Automatica 45 (2009) 2857–2863

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Interconnection topologies for multi-agent coordination under leader–follower framework[☆]

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ARTICLE INFO

Article history:
Received 10 November 2008
Received in revised form
25 June 2009
Accepted 7 August 2009
Available online 4 October 2009

Keywords:
Multi-agent systems
Controllability
Local interactions
Leader-follower structure

ABSTRACT

In this paper, the formation control problem of the network of multiple agents is studied in terms of controllability, where the network is of the leader–follower structure with some agents taking leaders role and others being followers interconnected via the neighbor-based rule. It is shown that the controllability of a multi-agent system can be uniquely determined by the topology structure of interconnection graph, for which the investigation comes down to that for a multi-agent system with the interconnection graph being connected. Based on these observations, two kinds of uncontrollable interconnection topologies are characterized, and a necessary and sufficient eigenvector-based condition is presented. Our studies also touch upon the selection of leaders.

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1. Introduction

Recently, the collective behavior in swarms of entities in the real world has inspired intensive study of networked systems (Bliman & Ferrari-Trecate, 2008; Hong, Gao, Cheng, & Hu, 2007; Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004; Ren, Beard, & Atkins, 2007). Understanding the cooperative and operational principles of such systems may facilitate the development of the formation control of unmanned air and underwater vehicles, satellite clusters etc. Formation control problem has been studied from various perspectives (see e.g. Fax & Murray, 2004; Ji, Lin, & Lee, 2008; Lin, Francis, & Maggiore, 2005; Lozano, Spong, Guerrero, & Chopra, 2008; Rahmani & Mesbahi, 2006; Tanner, 2004; Yu, Hendrickx, Fidan, & Anderson, 2007). In Tanner (2004), the concept of controllability was put forward for the first time for formation control of multi-agent systems. The main idea is to transform the formation control into a classical controllability problem for fixed topology as well as a switched controllability problem for switching topology. To date, few results have been available along these lines. In Ji and Egerstedt (2007), the controllability was characterized by graph theory. The controllability problem was studied under fixed and switching topologies in Liu, Xie, Chu, and Wang (2006) for the continuous-time case, and in Liu, Chu, Wang, and Xie (2008) for the discrete-time case. Different from the classical control, the dynamical behavior of networked systems relies heavily on how the network is connected, i.e., its topology structure. In particular, how the controllability is affected by the interconnection topology structure among agents remains a fundamental problem, and the corresponding investigation is at the very outset. Accordingly, the properties of interconnection topology structures call for specific investigation for the controllability problem. This motivates the present study.

In this paper, we consider a multi-agent system of the leader–follower structure, where some agents take the leaders' role and others are followers interconnected via the neighbor-based rule. The leaders are unaffected by followers and do not abide by the agreement protocol, whereas the followers are influenced by leaders directly or indirectly. We show that the controllability can be uniquely determined by the interconnection topology. A necessary and sufficient condition is then derived by dividing the overall system into several connected components. The result leads to the simplification of controllability for the investigation of that on a connected interconnection graph. Finally, two kinds of topology structures are constructed to identify the uncontrollability of networks. Note that the results in Ji et al. (2008) were expressed in terms of eigenvalues and eigenvectors

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hideaki Ishii under the direction of Editor Ian R. Peterson.

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doi:10.1016/j.automatica.2009.09.002

Proposition 1. *The controllability of multi-agent system (1) is invariant under any labeling of the nodes in interconnection graph $\mathcal{G}$ if the interconnection topology of $\mathcal{G}$ and the leader positions in $\mathcal{G}$ are fixed.*

Proof. Denote by $v_1, \ldots, v_{N+n_l}$ the nodes in $\mathcal{G}$, and $v_{N+1}, \ldots, v_{N+n_l}$ the leaders. Let $i_1, \ldots, i_{N+n_l}$ be a permutation of $1, \ldots, N+n_l$ and v_j be relabeled as v_{i_j} in $\mathcal{G}'$. Then $\mathcal{G}$ and $\mathcal{G}'$ have the same topology structure. Let $\mathcal{L}$ and $\mathcal{L}'$ be the Laplacian matrix of $\mathcal{G}$ and $\mathcal{G}'$, respectively. One has $\mathcal{L}' = P\mathcal{L}P^{\mathrm{T}}$, where $P = [e_{i_1}, \ldots, e_{i_N}, \ldots, e_{i_{N+n_l}}]^{\mathrm{T}}$, e_{i_j} is the i_jth identity vector of dimension $N+n_l$. By definition, the system matrices of $\mathcal{G}$ and $\mathcal{G}'$ are, respectively, $\mathcal{F} = E\mathcal{L}E^{\mathrm{T}}$, $\mathcal{R} = E\mathcal{L}T$; and $\mathcal{F}' = E'\mathcal{L}'E'^{\mathrm{T}}$, $\mathcal{R}' = E'\mathcal{L}'T'$, with $E = [e_1, \ldots, e_N]^{\mathrm{T}}$, $T = [e_{N+1}, \ldots, e_{N+n_l}]$; $E' = [e_{i_1}, \ldots, e_{i_N}]^{\mathrm{T}}$, $T' = [e_{i_{N+1}}, \ldots, e_{i_{N+n_l}}]$. So $\mathcal{F}' = E'P\mathcal{L}P^{\mathrm{T}}E'^{\mathrm{T}}$. Since $\mathcal{G}'$ has the same topology structure and leader positions as those of $\mathcal{G}$, there exist permutation matrices W of $N \times N$ and V of $n_l \times n_l$ such that $E = WE'P$, and $W\mathcal{R}' = \mathcal{R}V$. So $\mathcal{F} = WE'P\mathcal{L}P^{\mathrm{T}}E'^{\mathrm{T}}W^{\mathrm{T}} = W\mathcal{F}'W^{\mathrm{T}}$. This yields

$$\begin{aligned}\mathcal{C} &= W\left[W^{\mathrm{T}}\mathcal{R}V, \mathcal{F}'W^{\mathrm{T}}\mathcal{R}V, \ldots, \mathcal{F}'^{N-1}W^{\mathrm{T}}\mathcal{R}V\right] \\ &\quad \times \mathrm{diag}\left\{V^{\mathrm{T}}, \ldots, V^{\mathrm{T}}\right\} \\ &= W\mathcal{C}'\mathrm{diag}\left\{V^{\mathrm{T}}, \ldots, V^{\mathrm{T}}\right\},\end{aligned}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are controllability matrices of the multi-agent system associated with $\mathcal{G}$ and $\mathcal{G}'$, respectively. Since both W and V are permutation matrices, $\mathrm{rank}\,\mathcal{C} = \mathrm{rank}\,\mathcal{C}'$. This completes the proof. □

Let $\mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(\delta)}$ stand for the δ connected components of $\mathcal{G}$. Below is a requirement for leader selection.

Principle 1 (Ji et al., 2008). *For each connected component $\mathcal{G}^{(i)}$, the node set of the leader subgraph $\mathcal{G}_l$ contains at least one node of $\mathcal{G}^{(i)}$, $i = 1, \ldots, \delta$.*

If leaders are not selected as above, the necessary condition on controllability cannot be fulfilled (Ji et al., 2008). After the selection of leaders, each connected component $\mathcal{G}^{(i)}$ can be partitioned into two subgraphs: $\mathcal{G}_l^{(i)}$ and $\mathcal{G}_f^{(i)}$, with $\mathcal{G}_l^{(i)}$, $\mathcal{G}_f^{(i)}$ being, respectively, the induced leader and follower subgraph of $\mathcal{G}^{(i)}$. Denote by $\mathcal{L}_{i_1,\ldots,i_m}$ the principal submatrix obtained by selecting the i_1th, …, i_mth rows and columns of $\mathcal{L}$, and assume that $\mathcal{G}_f^{(i)}$ is on the node set $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$, with $n_0 = 0$, $n_\delta = N$, $i = 1, \ldots, \delta$. The following assertion is a combination of Lemmas 1, 2 in Ji et al. (2008).

Lemma 1. *$\mathcal{L}_{1,\ldots,N}$ is positive definite and*

$$\mathcal{L}_{1,\ldots,N} = \mathrm{diag}\left\{\mathcal{L}_{1,\ldots,n_1}, \mathcal{L}_{n_1+1,\ldots,n_2}, \ldots, \mathcal{L}_{n_{\delta-1}+1,\ldots,N}\right\},$$

where $\mathcal{L}_{1,\ldots,n_1}, \mathcal{L}_{n_1+1,\ldots,n_2}, \ldots, \mathcal{L}_{n_{\delta-1}+1,\ldots,N}$ are all positive definite submatrices too.

Theorem 1. *The multi-agent system (1) is controllable under fixed topology and leaders $x_{N+1}, \ldots, x_{N+n_l}$ if and only if each connected component $\mathcal{G}^{(i)}$ is controllable, $i = 1, \ldots, \delta$.*

Proof. Without loss of generality, we assume that $\delta = 3$, i.e., $\mathcal{G}$ consists of three connected components.

Let $\{v_1, \ldots, v_{n_3}\}$ and $\{v_{n_3+1}, \ldots, v_{n_6}\}$ represent the follower and leader node set of $\mathcal{G}$, respectively, with $n_3 = N$, $n_6 = N + n_l$. The follower node set can be partitioned into three parts in accordance with the three connected components, so is the leader node set. More specifically, since each component $\mathcal{G}^{(i)}$ consists of a leader subgraph $\mathcal{G}_l^{(i)}$ and a follower subgraph $\mathcal{G}_f^{(i)}$, $i = 1, 2, 3$; it can be assumed that $\{v_1, \ldots, v_{n_1}\}$ and $\{v_{n_3+1}, \ldots, v_{n_4}\}$ are, respectively, the follower and leader node set of $\mathcal{G}^{(1)}$; and $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$ and $\{v_{n_{i+2}+1}, \ldots, v_{n_{i+3}}\}$ are the follower and leader node set of $\mathcal{G}^{(i)}$, respectively, $i = 2, 3$; where the indices satisfy $1 < n_1 < n_2 < n_3$ and $n_3 < n_4 < n_5 < n_6$ with respect to the follower and leader node set, respectively. It follows from Lemma 1 that $\mathcal{F} = \mathrm{diag}\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$, where $\mathcal{F}_i = \mathcal{L}_{n_{i-1}+1,\ldots,n_i}$, $i = 1, 2, 3$; $n_0 = 0, n_3 = N$. Set $\widetilde{n}_i \triangleq n_i - n_{i-1}$, $i = 1, \ldots, 2\delta$; the control input matrix can be correspondingly partitioned as $\mathcal{R} = [\mathcal{R}_1^{\mathrm{T}}, \mathcal{R}_2^{\mathrm{T}}, \mathcal{R}_3^{\mathrm{T}}]^{\mathrm{T}}$, with $\mathcal{R}_1 = [\mathcal{R}_{11}, 0_{\widetilde{n}_1\times\widetilde{n}_5}, 0_{\widetilde{n}_1\times\widetilde{n}_6}]$, $\mathcal{R}_{11} : \widetilde{n}_1 \times \widetilde{n}_4$; $\mathcal{R}_2 = [0_{\widetilde{n}_2\times\widetilde{n}_4}, \mathcal{R}_{22}, 0_{\widetilde{n}_2\times\widetilde{n}_6}]$, $\mathcal{R}_{22} : \widetilde{n}_2 \times \widetilde{n}_5$; $\mathcal{R}_3 = [0_{\widetilde{n}_3\times\widetilde{n}_4}, 0_{\widetilde{n}_3\times\widetilde{n}_5}, \mathcal{R}_{33}]$, $\mathcal{R}_{33} : \widetilde{n}_3 \times \widetilde{n}_6$. The controllability matrix $\mathcal{C}$ can then be written as

$$\begin{aligned}\mathcal{C} &= \begin{bmatrix}\mathcal{R}_1 & \mathcal{F}_1\mathcal{R}_1 & \cdots & \mathcal{F}_1^{N-1}\mathcal{R}_1 \\ \mathcal{R}_2 & \mathcal{F}_2\mathcal{R}_2 & \cdots & \mathcal{F}_2^{N-1}\mathcal{R}_2 \\ \mathcal{R}_3 & \mathcal{F}_3\mathcal{R}_3 & \cdots & \mathcal{F}_3^{N-1}\mathcal{R}_3\end{bmatrix} \\ &= \begin{bmatrix}[\mathcal{R}_{11}, 0, 0] & [\mathcal{F}_1\mathcal{R}_{11}, 0, 0] & \cdots & [\mathcal{F}_1^{N-1}\mathcal{R}_{11}, 0, 0] \\ [0, \mathcal{R}_{22}, 0] & [0, \mathcal{F}_2\mathcal{R}_{22}, 0] & \cdots & [0, \mathcal{F}_2^{N-1}\mathcal{R}_{22}, 0] \\ [0, 0, \mathcal{R}_{33}] & [0, 0, \mathcal{F}_3\mathcal{R}_{33}] & \cdots & [0, 0, \mathcal{F}_3^{N-1}\mathcal{R}_{33}]\end{bmatrix}.\end{aligned}$$

The specific structure of the controllability matrix yields

$$\begin{aligned}\mathrm{rank}\,\mathcal{C} &= \mathrm{rank}[\mathcal{R}_{11}, \mathcal{F}_1\mathcal{R}_{11}, \ldots, \mathcal{F}_1^{N-1}\mathcal{R}_{11}] \\ &\quad + \mathrm{rank}[\mathcal{R}_{22}, \mathcal{F}_2\mathcal{R}_{22}, \ldots, \mathcal{F}_2^{N-1}\mathcal{R}_{22}] \\ &\quad + \mathrm{rank}[\mathcal{R}_{33}, \mathcal{F}_3\mathcal{R}_{33}, \ldots, \mathcal{F}_3^{N-1}\mathcal{R}_{33}].\end{aligned} \tag{4}$$

Denote $\mathcal{C}_i \triangleq [\mathcal{R}_{ii}, \mathcal{F}_i\mathcal{R}_{ii}, \ldots, \mathcal{F}_i^{\widetilde{n}_i-1}\mathcal{R}_{ii}]$, $i = 1, \ldots, \delta$; it follows from Cayley–Hamilton theorem that

$$\mathrm{rank}\,\mathcal{C}_i = [\mathcal{R}_{ii}, \mathcal{F}_i\mathcal{R}_{ii}, \ldots, \mathcal{F}_i^{N-1}\mathcal{R}_{ii}], \quad i = 1, \ldots, \delta.$$

By (4),

$$\mathrm{rank}\,\mathcal{C} = \mathrm{rank}\,\mathcal{C}_1 + \mathrm{rank}\,\mathcal{C}_2 + \mathrm{rank}\,\mathcal{C}_3. \tag{5}$$

On the other hand, let e_i stand for the ith identity vector with dimension $N + n_l$ and set

$$\begin{aligned}P = [&e_1, \ldots, e_{n_1}, e_{n_3+1}, \ldots, e_{n_4}, e_{n_1+1}, \ldots, e_{n_2}, \\ &e_{n_4+1}, \ldots, e_{n_5}, e_{n_2+1}, \ldots, e_{n_3}, e_{n_5+1}, \ldots, e_{n_6}]^{\mathrm{T}}.\end{aligned}$$

P is a permutation matrix. It can be verified that $P\mathcal{L}P^{\mathrm{T}} = \mathrm{diag}\left\{\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2, \widetilde{\mathcal{L}}_3\right\}$, where

$$\widetilde{\mathcal{L}}_i = \begin{bmatrix}\mathcal{F}_i & \mathcal{R}_{ii} \\ \mathcal{R}_{ii}^{\mathrm{T}} & *\end{bmatrix}. \tag{6}$$

Consider the ith connected component $\mathcal{G}^{(i)}$, $i = 1, \ldots, \delta$; with its follower subgraph $\mathcal{G}_f^{(i)}$ and leader subgraph $\mathcal{G}_l^{(i)}$ on the node sets $\{v_{n_{i-1}+1}, \ldots, v_{n_i}\}$ and $\{v_{n_{i+2}+1}, \ldots, v_{n_{i+3}}\}$, respectively. Concerning each connected component $\mathcal{G}^{(i)}$, we rename the nodes as follows: $w_1 \triangleq v_{n_{i-1}+1}, \ldots, w_{\widetilde{n}_i} \triangleq v_{n_i}$; $\widetilde{n}_i \triangleq n_i - n_{i-1}$; $w_{\widetilde{n}_i+1} \triangleq v_{n_{i+2}+1}, \ldots, w_{\widetilde{n}_i+\widetilde{n}_{i+3}} \triangleq v_{n_{i+3}}$, $\widetilde{n}_{i+3} \triangleq n_{i+3} - n_{i+2}$. It follows that there is a 'smaller' multi-agent system $(\mathcal{F}_i, \mathcal{R}_{ii})$ in association with an interconnection graph, denoted by $\widetilde{\mathcal{G}}^{(i)}$, whose follower and leader subgraphs are, respectively, on the node sets $\left\{w_1, \ldots, w_{\widetilde{n}_i}\right\}$ and $\left\{w_{\widetilde{n}_i+1}, \ldots, w_{\widetilde{n}_i+\widetilde{n}_{i+3}}\right\}$; and the linkages between agents in $\widetilde{\mathcal{G}}^{(i)}$ are the same as those in $\mathcal{G}^{(i)}$. Accordingly, the matrix $\widetilde{\mathcal{L}}_i$ shown in (6) is the Laplacian matrix of the interconnection graph $\widetilde{\mathcal{G}}^{(i)}$, and $\widetilde{\mathcal{G}}^{(i)}$ is controllable if and only if the connected component $\mathcal{G}^{(i)}$ is controllable. At the same time, it follows from (6) and the definition of $\mathcal{C}_i$ that $\mathcal{C}_i$ is the controllability matrix of the aforementioned 'smaller' multi-agent system. Furthermore, by (5), $\mathcal{C}$ is full row rank if and only if so is each $\mathcal{C}_i$, $i = 1, \ldots, \delta$. In other words, the original multi-agent system is controllable if and only if each $\widetilde{\mathcal{G}}^{(i)}$, and accordingly each $\mathcal{G}^{(i)}$, is controllable. □

Theorem 1 acts as a separation-like principle for controllability. It simplifies the controllability problem to the investigation of that for a connected graph since each $\mathcal{G}^{(i)}$ is connected. In view of this fact, we make, without loss of generality, the following assumption.

Assumption 1. The interconnection graph $\mathcal{G}$ is connected.

Next, we are to present a 'partition' for the connected interconnection graph $\mathcal{G}$. Recall that $\mathcal{G}_l$ and $\mathcal{G}_f$ are, respectively, the leader and follower subgraph of $\mathcal{G}$. Although $\mathcal{G}$ is connected as a whole, it is not always the case for $\mathcal{G}_f$. So it can be assumed that $\mathcal{G}_f$ consists of γ connected components $\mathcal{G}_{c_1}, \dots, \mathcal{G}_{c_\gamma}$. Let $\mathcal{G}(i)$ be an induced subgraph of $\mathcal{G}$, with its node set being the union of those of $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$, $i = 1, \dots, \gamma$. That is, $\mathcal{G}(i)$ can be viewed as a 'smaller' interconnection graph with its follower subgraph being $\mathcal{G}_{c_i}$ and leader subgraph still being $\mathcal{G}_l$. Then $\mathcal{G}(1), \dots, \mathcal{G}(\gamma)$ constitute a 'partition' of $\mathcal{G}$ in the sense that $\mathcal{G}$ is partitioned into γ induced subgraphs $\mathcal{G}(1), \dots, \mathcal{G}(\gamma)$, with each one having the same leader subgraph $\mathcal{G}_l$ and the union of them being $\mathcal{G}$.

Lemma 2 (*Theorem 1 of Ji et al. (2008)*). *If multi-agent system (1) with fixed topology is controllable, then the interconnection graph is leader–follower connected, and each subgraph $\mathcal{G}(i)$ is controllable, where $i \in \{1, \dots, \gamma\}$; γ is the number of connected components of $\mathcal{G}_f$.*

Lemma 3 (*Lemma 2.2 of Ji and Egerstedt (2007)*). *Suppose the interconnection graph $\mathcal{G}$ is connected, the multi-agent system (1) is controllable if and only if $\mathcal{L}$ and $\mathcal{F}$ do not share any common eigenvalues.*

In view of the above lemmas, we will pursue conditions under which Laplacian $\mathcal{L}$ has multiple eigenvalues. The following lemma is famous. The readers are referred to, for example, Theorem 9.1.1 of Godsil and Royle (2001) for detail.

Lemma 4 (*Interlacing*). *Suppose M and N are real symmetric matrices of order m and n with eigenvalues $\lambda_1(M) \geq \cdots \geq \lambda_m(M)$ and $\lambda_1(N) \geq \cdots \geq \lambda_n(N)$, respectively. If M is a principal submatrix of N, then the eigenvalues of M interlace those of N, that is,*

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N), \quad \text{for } i = 1, \dots, m.$$

To characterize the desirable topology structure, we give the following definition.

Definition 5. The κ nodes $v_{i_1}, \dots, v_{i_\kappa}$ in the graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ are said to have the same neighbor set if each of these nodes has the same set of neighbors $\{v_{i_{\kappa+1}}, \dots, v_{i_{\kappa+\varrho}}\}$, where $v_{i_j} \in \mathcal{V}$, $i_h \neq i_j$ for $\forall h \neq j$.

The definition is meaningless for a single node case, i.e., $\kappa = 1$. So κ should not be less than two whenever the concept of the same neighbor set is mentioned.

Lemma 5 (*Lemma 2.1 of Das (2004)*). *Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a graph with vertex subset $\mathcal{V}' = \{v_1, \dots, v_\kappa\}$ having the same set of neighbors $\{v_{\kappa+1}, \dots, v_{\kappa+\varrho}\}$, where $\mathcal{V} = \{v_1, \dots, v_\kappa, \dots, v_{\kappa+\varrho}, \dots, v_n\}$. Then the Laplacian matrix of the graph $\mathcal{G}$ has at least $\kappa-1$ equal eigenvalues and they are all equal to the cardinality ϱ of the neighbor set. Also the corresponding $\kappa - 1$ eigenvectors are*

$$[1, -1, 0, \dots, 0]^{\mathrm{T}}, [1, 0, -1, 0, \dots, 0]^{\mathrm{T}}, \dots, \underbrace{[1, 0, \dots, -1}_{\kappa}, 0, \dots, 0]^{\mathrm{T}}.$$

Theorem 2. *The multi-agent system (1) is uncontrollable if the following two conditions are fulfilled simultaneously:*

(i) *there are nodes with the same neighbor set in the interconnection graph $\mathcal{G}$;*
(ii) *leaders are selected as follows:*
- *when $\kappa = 2$, i.e., there are only two nodes with the same neighbor set, the leaders are to be selected from the remaining nodes in $\mathcal{G}$ other than the two nodes with the same neighbor set.*
- *when $\kappa \geq 3$, the number of leaders is not greater than $\kappa - 2$ and the leaders are to be selected arbitrarily.*

Proof. Since $\mathcal{G}$ is connected, any selection of leaders accords with the prerequisite of controllability, i.e., the leader–follower connectedness between leader and follower subgraphs. By Proposition 1, it can be assumed, without loss of generality, that $\{v_1, \dots, v_\kappa\}$ has the same set of neighbors $\{v_{\kappa+1}, \dots, v_{\kappa+\varrho}\}$.

When $\kappa = 2$, the two nodes with the same neighbor set can be indicated with v_1, v_2. It follows from Lemma 5 that $[1, -1, \underbrace{0, \dots, 0}_{N+n_l-2}]^{\mathrm{T}}$ is an eigenvector of Laplacian $\mathcal{L}$ associated with the eigenvalue ϱ. Since the n_l leaders are chosen from the remaining nodes $v_2, \dots, v_{N+n_l}$ and the system matrix $\mathcal{F}$ is obtained from $\mathcal{L}$ by deleting the rows and columns corresponding to the leader nodes, it can be verified by straightforward computation that $[1, -1, \underbrace{0, \dots, 0}_{N-2}]^{\mathrm{T}}$ is an eigenvector of $\mathcal{F}$ associated with the same eigenvalue ϱ. So $\mathcal{L}$ and $\mathcal{F}$ share a common eigenvalue ϱ. By Lemma 3, the multi-agent system (1) is uncontrollable.

When $\kappa \geq 3$, with the selected $n_l \leq \kappa - 2$ leaders, $\mathcal{G}$ can be 'partitioned', as mentioned above, into γ subgraphs $\mathcal{G}(1), \dots, \mathcal{G}(\gamma)$. Since $v_1, \dots, v_\kappa$ possess the same neighbor set, the induced subgraph on the node set $\{v_1, \dots, v_\kappa, v_{\kappa+1}, \dots, v_{\kappa+\varrho}\}$, indicated with $\widetilde{\mathcal{G}}$, is connected. As a consequence, $\widetilde{\mathcal{G}}$ must belong to a $\mathcal{G}(i)$ as long as the leaders are chosen in advance. In other words, it is a subgraph of this $\mathcal{G}(i)$, where the index i may vary with respect to differently selected leader set, $i \in \{1, \dots, \gamma\}$.

By Lemma 5, the Laplacian matrix $\mathcal{L}(i)$ associated with $\mathcal{G}(i)$ has an eigenvalue ϱ with its algebraic multiplicity at least $\kappa - 1$. For the convenience of presentation, we assume without loss of generality that $\varrho = \lambda_1 = \cdots = \lambda_{\kappa-1}$. Denote by $\mathcal{F}(i)$ the system matrix of the 'smaller' multi-agent system corresponding to $\mathcal{G}(i)$. Recall that $\mathcal{G}_{c_i}$ and $\mathcal{G}_l$ are, respectively, the follower and leader subgraph of $\mathcal{G}(i)$. If there are m_i nodes in $\mathcal{G}_{c_i}$, $\mathcal{F}(i)$ is $m_i \times m_i$ and $\mathcal{L}(i)$ is $(m_i + n_l) \times (m_i + n_l)$, where n_l is the number of leaders, i.e., the number of nodes in $\mathcal{G}_l$. It can be seen that $\mathcal{F}(i)$ is a principal submatrix of $\mathcal{L}(i)$ with order m_i. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m_i}$ be the eigenvalues of $\mathcal{F}(i)$. It follows from Lemma 4 that

$$\lambda_{n_l+1} \leq \mu_1 \leq \lambda_1$$

where $\lambda_1 \geq \cdots \geq \lambda_{m_i+n_l}$ are the eigenvalues of $\mathcal{L}(i)$. This, together with $n_l \leq \kappa - 2$, gives rise to

$$\mu_1 = \lambda_1 = \cdots = \lambda_{n_l+1} = \cdots = \lambda_{\kappa-1},$$

which means that $\mathcal{F}(i)$ and $\mathcal{L}(i)$ have at least one common eigenvalue $\mu_1 = \lambda_1 = \varrho$. In view of Lemma 3, the induced subgraph $\mathcal{G}(i)$ is uncontrollable. The multi-agent system is then uncontrollable by following Lemma 2. □

Example 1. The example is employed to illustrate Theorem 2. The interconnection graph (a) Fig. 1 corresponds to the situation $\kappa = 2$, where v_1, v_2 have the same neighbor set $\{v_3, v_4, v_6\}$. If v_5, v_6 are chosen to be leaders, computations show that $[1, -1, 0, 0, 0, 0]^{\mathrm{T}}$ and $[1, -1, 0, 0]^{\mathrm{T}}$ are, respectively, the eigenvector of $\mathcal{L}$ and $\mathcal{F}$, associated with the same eigenvalue $\varrho = 3$. The corresponding multi-agent system is uncontrollable. The interconnection graph (b) corresponds to the situation $\kappa \geq 3$, where each node in the

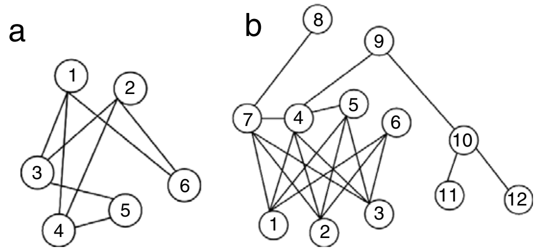

Fig. 1. Two interconnection graphs.

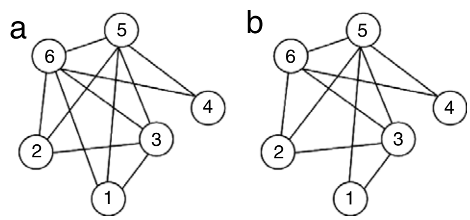

Fig. 2. Graph (a) is uncontrollable, while (b) is controllable.

subset $\{v_1, v_2, v_3\}$ has the same set of neighbors $\{v_4, v_5, v_6, v_7\}$. So $\kappa = 3, \varrho = 4$. Since $\kappa = 3$, one only need consider the case of single leader. The leader is indicated with v_l and falls into one of the following three cases: (a) $v_l \in \{v_1, v_2, v_3, v_5, v_6, v_8, v_{11}, v_{12}\}$; (b) $v_l \in \{v_4, v_7, v_9\}$; (c) $v_l = v_{10}$. In case of (a), the follower subgraph $\mathcal{G}_f$ is connected. Accordingly, $\gamma = 1$. In case of (b), $\mathcal{G}_f$ consists of two connected components, and then $\gamma = 2$. In case of (c), $\mathcal{G}_f$ consists of three connected components, which are denoted, respectively, by $\mathcal{G}_{c_1}$, $\mathcal{G}_{c_2}$, and $\mathcal{G}_{c_3}$, where $\mathcal{G}_{c_1}$ is on the note set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$, $\mathcal{G}_{c_2}$ on the node v_{11} and $\mathcal{G}_{c_3}$ on the node v_{12}. In any case, it can be verified that the Laplacian $\mathcal{L}$ and the system matrix $\mathcal{F}$ share the common eigenvalue 4. Accordingly, the multi-agent system with the single leader and the topology structure depicted in (b) of Fig. 1 is uncontrollable regardless how the leader is selected.

Lemma 6 (*Corollary 2.3 Merris (1998)*). *Let $\mathcal{G}$ be a graph on n vertices. If $0 \neq \mu < n$ is an eigenvalue of Laplacian matrix of $\mathcal{G}$, then any eigenvector associated with μ takes the value 0 on every vertex of degree $n - 1$.*

Let χ stand for the number of nodes in the interconnection graph $\mathcal{G}$. Assume that there are m nodes, say $v_{\chi-(m-1)}, \ldots, v_\chi$, in the interconnection graph $\mathcal{G}$, with each one having the degree $\chi - 1$.

Theorem 3. *The multi-agent system (1) is uncontrollable if leaders are chosen from the node set $\{v_{\chi-(m-1)}, \ldots, v_\chi\}$, with each one in the set having degree $\chi - 1$, and there is an eigenvalue of $\mathcal{G}$ not equal to 0 and χ. In this case, the dimension of the controllable subspace is one.*

Proof. Let $\{v_1, \ldots, v_N\}$ and $\{v_{N+1}, \ldots, v_{N+n_l}\}$ stand for the follower and leader node set, respectively, where $\chi \triangleq N + n_l$, with leaders chosen from $\{v_{\chi-(m-1)}, \ldots, v_\chi\}$, $1 \leq n_l \leq m$. Let μ be an eigenvalue of $\mathcal{G}$ not equal to 0 and χ. By Lemma 6, any eigenvector associated with μ takes value 0 on the $(N + i)$th element, $i = 1, \ldots, n_l$. Accordingly, any eigenvector ξ associated with μ can be denoted by $[\xi^{\mathrm{T}}_{1,\ldots,N}, \underbrace{0, \ldots, 0}_{n_l}]^{\mathrm{T}}$, where $\xi_{1,\ldots,N}$ is the vector consisting of the first N elements of ξ. Since $\mathcal{F}$ is a principle submatrix of the Laplacian $\mathcal{L}$, obtained by deleting the last n_l rows and n_l columns of $\mathcal{L}$, a straightforward calculation shows that $\xi_{1,\ldots,N}$ is an eigenvector of $\mathcal{F}$ corresponding to the eigenvalue μ. So, $\mathcal{L}$ and $\mathcal{F}$ share a common eigenvalue μ. By Lemma 3, the multi-agent system (1) is uncontrollable. Direct computation shows that under the given conditions, each row of the controllability matrix is

$$[\underbrace{1, \ldots, 1}_{n_l}, \underbrace{-n_l, \ldots, -n_l}_{n_l}, \underbrace{(-n_l)^2, \ldots, (-n_l)^2}_{n_l}, \ldots,$$
$$\underbrace{\cdot(-n_l)^{N-1}, \ldots, (-n_l)^{N-1}}_{n_l}].$$

Accordingly, the dimension of the controllable subspace is one. $\square$

Corollary 1. *A complete graph is uncontrollable.*

The corollary holds because each node in a complete graph has a degree of $\chi - 1$. Corollary 1 is Proposition V.1 in Tanner (2004), employed therein to show that increased connectivity has an adverse effect on controllability. Here, Theorem 3 implies that, rather than completeness of the overall graph, the existence of one node with the degree (of connectivity) $\chi - 1$ is enough to destroy the controllability only if it is chosen as a leader.

Example 2. The example is used to verify Theorem 3. The nodes number χ of the interconnection graph (a) of Fig. 2 is 6. v_5 and v_6 have the same degree $\chi - 1$. The eigenvalues of $\mathcal{L}$ are 0, 2, 3, 5, 6, 6. If v_5 and v_6 are chosen to play leaders role, calculations show that the eigenvalues of $\mathcal{F}$ are 2, 2, 3, 5. In this case, 2, 3, 5 are common eigenvalues of $\mathcal{F}$ and $\mathcal{L}$. If the leader is single, say v_5, it can be verified that $\mathcal{F}$ and $\mathcal{L}$ also share the common eigenvalues 2, 3, 5.

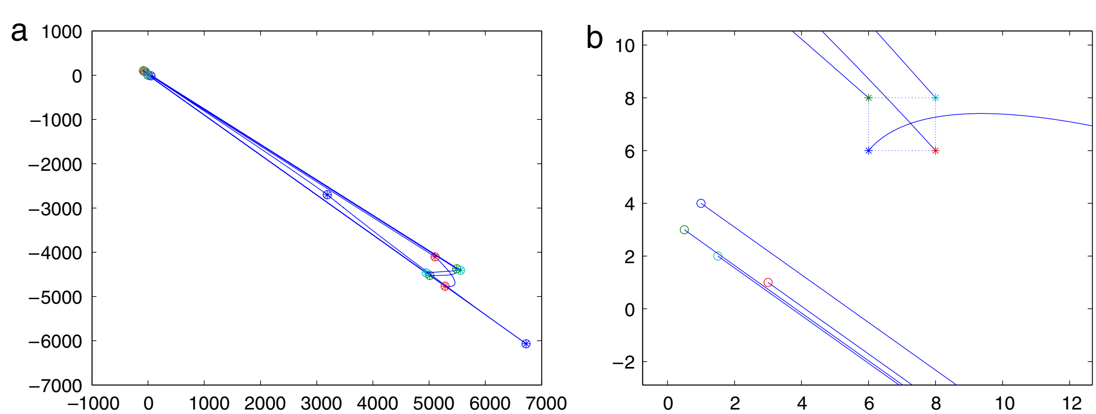

Fig. 3. (a) is the system trajectory in plane, with the associated graph depicted in (b) of Fig. 2. (b) depicts the initial state and the final desired configuration, which is the magnification of the corresponding part of (a).

In both cases, the rank of the controllability matrix is one. Although the controllable subspace dimension is just one, the system can be turned to be controllable by a slight modification of original graph. For example, if the connection/edge between nodes 1 and 6 is removed, see (b) of Fig. 2, the original system turns to be controllable. Fig. 3 depicts the trajectories of the four controllable followers in the plane, where Fig. 3(a) is magnified in apart in Fig. 3(b) to observe clearly the initial state and the final desired configuration of the system.

Theorem 4. *The multi-agent system is controllable if and only if there is no eigenvector of $\mathcal{G}$ taking* 0 *on the elements corresponding to the leaders.*

Proof. The theorem can be reformulated as stating that the system is uncontrollable if and only if there exists an eigenvector of $\mathcal{G}$ taking 0 on the elements corresponding to the leaders.

(*Sufficiency*) Suppose $\{v_{i_1}, \ldots, v_{i_N}\}$ and $\{v_{i_{N+1}}, \ldots, v_{i_{N+n_l}}\}$ are, respectively, the follower and leader node set. Set $E \triangleq [e_{i_1}, \ldots, e_{i_N}]^{\mathrm{T}}$, $T \triangleq \left[e_{i_{N+1}}, \ldots, e_{i_{N+n_l}}\right]$, where e_{i_j} is the i_jth identity vector with dimension $N + n_l$. Then $\mathcal{F} = E\mathcal{L}E^{\mathrm{T}}$, $\mathcal{R} = E\mathcal{L}T$. Let y be an eigenvector of $\mathcal{L}$ associated with the eigenvalue λ, with the i_jth component of y, i.e. y_{i_j}, being zero, $j = N+1, \ldots, N+n_l$. It can be directly verified that $E^{\mathrm{T}}Ey = y$. Then, from $\mathcal{L}y = \lambda y$, one has $E\mathcal{L}E^{\mathrm{T}}Ey = \lambda Ey$, $T^{\mathrm{T}}\mathcal{L}E^{\mathrm{T}}Ey = 0$. That is, $\mathcal{F}y_1 = \lambda y_1$, $\mathcal{R}^{\mathrm{T}}y_1 = 0$, where $y_1 \triangleq Ey = \left[y_{i_1}, \ldots, y_{i_N}\right]^{\mathrm{T}}$. By the controllability PBH criteria, the multi-agent system $(\mathcal{F}, \mathcal{R})$ is uncontrollable.

(*Necessity*) Since $\mathcal{F}$ is symmetric, its left eigenvectors are equal to the right ones. Suppose the system is uncontrollable. Then, by the PBH criteria of controllability, there exists a vector $x \in \mathbb{R}^N$ such that $\mathcal{F}x = \lambda x$ for some $\lambda \in \mathbb{R}$, with $\mathcal{R}^{\mathrm{T}}x = 0$. Let

$$P \triangleq [e_{i_1}, \ldots, e_{i_N}, e_{i_{N+1}}, \ldots, e_{i_{N+n_l}}]^{\mathrm{T}} = \begin{bmatrix} E \\ T^{\mathrm{T}} \end{bmatrix},$$

where E, T are matrices defined as above. It follows that P is a permutation matrix and

$$P\mathcal{L}P^{\mathrm{T}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{F} & \mathcal{R} \\ \mathcal{R}^{\mathrm{T}} & T^{\mathrm{T}}\mathcal{L}T \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Accordingly, $y \triangleq P^{\mathrm{T}} \begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector of $\mathcal{L}$, with the components corresponding to the leaders being zero. This completes the proof. □

Remark 1. Theorem 4 characterizes the controllability from the viewpoint of eigenvector of Laplacian matrix, while Lemma 3 from the viewpoint of eigenvalue. The result means that construction of controllable topologies may benefit from the graphical implications of eigenvectors of Laplacian matrix. Also, the verification of controllability and the selection of leaders can be facilitated by checking the eigenvectors of Laplacian matrix.

4. Conclusions

We have studied connections between controllability of multi-agent systems and topology structures of the interconnection graph. Controllability has been shown to be uniquely determined by the topology structure as long as leaders are designated. Two kinds of uncontrollable topologies have been characterized, and necessary and sufficient conditions have been proposed. One advantage of the results is that controllability, and then the feasibility of formation control, can be determined directly from the graph topology itself. This adds to the understanding of formation control by means of the classical concept of controllability.

Acknowledgments

This work was supported by the Royal Society K.C. Wong Education Foundation Postdoctoral Fellowship of the United Kingdom and the National Natural Science Foundation of China (No. 60604032, 10601050, 60704039).

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