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A new perspective on criteria and algorithms for reachability of discrete-time switched linear systems☆

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ABSTRACT

The paper presents a unified perspective on geometric and algebraic criteria for reachability and controllability of controlled switched linear discrete-time systems. Direct connections between geometric and algebraic criteria are established as well as that between the subspace based controllability/reachability algorithm and Kalman-type algebraic rank criteria. Also the existing geometric criteria is simplified and new algebraic conditions on controllability and reachability are given.

1. Introduction

Switched systems are control systems that consist of a finite number of subsystems and a logical rule that orchestrates switchings among them. The last decade has witnessed a growing interest in the study of such systems because the study is significant from both practical and theoretical point of view (DeCarlo, Branicky, Pettersson, & Lennartson, 2000; Liberzon & Morse, 1999; Sun & Ge, 2005). A challenging topic in switched systems is to evaluate the effect of switched control on the system operation, which is usually formulated as the controllability problem (Krastanov & Veliov, 2005; Petreczky, 2006a; Yang, 2002). The switching mechanism involved in the controllability and reachability was analyzed in Ji, Feng, and Guo (2007), Ji, Wang, and Guo (2007), Ji, Wang, and Guo (2008), Sun (2004), Sun, Ge, and Lee (2002) and Xie and Wang (2003a). Most results along this line were expressed in terms of geometric symbols (e.g. Cheng, Lin, and Wang (2006), Ge, Sun, and Lee (2001), Sun and Ge (2005), Sun et al. (2002), Sun and Zheng (2001), Xie and Wang (2003a)), while a few others algebraic (e.g. Stikkel, Bokor, Szabó (2004) and Yang (2002)). The geometric criteria have the advantage of a straightforward characterization of the reachable/controllable subspace, while the algebraic criteria can be checked and manipulated more conveniently. It is worth noting that there is a lack of systematic perspective on the connections between these two kinds of results as well as the relevant subspace-based algorithms. This motivates the study in this note. Also, the study is fueled by providing computational tools for reachable/controllable subspace of switched linear discrete-time systems. We present not only the aforementioned connections but also some improved geometric and algebraic criteria. Also the relationship between the existing subspace-based algorithms is revealed, which leads to a simplified computation method for controllable subspace. It should be noted that there is a strong relationship between reachability and minimality of linear switched systems (Petreczky, 2006b, 2007). In fact, the presented characterizations of reachability in this note can also be used for devising characterization of minimality of switched linear systems.

The paper is organized as follows: Section 2 presents some preliminary definitions and supporting lemmas. A unified perspective on reachability and controllability criteria is given in Section 3. A brief conclusion is made in Section 4.

2. Definitions and supporting lemmas

A switched linear discrete-time system is described by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1}$$

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where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ the input, $\sigma(k) : \{0, 1, \ldots\} \to \Lambda := \{1, \ldots, m\}$ is the switching path to be designed, and matrix pairs (A_k, B_k) for $k \in \Lambda$ are referred to as the subsystems of (1). Moreover, $\sigma(k) = i$ implies that the ith subsystem (A_i, B_i) is activated. Throughout the paper, we assume that the discrete-time switched system (1) is *reversible*, i.e., A_i is nonsingular for all $i \in \Lambda$. The derivation of the following Lemma 2 is based on this assumption (see, e.g. Ge et al. (2001) and Xie and Wang (2003b)).

For any positive integer k, set $\underline{k} = \{0, \ldots, k-1\}$. Given a switching sequence $\pi = \{(i_0, h_0) \cdots (i_{s-1}, h_{s-1})\}$, a corresponding switching path $\sigma(k) : \underline{k} \to \Lambda$ is determined by

$$\sigma(0) = \sigma(1) = \cdots = \sigma(h_0 - 1) = i_0$$
$$\sigma(h_0) = \sigma(h_0 + 1) = \cdots = \sigma(h_0 + h_1 - 1) = i_1$$
$$\vdots$$
$$\sigma\left(\sum_{j=0}^{s-2} h_j\right) = \sigma\left(\sum_{j=0}^{s-2} h_j + 1\right) = \cdots = \sigma\left(\sum_{j=0}^{s-1} h_j - 1\right) = i_{s-1}.$$

Definition 1. State x is reachable, if there exist a time instant $k > 0$, a switching path $\sigma : \underline{k} \to \Lambda$, and inputs $u : \underline{k} \to \mathbb{R}^p$, such that $x(0) = 0$, and $x(k) = x$. The reachable set of system (1) is the set of states which are reachable. System (1) is said to be (completely) reachable, if its reachable set is $\mathbb{R}^n$.

The controllability counterpart of Definition 1 can be given by replacing '$x(0) = 0$, and $x(k) = x$' with '$x(0) = x$, and $x(k) = 0$'. Given a matrix $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$, we denote $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{n} A^{i-1}\mathscr{W}$. It follows that $\langle A|\mathscr{W}\rangle$ is a minimum A-invariant subspace that contains $\mathscr{W}$. Define the subspace sequence $\mathscr{P}_j = \sum_{i=1}^{j} A^{i-1}\mathscr{W}$, $j = 1, 2, \ldots$. Clearly, $\langle A|\mathscr{W}\rangle = \mathscr{P}_n$. Let ϑ be the integer such that $\vartheta = \min\{j \mid \mathscr{P}_j = \mathscr{P}_{j+1},\ j = 1, 2, \ldots\}$. In association with A, we denote by $\rho(A)$ the degree of its minimal polynomial.

Lemma 1 (*Chen, Desoer, Niederlinski, & Kalman, 1966*). *Given a matrix $A \in \mathbb{R}^{n\times n}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^n$, $\mathscr{P}_j = \mathscr{P}_\vartheta$ holds for all $j \geq \vartheta$, with ϑ satisfying $\vartheta \leq \min\{n - \dim \mathscr{W} + 1, \rho(A)\}$.*

An immediate consequence of this lemma is $\langle A|\mathscr{W}\rangle = \sum_{i=1}^{\vartheta} A^{i-1}\mathscr{W}$. For the convenience of statement, we hereafter call ϑ the *$(A, \mathscr{W})$-invariant subspace index*. For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, set $\mathscr{B} := \operatorname{Im} B$. To study the reachability and controllability of discrete-time switched linear systems, the following recursively defined subspace sequence was introduced in Ge et al. (2001) and Sun et al. (2002).

$$\mathscr{V}_1 = \sum_{s=1}^{m} \mathscr{B}_s, \qquad \mathscr{V}_i = \sum_{s=1}^{m} \langle A_s|\mathscr{V}_{i-1}\rangle, \quad i = 2, 3, \ldots. \tag{2}$$

The subspace $\mathscr{V}$ is defined by $\mathscr{V} = \sum_{i=1}^{\infty} \mathscr{V}_i$. Furthermore, the following elegant result holds.

Lemma 2 (*Ge et al., 2001; Xie & Wang, 2003b*). *For discrete-time switched linear systems (1), $\mathscr{T} = \mathscr{V} = \mathfrak{C}$, where $\mathscr{T}$ is the set of all reachable states of system (1) and $\mathfrak{C}$ is the set of all controllable states.*

Denote by $\mathbb{d}_i$ the dimension of $\mathscr{V}_i$, i.e., $\mathbb{d}_i = \dim \mathscr{V}_i$. Let $\mu = \min\{i| \mathscr{V}_i = \mathscr{V}_{i+1}, i = 1, 2, \ldots\}$. It can be readily seen that $\mathscr{V}_1 \subset \mathscr{V}_2 \subset \cdots \subset \mathscr{V}_\mu$, and $\mathscr{V} = \mathscr{V}_\mu$. Obviously, μ is fixed once the switched system (1) is given. Furthermore $\mu \leq n - \mathbb{d}_1 + 1$. Hereafter, we call μ the *joint invariant subspace index of $(A_1, \ldots, A_m;\ B_1, \ldots, B_m)$*.

3. A unified perspective on reachability and controllability criteria

3.1. Geometric and algebraic criteria

In this subsection we derive at first a simplified geometric criterion for reachability and controllability. Then the corresponding algebraic criterion is given. Finally the geometric and algebraic criteria are discussed from a unified point of view.

Let $\omega_{i,j}$ be the $(A_i, \mathscr{B}_j)$-invariant subspace index, $i, j = 1, \ldots, m$; and $\vartheta_{i,j}$ be the $(A_i, \mathscr{V}_j)$-invariant subspace index, $i = 1, \ldots, m; j = 1, \ldots, \mu - 1$. Set $\vartheta_i = \max\{\omega_{i,s}, \vartheta_{i,j};\ s = 1, \ldots, m; j = 1, \ldots, \mu - 1\}$; and define $\underline{\vartheta_i} \triangleq \{0, 1, \ldots, \vartheta_i - 1\}$, $i = 1, \ldots, m$. We have the following result.

Theorem 1. *The switched linear discrete-time system (1) is reachable if and only if $\mathfrak{M} = \mathbb{R}^n$, where*

$$\mathfrak{M} \triangleq \sum_{i_0,\ldots,i_{\mu-1}\in\Lambda}^{j_1\in\underline{\vartheta_{i_1}},\ldots,j_{\mu-1}\in\underline{\vartheta_{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathscr{B}_{i_0}. \tag{3}$$

Proof 1. Denote by $\rho(A_i)$ the degree of the minimal polynomial of A_i. It follows from Lemma 1 that $\mathscr{V}_i$, $i = 2, \ldots, \mu$, can be written in the form

$$\mathscr{V}_i = \sum_{s=1}^{m} \langle A_s|\mathscr{V}_{i-1}\rangle = \sum_{s=1}^{m}\sum_{j=1}^{\vartheta_{s,i-1}} A_s^{j-1}\mathscr{V}_{i-1}, \tag{4}$$

where $\vartheta_{s,i-1}$ is the $(A_s, \mathscr{V}_{i-1})$-invariant subspace index, satisfying

$$\vartheta_{s,i-1} \leq \min\{n - \mathbb{d}_{i-1} + 1, \rho(A_s)\}, \tag{5}$$

with $s = 1, \ldots, m$; $i = 2, \ldots, \mu$; and $\omega_{i,j}$ satisfying

$$\omega_{i,j} \leq \min\{n - \dim \mathscr{B}_j + 1, \rho(A_i)\}. \tag{6}$$

Denote $\beta \triangleq \min\{\dim \mathscr{B}_j, j = 1, \ldots, m\}$, $\rho \triangleq \max\{\rho(A_s), s = 1, \ldots, m\}$. By (5) and (6), and $1 \leq \beta \leq \mathbb{d}_1 < \mathbb{d}_2 < \cdots < \mathbb{d}_{\mu-1} < \mathbb{d}_\mu = \dim \mathscr{V}$, one has $\vartheta_i \leq \min\{n - \beta + 1, \rho\}$, $i = 1, \ldots, m$. We then associate with each subsystem matrix A_i a nonnegative integer set $\underline{\vartheta_i}$. Since $\vartheta_s \geq \vartheta_{s,i-1}$, it follows from Lemma 1 that

$$\sum_{j=1}^{\vartheta_{s,i-1}} A_s^{j-1}\mathscr{V}_{i-1} = \sum_{j=1}^{\vartheta_s} A_s^{j-1}\mathscr{V}_{i-1}. \tag{7}$$

On the other hand, a nested subspace is defined by (2). It can be verified directly from (2) that $\mathscr{V}_\mu$ is a sum of various adding terms with each one in the form of $A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1}\mathscr{B}_{i_0}$. Equalities (4) and (7) imply that $0 \leq j_s \leq \vartheta_s - 1$, $s = 1, \ldots, \mu - 1$. Accordingly, computations according to (2) give rise to $\mathscr{V}_\mu = \mathfrak{M}$, where $\mathfrak{M}$ is given by (3). Since $\mathscr{V} = \mathscr{V}_\mu$, the result then follows from Lemma 2. □

Remark 1. The contribution of Theorem 1 consists in providing a simplified geometric characterization for the reachability subspace $\mathscr{T}$, i.e. $\mathscr{T} = \mathfrak{M}$. More specifically, $\mathscr{T}$ was written in Sun and Ge (2005) in the form

$$\mathscr{T} = \sum_{i_0,\ldots,i_{n-1}\in\Lambda}^{j_1,\ldots,j_{n-1}\in\{0,\ldots,n-1\}} A_{i_{n-1}}^{j_{n-1}} \cdots A_{i_1}^{j_1}\mathscr{B}_{i_0}, \tag{8}$$

The difference between (3) and (8) lies in: (i) The number of multiplying matrices in each adding term in (3) is μ ($\leq n - \mathbb{d}_1 + 1$), which is not greater than n, the same kind of number in (8) as μ in (3). Hence, the number of adding terms in (8) is greatly reduced in (3), especially when μ is much less than n; (ii) The maximum amount of power in association with each multiplying system matrix A_{i_s} in (8) is $n - 1$, which is reduced to $\vartheta_{i_s} - 1$ in (3), $i_s \in \Lambda$. Note that by (5) and (6), $\vartheta_{i_s} \leq \min\{n -$

$\dim \mathcal{B}_{i_s} + 1, \rho(A_{i_s})\}$. These arguments indicate that Theorem 1 presents a simplified geometric criteria for system (1) and the concept of $(A, \mathscr{W})$-invariant subspace index plays an important role in characterization of the reachable subspace.

Next, we demonstrate the corresponding algebraic criteria for Theorem 1. Let $i_1, \ldots, i_{\mu-1} \in \Lambda$ be given. Define at first the following m matrices for $s = 1, \ldots, m$

$$\mathbb{E}^{\mu-1}(s, i_1, \ldots, i_{\mu-1}) \triangleq \left[A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} B_s\right]_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_{\mu-1} \in \vartheta_{\underline{i_{\mu-1}}}}. \quad (9)$$

Then define

$$\mathfrak{A}^{\mu-1}(s) \triangleq \left[\mathbb{E}^{\mu-1}(s, i_1, \ldots, i_{\mu-1})\right]_{i_1, \ldots, i_{\mu-1} \in \Lambda}, \quad (10)$$

where $s = 1, \ldots, m$. Let

$$\mathbb{M} = \left[\mathfrak{A}^{\mu-1}(1) \quad \mathfrak{A}^{\mu-1}(2) \quad \cdots \quad \mathfrak{A}^{\mu-1}(m)\right]. \quad (11)$$

The following is a Kalman-type rank criterion.

Theorem 2. *The switched linear discrete-time system (1) is reachable if and only if the controllable matrix $\mathbb{M}$ is of full row rank, i.e. rank $\mathbb{M} = n$.*

Proof 2. From (3), $\mathscr{V}_\mu$ can be written in the form

$$\mathscr{V}_\mu = \sum_{i_1, \ldots, i_{\mu-1} \in \Lambda} \sum_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_{\mu-1} \in \vartheta_{\underline{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_1 + \cdots + \sum_{i_1, \ldots, i_{\mu-1} \in \Lambda} \sum_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_{\mu-1} \in \vartheta_{\underline{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_m. \quad (12)$$

By (9), it can be seen that for a group of given $i_1, \ldots, i_{\mu-1}$ and $s = 1, \ldots, m$

$$\operatorname{Im} \mathbb{E}^{\mu-1}(s, i_1, \ldots, i_{\mu-1}) = \sum_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_{\mu-1} \in \vartheta_{\underline{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_s.$$

Furthermore, it follows from (10) that for $s = 1, \ldots, m$

$$\operatorname{Im} \mathfrak{A}^{\mu-1}(s) = \sum_{i_1, \ldots, i_{\mu-1} \in \Lambda} \sum_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_{\mu-1} \in \vartheta_{\underline{i_{\mu-1}}}} A_{i_{\mu-1}}^{j_{\mu-1}} \cdots A_{i_1}^{j_1} \mathcal{B}_s.$$

Combining this with (11) and (12) yields $\operatorname{Im} \mathbb{M} = \mathscr{V}_\mu$. The result then follows from Lemma 2. □

The algebraic conditions on controllability were recently studied by Stikkel et al. (2004) and Yang (2002) by employing a concept of *joint controllability matrices* of switched linear systems. To proceed, let us revisit this concept used by them. Define

$$\text{æ}^k(i_1, \ldots, i_k) \triangleq \left[A_{i_k}^{j_k} \cdots A_{i_2}^{j_2} A_{i_1}^{j_1} B_{i_1}\right]_{j_1, \ldots, j_k \in \{0, 1, \ldots, n-1\}}.$$

Then let $\text{Æ}^0(i) = \text{æ}^1(i), \ldots$, and

$$\text{Æ}^k(i) = [\text{æ}^{k+1}(i, i_1, \ldots, i_k)]_{i_1, \cdots, i_k \in \Lambda}.$$

The joint controllability matrices can be iteratively defined as $W^0 = [\text{Æ}^0(1)\text{Æ}^0(2) \cdots \text{Æ}^0(m)], \ldots, W^k = [\text{Æ}^k(1)\text{Æ}^k(2) \cdots \text{Æ}^k(m)]$. There exists a *joint controllability coefficient* k_r of the system, defined in Yang (2002) by $k_r = \arg\min_l\{\text{rank}\, W^l = \text{rank}\, W^{l+1}\}$. Yang proved that a necessary condition for the controllability is $\text{rank}\, W^{k_r} = n$. Then Stikkel, Bokor and Szabó showed that this condition is also sufficient provided the persistency of excitation assumption on switching signals. So the algebraic criterion on controllability has not been solved completely. In particular, few properties are known on k_r, especially the exact value. So we want to know whether there are any other characterizations for k_r. To analyze this problem, we present a modified version of joint controllability matrices. Let $\mathbb{E}^0(i) \triangleq B_i$, $i = 1, \ldots, m$; and

$$\mathbb{E}^k(i, i_1, \ldots, i_k) \triangleq \left[A_{i_k}^{j_k} \cdots A_{i_1}^{j_1} B_i\right]_{j_1 \in \vartheta_{\underline{i_1}}, \ldots, j_k \in \vartheta_{\underline{i_k}}}.$$

Define $\mathfrak{A}^0(i) = \mathbb{E}^0(i), \ldots,$

$$\mathfrak{A}^k(i) = \left[\mathbb{E}^k(i, i_1, \ldots, i_k)\right]_{i_1, \ldots, i_k \in \Lambda}$$

and $W^0 = \left[\mathfrak{A}^0(1)\, \mathfrak{A}^0(2) \ldots \mathfrak{A}^0(m)\right], \cdots,$

$$W^k = \left[\mathfrak{A}^k(1)\, \mathfrak{A}^k(2) \ldots \mathfrak{A}^k(m)\right]. \quad (13)$$

It can be seen that the joint controllability matrices defined in this way have the property of $\operatorname{Im} W^k = \mathscr{V}_{k+1}$, $k = 0, 1, \ldots$. An immediate consequence of this observation is the following result.

Theorem 3. *The relationship between the joint controllability coefficient k_r and the joint invariant subspace index of system (1) is $k_r = \mu - 1$.*

Remark 2. The modified version (13) of joint controllability matrices allows one to take advantage of the nested subspace sequence (2) to get Theorem 3. This characterization enables people to understand k_r via a geometric rather than only an algebraic point of view.

Remark 3. Theorems 1 and 2 exhibit a direct connection and correspondence between the geometric and algebraic criteria. Theorems 1–3 not only present simplified geometric and algebraic criteria for controllability and reachability of switched linear discrete-time systems, but also demonstrate these two criteria in a systematic and unified way for the first time. At the same time the relationship between the joint controllability coefficient and the joint invariant subspace index is revealed.

3.2. Computational issues and other algebraic rank conditions

To calculate $\operatorname{Im} W^{k_r}$, Stikkel et al. introduced the following subspace algorithm

$$\mathscr{W}_0 = \sum_{j=1}^m \mathcal{B}_j, \qquad \mathscr{W}_{k+1} = \mathscr{W}_0 + \sum_{j=1}^m A_j \mathscr{W}_k. \quad (14)$$

Let $\mathscr{W}^* = \lim_{k \to \infty} \mathscr{W}_k$, it is proved in Stikkel et al. (2004) that $\operatorname{Im} W^{k_r} = \mathscr{W}^*$. With respect to the subspace sequence (14), we have the following observation.

Proposition 1. *The subspace sequence (14) can be equivalently written as*

$$\mathscr{W}_0 = \sum_{j=1}^m \mathcal{B}_j, \qquad \mathscr{W}_{k+1} = \mathscr{W}_k + \sum_{j=1}^m A_j \mathscr{W}_k. \quad (15)$$

As a consequence, if a nonnegative number γ is defined by $\gamma = \min\{k \mid \mathscr{W}_k = \mathscr{W}_{k+1}, k = 0, 1, \ldots\}$, then

$$\mathscr{W}_0 \subset \mathscr{W}_1 \subset \cdots \subset \mathscr{W}_\gamma = \mathscr{W}_{\gamma+1} = \cdots = \mathscr{W}^* = \mathscr{T}. \quad (16)$$

Proof 3. We show it by induction. Clearly, $\mathscr{W}_1 = \mathscr{W}_0 + \sum_{j=1}^m A_j \mathscr{W}_0$. Suppose $\mathscr{W}_{k+1} = \mathscr{W}_k + \sum_{j=1}^m A_j \mathscr{W}_k$. We have

$$\begin{aligned}
\mathscr{W}_{k+2} &= \mathscr{W}_0 + \sum_{j=1}^m A_j \mathscr{W}_{k+1} \\
&= \mathscr{W}_0 + \sum_{j=1}^m A_j \left(\mathscr{W}_k + \sum_{l=1}^m A_l \mathscr{W}_k\right) \\
&= \mathscr{W}_0 + \sum_{j=1}^m \left(A_j \mathscr{W}_k + A_j \sum_{l=1}^m A_l \mathscr{W}_k\right)
\end{aligned}$$

$$= \left(\mathscr{W}_0 + \sum_{j=1}^{m} A_j \mathscr{W}_k\right) + \sum_{j=1}^{m} A_j \left(\mathscr{W}_k + \sum_{l=1}^{m} A_l \mathscr{W}_k\right)$$
$$= \mathscr{W}_{k+1} + \sum_{j=1}^{m} A_j \mathscr{W}_{k+1}.$$

So (15) holds. By (15) and the definition of γ, $\mathscr{W}_0 \subset \mathscr{W}_1 \subset \cdots \subset \mathscr{W}_\gamma = \mathscr{W}_{\gamma+1} = \cdots = \mathscr{W}^*$. The equality $\mathscr{W}^* = \mathscr{T}$ follows by combining (8), Lemma 2 and the proof of Proposition 1 in Stikkel et al. (2004). The proof is completed. □

Remark 4. The subspace sequence (15) is exactly the one used by Sun et al. in Sun et al. (2002). Proposition 1 tells us that the subspace sequences (14) and (15) are actually equivalent to each other. The advantage of (14) lies in its simple form while the subspace (15) possesses good Proposition (16). By Proposition 1, these two advantages can be combined together when one tries to calculate the reachable/controllable subspace for switched linear systems. In other words, one can start computing $\mathscr{W}_k, k = 0, 1, 2, \ldots$, according to (14) which is simpler than (15), and stop the algorithm at most within $n - \dim \mathscr{W}_0$ steps because according to (16), $\gamma \le n - \dim \mathscr{W}_0$.

Now we state another algebraic criterion for reachability and controllability. Let

$$\begin{aligned}\Gamma \triangleq [&B_1, \ldots, B_m, A_1B_1, \ldots, A_1B_m, \ldots, A_mB_1, \ldots,\\ &A_mB_m, \ldots, A_1^{\gamma}B_1, \ldots, A_1^{\gamma}B_m, A_1^{\gamma-1}A_2B_1, \ldots,\\ &A_1^{\gamma-1}A_2B_m, \ldots, A_m^{\gamma}B_1, \ldots, A_m^{\gamma}B_m].\end{aligned}$$

That is, Γ consists of block matrices $A_{i_l} \cdots A_{i_1} B_{i_0}$ with $0 \le l \le \gamma$; $i_0, i_1, \ldots, i_l \in \{1, \ldots, m\}$, and $i_0, i_1, \ldots, i_l$ are not necessarily distinct discrete modes.

Theorem 4. *The switched linear discrete-time system (1) is reachable if and only if the matrix Γ is of full row rank, i.e. rank $\Gamma = n$.*

Proof 4. It follows from (14)(refer to the proof of Proposition 1 in Stikkel et al. (2004) for detail) that an arbitrary subspace $\mathscr{W}_k$ can be written as

$$\mathscr{W}_k = \sum_{j=1}^{m} \mathscr{B}_j + \sum_{l=1}^{k} \sum_{i_0,\ldots,i_l \in \Lambda} A_{i_l} \cdots A_{i_1} \mathscr{B}_{i_0}.$$

In particular, with respect to $\mathscr{W}_\gamma$, one has $\mathscr{W}_\gamma = \operatorname{Im} \Gamma$. This, together with (16) gives rise to the result. □

Since $\gamma \le n - \dim \mathscr{W}_0$, Theorem 4 still holds when γ in Γ is replaced by n.

4. Conclusions

The paper contributes to the field by providing a unified perspective for controllability and reachability algebraic and geometric criteria as well as the corresponding subspace based algorithms. Connections between the algebraic and geometric criteria are revealed as well as those between the algorithms and algebraic rank conditions, which gain an insight into the significance of some existing criteria for controllability and reachability of controlled switched linear discrete-time systems. The contribution also includes simplified geometric criteria and new Kalman-type algebraic rank criterion.

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