’s closest approach to them. E and M designate the satellites Enceladus and Mimas respectively. The negative hours listed along the trajectory indicate the time from that position to the point of Voyager’s closest approach, 0 labels the instant of closest approach, and the positive hours indicate the elapsed time from the point of closest approach to that position.

63. Look up the relevant data for the Saturn flybys of the Voyagers and repeat Exercises 56 to 61.
Tossing the Hammer

The hammer throw has been an Olympic event since 1896. A 16 pound metal ball is attached to a 4 ft piece of wire and the wire in turn is attached to a handle. (The weights of the wire and the handle are negligible compared to the ball.) The thrower grasps the handle with both hands and swings the ball in a circular arc with ever increasing velocity, extending his arms and swinging his body around in the process. This is done in such a way that the plane of the circle traced out by the ball rises in the direction of the throw. All this occurs within a throwing circle of 7 feet in diameter. The figure shows the thrower in action with the hammer in the throwing circle. When the thrower thinks that he has imparted on the ball the largest possible velocity and an appropriate angle of elevation, he releases the handle, and the ball (along with the wire and the handle) flies off. For a throw to be valid, the thrower has to remain within the circle throughout and the ball has to land within a specified wedge shaped area. The official distance of a throw is the distance from the point of impact of the ball to the circle (the circle, not its center). The figure above depicts the throwing circle, the throw and the distance measurement, as well as the point $P$ from which the ball flies off, the point $I$ of impact of the ball, and the measured distance $IC$.

64. The world record in the hammer throw is currently 284 feet and 7 inches and was set by the Russian Yuri Sedykh in 1986. (This was the state of affairs as of March 2002.) Estimate the outward force that the ball exerts on Yuri just before he releases the ball. [Hint: You might proceed as follows. Use the estimate (supplied by an expert in the College of Engineering at Notre Dame) that the ball would travel about 5% farther in an environment without air resistance. Next, provide reasonable data for the instant Yuri releases the ball on his record setting throw: the radius of the ball’s circular arc, the height of the ball above the ground, and the angle of the ball’s initial trajectory with the horizontal. Once you have thought about this and made your choices, use information from Section 6.5 to estimate the speed of the ball at the beginning of its flight. Finally, refer to the Solution of Exercise 25.]
65. The Olympic record of 278 feet and 2 inches was set by Sergei Litvinov in 1992. (This was the state of affairs before the 2004 Olympics.) Repeat Exercise 64 for this gold medal performance.

66. The best throw of the year 2001 was the 273 feet and 10 inches recorded by Koji Murofushi of Japan. Repeat Exercise 64 for it.

Questions: Was gravity considered in Exercises 64 to 66? If not, should it have been? See the discussion at the end of the For the Instructor page for Chapter 7. How should this discussion be modified to capture the current situation?