Graphic Interpretations of Structural Controllability for Switched Linear Systems

Xiaomeng Liu, Hai Lin and Ben M. Chen

***Abstract*— This paper considers the controllability problem for switched linear systems. In particular, the structural controllability of switched linear systems is investigated. The structural controllability of switched linear systems is a generalization of the traditional controllability concept for dynamical systems, and purely based on the graphic topologies among state and input nodes. First, two kinds of graphic representations of switched linear systems are proposed. Second, several graph-theoretic characterizations of the structural controllability for switched linear systems are presented based on these two newly introduced graphs. Finally, the paper concludes with several illustrative examples and discussions of the results and future work.**

I. INTRODUCTION

As a special class of hybrid control systems, basically, a switched linear system consists of several linear subsystems and a rule that orchestrates the switching among them. Due to the significant importance in both theoretical research and practical applications, switched linear systems have attracted considerable attention during the last decade [1]-[6]. Switching between different subsystems or different controllers can greatly enrich the control strategies and may accomplish certain control object which can not be achieved by conventional dynamical systems. For example, it provided an effective mechanism to cope with highly complex systems and/or systems with large uncertainties [4][9]. [10][11] presented good examples that switched controllers could provide a performance improvement over a fixed controller. Besides, Switched linear systems also have promising applications in control of mechanical systems, aircrafts and satellites and kinds of multi-agents systems, such as unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs) and so on.

Much work has been done on the controllability of switched linear systems. For example, the controllability and reachability for low-order switched linear systems have been presented in [12][13]. Under the assumption that the switching sequence is fixed, [14][15] introduced some sufficient conditions and necessary conditions for controllability of switched linear systems. Complete geometric criteria for controllability and reachability were established in [3][16]. [17] studied the controllability of switched bilinear systems using Lie algebraic technique. In [5]-[8] the controllability, reachability and switching sequence design problem of switched systems were deeply investigated.

Xiaomeng Liu is with Dept. of Electrical and Computer Engineering, National University of Singapore, Singapore. liuxiaomeng@nus.edu.sg

Hai Lin is with Dept. of Electrical and Computer Engineering, National University of Singapore, Singapore. elelh@nus.edu.sg

Ben M. Chen is with Dept. of Electrical and Computer Engineering, National University of Singapore, Singapore. bmchen@nus.edu.sg

Up to now, all the previous work mentioned above is based on the traditional controllability concept of switched linear systems. In this paper, we propose a new notation for the controllability of switched linear system: structural controllability, which may present more practical significance. Actually, when people try to obtain the models of physical processes, a more realistic situation is that most of system parameter values are known only with the approximation of some errors of measurement. Only the zero elements that are fixed either by coordination or by the absence of physical connections between certain parts of the system can be known with 100 percent precision. Thus we will assume here that all the elements of matrices of switched linear systems to be fixed zeros or free parameters. Such kind of switched linear systems would represent a large class of parameter dependent switched linear systems. Furthermore, the switched linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding switched linear system is controllable in the classical sense. For such structured systems, generic properties including structural controllability have been studied deeply and it turns out that properties generic properties including structural controllability are true for almost all values of the parameters [18]-[22].That is one of the reasons why this kind of structural controllability is so valuable and attract our great interest.

No matter the traditional controllability or the structural controllability of switched linear systems, all the results achieved were algebraic conditions. However, it remains elusive on what exactly is the graphical meaning of these algebraic conditions. Graphical conditions can help to understand how the graphic topologies of dynamical systems influence the corresponding generic properties, here especially for the structural controllability. This would be of great significance in many practical applications. For example, in multi-agent systems, graphical interpretations for structural controllability help us to understand the necessary information exchange between agents to make the whole team controllable. Therefore, this motivates our pursuit on illuminating the structural controllability of switched linear systems from a graph theoretical point of view. In this paper, we propose two graphic representations of switched linear systems and finally, it turns out that the structural controllability of switched linear systems only depends on the graphic topologies of the corresponding systems. Especially, in multi-agent systems [23], it was proved that the multi-agent system with switching topology is structurally controllable if and only if the union graph is connected.

The outline of this paper is as follows: In Section II, we introduce some basic preliminaries, followed by structural controllability study in Section III, where several graphic necessary and/or sufficient conditions for the structural controllability are given. In Section IV, some examples are presented to illustrate the theoretical results. Finally, some concluding remarks are drawn in the paper.

II. Preliminaries and Problem Formulation

A. Graph Theory Preliminaries

Consider a linear control system:

$$\dot{x} = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$. The matrices A and B are structured matrices, which means that their elements are either fixed zeros or independent free parameters.

The graphic representation of the pair (A, B) is described as follows: Consider one directed graph $\mathcal{G}$ with N vertices consisting of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{I} = \{e_1, e_2, \ldots, e_{N'}\}$, which are the interconnection links among the vertices. The vertex set $\mathcal{V}$ consists of node sets $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, which is called $state\ nodes\ set$ and $\mathcal{U} = \{u_1, u_2, \ldots, u_r\}$, which is called $input\ nodes\ set$. Place an oriented edge (x_i, x_j) (an arrow going from node x_j to node x_i) of weight a_{ij} if and only if the (i, j) entry of A is a free parameter . Similarly, place an oriented edge (x_i, u_j) of weight b_{ij} if and only if b_{ij} is a free parameter of B. This directed graph $\mathcal{G}$ is called the graph of the pair (A, B) and denoted by $\mathcal{G}(A, B)$.

For any oriented edge, one node is called the initial (terminal) node of the edge if the oriented edge starts (ends) with this node. An alternating sequence of distinct vertices and oriented edges in the graph $\mathcal{G}$ is called a *path*. Furthermore, a path whose initial node is in $\mathcal{U}$ is called a *stem*. Generally, a node (other than the input nodes) is called $nonaccessible$ if and only if there is no possibility of reaching this node through any stem of the graph $\mathcal{G}$.

In the directed graph $\mathcal{G}(A, B)$, a bud is an elementary cycle in X with an an additional edge that ends, but not begins, in a vertex of the cycle. The begin vertex of the edge is then also said to be the begin vertex of the bud. A $cactus$ is a subgraph that is defined as follows: A cactus $\wp$ is either a stem, or is obtained from a smaller cactus $\wp\prime$, to which a bud has been added that is vertex disjoint from $\wp\prime$ apart from the begin vertex of the bud, which can be any vertex of $\wp\prime$, except for the last vertex of the stem that is contained in $\wp\prime$.

Consider one vertex set S formed by the nodes from the input nodes set $\mathcal{X}$ and determine another vertex set $T(S)$, which contains all the nodes that has an oriented edge pointing to any vertex in S. Then the graph $\mathcal{G}(A, B)$ contains a '$dilation$' if and only if there is a set S of k nodes in the vertex set of the graph such that there are no more than $k-1$ nodes in $T(S)$[18].

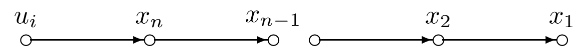

Fig. 1. One stem in one directed graph

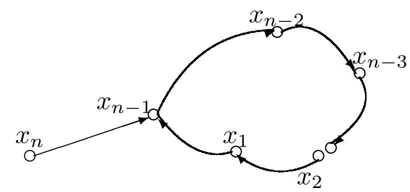

Fig. 2. One bud in one directed graph

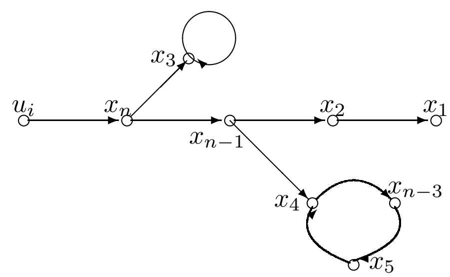

Fig. 3. One cactus in one directed graph

B. Switched Linear System, Controllability and Structural Controllability

In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

$$\dot{x}(t) = \quad A_\sigma x(t) + B_\sigma u(t), \tag{2}$$

where $x \in \mathbb{R}^n$ are the states, $u_k \in \mathbb{R}^{r_k}$, $k = 1, \ldots, m$ are piecewise continuous input, σ is the control signal and a piecewise constant signal taking value from an index set $M \triangleq \{1, \ldots, m\}$. Suppose that there are m subsystems (A_i, B_i), $i \in \{1, \ldots, m\}$. Moreover, $\sigma(t)= i$ implies that the ith subsystem (A_i, B_i) is activated. Given a switching sequence $\sigma : [t_o, t_f] \to M$, we refer to the sequence $t_0, t_1, \ldots, t_{s-1}$ with $t_0 < t_1 < \cdots < t_{s-1}$ as switching time sequence, and the sequence $\sigma(t_0) = i_0, \sigma(t_1) = i_1, \cdots, \sigma(t_{s-1}) = i_{s-1}$ as switching index sequence. Let $h_i := t_{i+1} - t_i$, $i = 0, 1, \ldots, s-1$, and $t_s := t_f$. We denote a switching sequence by $\pi = \{(i_0, h_0), (i_1, h_1), \ldots, (i_{s-1}, h_{s-1})\}$.

Definition 1. A nonzero state $x \in \mathbb{R}^n$ is controllable, if there exist a switching sequence π and input $u(t)$ such that $x(0) = x$ and $x(t_f) = 0$. Switched linear system (2) is said to be (completely) controllable if any nonzero state x is controllable; A nonzero state $x \in \mathbb{R}^n$ is reachable, if there exist a switching sequence π and input $u(t)$ such that $x(0) = 0$ and $x(t_f) = x$. Switched linear system (2) is said to be (completely) reachable if any nonzero state x is reachable.

In [3], it is proved that the controllable set and reachable set are always identical and here we use controllability to represent.

For the controllability of switched linear system, one matrix condition has been given in [3]:

Lemma 1.([3]) If the matrix:

$$\begin{aligned}&[B_1, B_2, B_3, \ldots, B_m,\\&A_1B_1, A_2B_1, A_3B_1, \ldots, A_mB_m,\\&\ldots,\\&A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m]\end{aligned} \quad (3)$$

has full row rank n, the switched linear system (2) is controllable, and vice versa.

This matrix is called the controllability matrix of the corresponding switched linear system (2).

This paper mainly focuses on the structural controllability of switched linear systems and here all the matrix pairs $(A_i, B_i), i \in M$ consist of free parameters and zero elements. Consequently:

Definition 2. The switched linear system (2) is structurally controllable if after assigning values to the parameters in the matrices $[A_i, B_i]$, $i \in \{1, \ldots, m\}$, there exists a switched linear system which is controllable in the usual sense.

Consequently:

Lemma 2. If the controllability matrix (3) has $g\text{-}rank$ n, which is the maximum rank achievable by a matrix as a function of the free parameters, then the switched linear system (2) is structurally controllable.

III. Structural Controllability

A. Union Graph and Colored Union Graph

Here, Consider a switched linear system:

$$\dot{x}(t) = \quad A_\sigma x(t) + B_\sigma u(t), \quad (4)$$

where $x \in \mathbb{R}^n$ are the states, $u_k \in \mathbb{R}^{r_k}$, $k = 1, \ldots, m$ are piecewise continuous input, $\sigma \in \{1, \ldots, m\}$. The elements in every subsystem matrix pair (A_i, B_i) are independent parameters or fixed zeros. Use $\mathcal{G}_i$ with vertex set $\mathcal{V}_i$ and edge set $\mathcal{I}_i$ to represent the underlaying graph of subsystem (A_i, B_i). Notice that here the B_i, $i \in \{1, \ldots, m\}$ may have different dimensions. However, fortunately, if we add zero columns to B_i matrices to make all the B_i matrices have the same dimension, it turns out that it makes no difference on the controllability properties with the original switched linear systems (4), and in the following discussion, it also has no influence on building the union graph and the colorfed union graph. Then here, without loss of generality, the B_i, $i \in \{1, \ldots, m\}$ are assumed to have the same dimension and all the results in this paper hold for the situations that the B_i matrices have different dimensions.

As to the whole switched system, one kind of the representation graph, which is called union graph, is defined as follows:

Notation 1. The switched linear system (4) can be represented by a union digraph, defined as a flow structure $\mathcal{G}$. Mathematically, $\mathcal{G}$ is defined as

$$\begin{aligned}\mathcal{G}_1 \bigcup \mathcal{G}_2 \bigcup \mathcal{G}_3 \bigcup \ldots \bigcup \mathcal{G}_m = \{&\mathcal{V}_1 \bigcup \mathcal{V}_2 \bigcup \mathcal{V}_3 \ldots\\&\bigcup \mathcal{V}_m; \mathcal{I}_1 \bigcup \mathcal{I}_2 \bigcup \mathcal{I}_3 \ldots \bigcup \mathcal{I}_m\}\end{aligned}$$

For the union graph $\mathcal{G}$, the vertex set is the same with the vertex set of every subgraph $\mathcal{G}_i$. The edge set of $\mathcal{G}$ equals to the union of the edge sets of the subgraphs.

Remark 1. Actually, it turns out that the union graph $\mathcal{G}$ is the representation of the linear system:
$(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$.

To derive one of our main results for the structural controllability problem, a new notation is proposed here: $colored\ union\ graph$.

Notation 2. The switched linear system (4) can be represented by a colored union digraph, defined as a flow structure $\tilde{\mathcal{G}}$, identified according to difference with union graph $\mathcal{G}$. For $\tilde{\mathcal{G}}$, the vertex set is also the same with the vertex set of every subgraph $\mathcal{G}_i$. However the the edge set of $\tilde{\mathcal{G}}$ is no longer the the union of the edge sets of the subgraphs. Different colors are assigned to the edges from different subgraphs, then all of them are put in the colored union graph without deleting any edges. Now $\tilde{\mathcal{G}}$ actually has multi edges between any two vertices with different colors and other edges in $\tilde{\mathcal{G}}$ coming from different subgraphs also have different colors.

In the following discussion, the notations of union graph $\mathcal{G}$ and colored union graph $\tilde{\mathcal{G}}$ are employed to propose the necessary and the sufficient conditions for the structural controllability.

B. Main Results on Structural Controllability

Before considering the structural controllability of switched linear system (4), we first introduce two definitions for linear system which were proposed in [18]:

Definition 3.(Definition in [18]) The pair (A, B) is said to be reducible or have form I if they can be written in the following form:

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad (5)$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_{22} \in \mathbb{R}^{(n-p)\times r}$.

Remark 2. Whenever the matrix pair (A, B) is in form I, the system is structurally uncontrollable and meanwhile, the controllability matrix

$$Q = \left[B, AB, \ldots, A^{n-1}B\right],$$

will now have at least one row which is identically zero for all parameter values.

Definition 4. (Definition in [18]) The pair (A, B) is said to be of form II if they can be written in the following form:

$$[A, B] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad (6)$$

where $P_2 \in \mathbb{R}^{(n-k)\times(n+r)}$, $P_1 \in \mathbb{R}^{k\times(n+r)}$ with no more than $k-1$ nonzero columns (all the other columns of P_1 have only fixed zero entries).

For the structural controllability and its graphic interpretation of a linear system (A, B), the following results have been proved in [18]-[21]:

Theorem 1.([18]-[21]) For a linear system (A, B), the following several statements are equivalent.

a) The pair (A, B) is structurally controllable.
b) i)$[A, B]$ is irreducible or not of form I
   ii)$[A, B]$ is not of form II or g-rank$[A, B] = n$
c) i)There is no nonaccessible node in $\mathcal{G}(A, B)$
   ii)There is no 'dilation' in $\mathcal{G}(A, B)$.
d) There exists a cacti which spans $\mathcal{G}(A, B)$, where cacti is a set of mutually disjoint cactus.

This theorem proposed interesting graphic conditions for structural controllability of linear systems and revealed that the structural controllability is totally determined by the underlaying graph topology. However, how about in switched linear systems? According to *Lemma 2*, once we impose proper scalars for the parameters of the system matrix (A_i, B_i) to satisfy the full rank condition, the switched linear system (4) is structurally controllable. However, this only proposed an algebraic condition. Can we still find some kinds of graph which can totally determine the structural controllability properties of switched linear systems? The following results will answer this question and provide several graphic interpretations for structural controllability of switched linear systems.

The following is our first main result, which is actually one graphic sufficient condition for the structural controllability of switched linear systems:

Theorem 2.(Sufficient Condition) The switched linear system (4) with graphic topologies $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ is structurally controllable if the union graph $\mathcal{G}$ is spanned by a cacti or equivalently, $\mathcal{G}$ satisfies:

i) There is no nonaccessible node in $\mathcal{G}$
ii) There is no 'dilation' in $\mathcal{G}$.

Proof: Here, the union graph $\mathcal{G}$: $\mathcal{G}(A_1, B_1) \bigcup \mathcal{G}(A_2, B_2) \bigcup \mathcal{G}(A_3, B_3) \bigcup \ldots \bigcup \mathcal{G}(A_m, B_m)$ is spanned by a cacti or satisfies the two conditions mentioned in the theorem. According to *Remark 1* and *Theorem 1*, the corresponding linear system $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$ is structurally controllable. Then there exist some scalars for the parameters in the A_i and B_i matrices that make the controllability matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&(A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),\\
&(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m),\\
&,\ldots,\\
&(A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)],
\end{aligned}$$

has full row rank n. Expanding the matrix, it follows that the matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\
&,\ldots,\\
&A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_m^{n-1}B_m],
\end{aligned}$$

has full rank n. Next, we add some column vectors to the above matrix and get

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m, B_2, \ldots, B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m + \ldots + A_mB_m,\\
&A_2B_1, \ldots, A_mB_m\\
&,\ldots,\\
&A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots\\
&+ A_m^{n-1}B_m, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m].
\end{aligned}$$

This matrix still have n linear independent column vectors, so it has full row rank. Next, subtract $B_2, B_3, \ldots, B_m$ from $B_1 + B_2 + B_3 + \ldots + B_m$; subtract $A_2B_1, A_3B_1, \ldots, A_mB_m$ from $A_1B_1 + A_2B_1 + A_3B_1 + \ldots + A_mB_1 + \ldots + A_1B_m + \ldots + A_mB_m$ and subtract $A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m$ from $A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots + A_m^{n-1}B_m$. Because this column fundamental transformation will not change the matrix rank, the matrix still has full row rank n. Now the matrix becomes

$$\begin{aligned}
&[B_1, B_2, , \ldots, B_m,\\
&A_1B_1, A_2B_1, \ldots, A_mB_m\\
&,\ldots,\\
&A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m],
\end{aligned}$$

which is the controllability matrix for switched linear systems (4) and has full row rank n. Therefore, the switched linear system is controllable and therefore structurally controllable. And finally, we get that if the union graph $\mathcal{G}$ is spanned by a cacti or has no nonaccessible and no 'dilation' in it, the switched linear system (4) is structurally controllable. ■

As a consequence of this theorem, another sufficient condition can be described as follows:

Corollary 1.(Sufficient Condition) The switched linear system (4) with graphic topologies $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ is structurally controllable if any subgraph $\mathcal{G}_i$ is spanned by a cacti or has no nonaccessible and no 'dilation' in it.

It is easy to obtain this result by noting that, by definition of union graph $\mathcal{G}$, if any subgraph $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ is spanned by a cacti or has no nonaccessible and no 'dilation', then the union graph $\mathcal{G}$ also has this kind of property.

Until now, the results are based on the union graph $\mathcal{G}$ of switched linear systems. Our following results are all based on the colored union graph $\tilde{\mathcal{G}}$. Firstly, we will investigate several graphic properties of union colored graph and their relationship with the switched linear system matrices.

Lemma 3. There is no nonaccessible node in the colored union graph $\tilde{\mathcal{G}}$ of the switched linear system (4) if and only if the following matrix

$$[A_1 + A_2 + A_3 + \cdots + A_m, B_1 + B_2 + B_3 + \cdots + B_m]$$

is irreducible or not of form I.

Proof: One node is accessible if and only if there is an oriented path starting from one of the input nodes and ending in this node. According to the definitions of the union graph and colored union graph, it can be concluded that there is no nonaccessible node in the colored union graph if and only if there is no nonaccessible node in the union graph. And from *remark 1*, we know that the matrix representation of the union graph is

$$(A_1 + A_2 + A_3 + \cdots + A_m, B_1 + B_2 + B_3 + \cdots + B_m).$$

Then according to *theorem 1* which talks about the structural controllability of linear systems, we can easily get this lemma proved. ■

Definition 5. In the colored union graph $\tilde{\mathcal{G}}$, we propose one new definition: S-$dilation$. 'Dilation' in the graph of linear systems was proposed in [18]. If we choose a vertex set S formed by the noninput nodes and let $T(S)$ define the vertex set that any node in $T(S)$ has one edge pointing to one node in S. If $|T(S)| < |S|$, then we say there is a dilation in the graph. Now in the union color graph, $|T(S)|$ is calculated as the summation of $|T_i(S)|$ $i \in \{1, \ldots, m\}$ in every subgraph. If $|T(S)| < |S|$, we say there is a S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$.

Based on this new graph property, we get this following lemma:

Lemma 4. There is S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (4) if and only if the following matrix

$$[A_1, B_1, A_2, B_2, \ldots, A_m, B_m] \tag{7}$$

is of form II. It means that there exist some permutation of matrix pair (A_i, B_i) $i \in \{1, \ldots, m\}$, (7) can be written into:

$$[A_1, B_1, A_2, B_2, \ldots, A_m, B_m] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{8}$$

where $P_1 \in \mathbb{R}^{p \times k}$ with no more than $p-1$ nonzero columns (all the other columns of P_1 have only fixed zero entries).

Proof: From [18][19] or *theorem 1*, it is known that in linear systems, there is no 'dilation' in the corresponding graph if and only if the matrix pair $[A, B]$ can not be of form II or has g-rank n. From the explanation of this result in [18] and *definition 4*, P_1 in $[A, B]$ has p rows, which actually represent the p nodes of node set S (defined for dilation) and each nonzero element of each row of P_1 represents that there is one node pointing to the node presented by this row. Therefore, the number of nonzero columns in P_1 is the number of nodes pointing to some node in S, and actually equals to $|T(S)|$. Furthermore, by the definition of S-$dilation$, $|T(S)|$ is now the summation of $|T_i(S)|$ $i \in \{1, \ldots, m\}$ in every subgraph. Consequently, it can be easily got that there is S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$ if and only if matrix (7) in not of form II. ■

Before we going further to give another algebra explanation of S-$dilation$, one definition and theorem proposed in [20] must be introduced first:

Definition 6.(Definition 1 in [20]) A structured $n \times m'$ $(n \leq m')$ matrix A is of form (t) for some t, $1 \leq t \leq n$, if for some k in the range $m' - t < k \leq m'$, A contains a zero submatrix of order $(n + m' - t - k + 1) \times k$.

Theorem 3.(Theorem 1 in [20]) g-rank $A = t$

i) for $t = n$ if and only if A is not of form (n).
ii) for $1 \leq t < n$ if and only if A is of form $(t+1)$ but not of form (t).

Starting from the above definition and theorem, another lemma is proposed here:

Lemma 5. There is no S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (4) if and only if the following matrix

$$[A_1, B_1, A_2, B_2, \ldots, A_m, B_m] \tag{9}$$

has g-rank n.

Proof: Necessity: If matrix (9) has g-rank $< n$, from *theorem 3*, it follows that matrix (9) is of form (n). Then referring to *definition 6*, (9) must have a zero submatrix of order $(n+m'-t-k+1) \times k$. Here, t can be chosen as n, then (9) has a zero submatrix of order $(m' - k + 1) \times k$. For this $(m' - k + 1)$ rows, there are only $(m' - k)$ nonzero columns. Consequently, matrix (9) is of form II and by *lemma 4*, there is S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (4).

Sufficiency: If there is S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$, by *lemma 4*, matrix (9) is of form II, then obviously P_1 in (9) can not achieve row rank equal to k and furthermore, (9) can not have g-rank = n. ■

Next is another main result of this paper, which is actually one graphic necessary condition for the switched linear system (4) to be structurally controllable:

Theorem 4.(Necessary Condition) If the switched linear system (4) with graphic representations $\mathcal{G}_i$,$i \in \{1, \ldots, m\}$ is structurally controllable, the colored union graph $\tilde{\mathcal{G}}$ should always satisfy the following two conditions:

i) there is no nonaccessible node in the colored union graph $\tilde{\mathcal{G}}$;
ii) there is no S-$dilation$ in the colored union graph $\tilde{\mathcal{G}}$.

Proof: *(i)* If there exist nonaccessible nodes in $\tilde{\mathcal{G}}$, by *lemma 3*, the matrix

$$[A_1 + A_2 + A_3 + \cdots + A_m, B_1 + B_2 + B_3 + \cdots + B_m]$$

is reducible or of form I. It follows that the controllability matrix

$$\begin{aligned} &[B_1 + B_2 + \ldots + B_m, \\ &(A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m), \\ &(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m), \\ &, \ldots, \\ &(A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)], \end{aligned}$$

always has at least one row that is identically zero (*remark 2*). We can know that every component of the matrix, such as B_i, A_iB_j and $A_i^pA_j^qB_r$ has the same row always to be zero. As a result, the controllability matrix

$$\begin{aligned}&[B_1, B_2, B_3, \ldots, B_m,\\&A_1B_1, A_2B_1, A_3B_1, \ldots, A_mB_m,\\&, \ldots,\\&A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m],\end{aligned}$$

always has one zero row and can not achieve full rank n. Therefore, the switched linear system (4) is not controllable and not structurally controllable.

(ii) If there is $S\text{-}dilation$ in the colored union graph $\tilde{\mathcal{G}}$ of switched linear systems, by *lemma 5*, the matrix

$$[A_1, A_2, A_3, \ldots, A_m, B_1, B_2, B_3, \ldots, B_m]$$

has g-rank less than n. It means that we can not find any set of parameters that can make this matrix has full rank n. Then for the controllability matrix of switched linear system (4):

$$\begin{aligned}&[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_1B_m, \ldots, A_mB_m,\\&A_1^2B_1, \ldots, A_mA_1B_1, \ldots, A_1^2B_m, \ldots, A_mA_1B_m, \ldots,\\&A_1^{n-1}B_1, \ldots, A_mA_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m],\end{aligned} \tag{10}$$

we know that the columns space of (10) is the summation of every component $A_i^pA_j^qB_r$'s column space and obviously the column space of $A_i^pA_j^qB_r$ is contained in the column space of A_i. It follows that the column space of controllability matrix (10) is contained in the column space of $[A_1, A_2, A_3, \ldots, A_m, B_1, B_2, B_3, \ldots, B_m]$. Then it follows that the g-rank of (10) is less than n. Consequently the switched linear system (4) is not structurally controllable. ■

C. Structural Controllability of Multi-Agent Systems as the Special Case of Switched Linear Systems

Two graphic sufficient conditions and one graphic necessary condition have been proposed in the above discussion. The union graph does not differentiate the information flows from different subsystems. But in colored union graph, the information which subsystems specific edges come from is provided. It turns out that the conditions based on the union graph $\mathcal{G}$ are much stronger than the conditions based on the colored union graph $\tilde{\mathcal{G}}$. If there is no nonaccessible node and no 'dilation' in $\mathcal{G}$, the colored union graph $\tilde{\mathcal{G}}$ does not have nonaccessible node and $S\text{-}dilation$. The sufficient conditions based on union graph will be illustrated that they are not necessary conditions for the switched linear systems to be structurally controllable. Furthermore, We still need one complete necessary and sufficient condition for structural controllability of switched linear systems. Fortunately, as a special case of switched linear systems, our previous work on multi-agent systems with switching topology [23], in which very good graphic necessary and sufficient condition was introduced, gives us perspective to find such kind of graphic condition for general switched linear systems.

In [23], the multi-agent system with switching topology was modeled as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{aq_i} & B_{aq_i} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix} \quad . \tag{11}$$

where $i \in \{1, \ldots, m\}$, $A_{aq_i} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B_{aq_i} \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $\mathcal{L}$. The matrix A_{aq_i} reflects the interconnection among followers, and the column vector B_{aq_i} represents the relation between followers and the leader under corresponding subsystems. Since the communication topologies among agents are time-varying, so the matrices A_{aq_i} and B_{aq_i} are also varying as a function of time. Therefore, the dynamical system described in (11) can be naturally modeled as a switched linear system.

For each subsystem of this special switched linear system, the matrix pair (A_{aqi}, B_{aqi}), $i \in \{1, \ldots, m\}$ can be represented by a digraph G_i. For the whole multi-agent system, a union graph G was also defined in the same way with it is defined this paper. Then it turns out that the multi-agent system with switching topology is structurally controllable if and only if the union graph G of the underlaying communication topologies is connected. It means that we can always assign proper communication weights between agents to make this whole multi-agent system controllable if and only if the connectivity of union graph is kept.

Multi-agent system is a special case of switched linear system, which requires structured symmetry (symmetric elements are free parameters or zeros simultaneously) and free parameters on diagonal elements. For this special case, we can get graphic necessary and sufficient condition for structural controllability.

IV. Numerical Examples

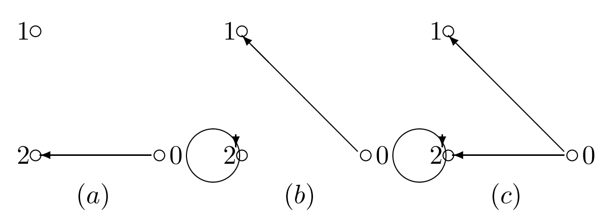

Fig. 4. Switched linear system with two subsystems

To illustrate our main results, we consider here several switched linear systems with two subsystems and single input. Switched linear system 1 is described by the graphs in Fig. 4(a)-(b), where the 0 node represent the input and the rest are state nodes. Overlay the subgraphs together to get the union graph $\mathcal{G}$ of this example shown in Fig. 4 (c). It turns out that the union graph of the switched system has no nonaccessible node and no 'dilation' (actually a cactus). By one of our main results *Theorem 2*, we get that the switched linear system is structurally controllable.

Next, we will check the rank condition of this switched system to see whether it is structurally controllable.

From Fig. 4, we can compute the system matrices from subgraph of each subsystem to be:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_3 \\ 0 \end{bmatrix}.$$

According to *Lemma 1*, we have the controllability matrix for this switched linear system here:

$$[B_1, B_2, A_1B_1, A_2B_1, A_1B_2, A_2B_2]. \tag{12}$$

Apply (12) to this example, we can easily find there are only three nonzero column vectors here:

$$\begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \lambda_1\lambda_2 \end{bmatrix}.$$

We impose all the parameters scalar 1.It turns out that the three column vectors have rank 2. As a result, the matrix has full row rank 2. Finally it shows that if the union graph $\mathcal{G}$ is spanned by a cacti or with no nonaccessible and no 'dilation', the switched linear system is structurally controllable.

For the next example, we still consider a switched linear system with two subsystems described by the graphs in Fig. 5(a)-(b). Overlay the subgraphs together to get the union graph $\mathcal{G}$ of this example shown in Fig. 5 (c). Easily we can see that if node 1 and 3 are chosen to compose S and now $T(S)$ only has node 0, therefore, there is 'dilation' in the union graph $\mathcal{G}$. The colored union graph $\tilde{\mathcal{G}}$ is shown in Fig. 5 (d), where the thick lines represent the edges coming from subgraph (b). It turns out that the colored union graph $\tilde{\mathcal{G}}$ has no nonaccessilbe node and no $S\text{-}dilation$.

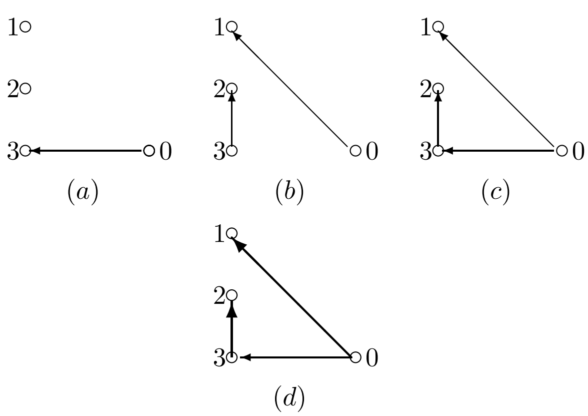

Fig. 5. Another switched linear systems with two subsystems

Next, we will check the rank condition of this switched linear system to see whether it is structurally controllable or not.

From Fig. 5, we can compute the system matrices of subgraphs of corresponding subsystems to be:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_3 \\ 0 \\ 0 \end{bmatrix}.$$

we can easily find three nonzero column vectors (there also other nonzero columns) in the controllability matrix for this switched linear system:

$$\begin{bmatrix} 0 \\ 0 \\ \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_3 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \lambda_1\lambda_2 \\ 0 \end{bmatrix}.$$

We impose all the parameters scalar 1. It follows that these three column vectors are linear independent. As a result, the matrix has full row rank and by $Lemma$ 1, the switched linear system is controllable and therefore structurally controllable. Then it follows that the condition in *Theorem 2* is not necessary for the switched linear system to be structurally controllable. Besides, this switched linear system is structurally controllable and here the colored union graph $\tilde{\mathcal{G}}$ has no nonaccessible node and no $S\text{-}dilation$.

The last example is presented to illustrate the necessary condition in *theorem 4* with colored union graph. The representation subgraph for each subsystem is depicted in Fig. 6(a)-(b).

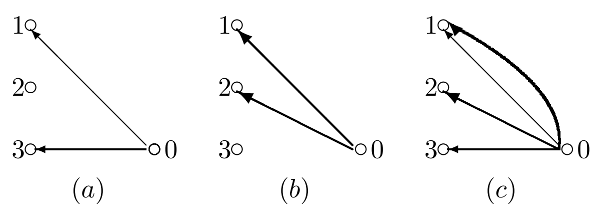

Fig. 6. Another switched linear systems with two subsystems

Fig. 6 (c) is the colored union graph of this switched linear system and the thicklines represent the edges coming from the second subsystem. If we choose node 1,2,3 as the nodes in S, then the set $T(S)$ contains 2 nodes (two 0 nodes from different subgraphs) and $|T(S)| < |S|$. It follows that there is $S\text{-}dilation$ in the colored union graph $\tilde{\mathcal{G}}$ and according to *theorem 4*, the corresponding switched linear system is not structurally controllable.

Similarly, we need to check the controllability matrix for this switched linear system and we write the system matrices first:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_2 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_3 \\ \lambda_4 \\ 0 \end{bmatrix}.$$

Compute the controllability matrix for this example:

$$\begin{bmatrix} \lambda_1 & \lambda_3 & 0 & \dots & 0 \\ 0 & \lambda_4 & 0 & \dots & 0 \\ \lambda_2 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Obviously, no mater what kind of values are assigned to the free parameters, the controllability matrix always has rank less than 3. Consequently, the switched linear system is not controllable and not structural controllable and this finally illustrate our result in *theorem 4*.

From the above several examples, we illustrate our main results and present an intuitive interpretation that the switched linear system is structurally controllable if the union graph $\mathcal{G}$ is spanned by a cacti (or no nonaccessible node and no 'dilation') and the colored union graph should have no nonaccessible nodes and no S-$dilation$ if the switched linear system is structurally controllable.

V. Conclusions and Future Work

In this paper, a more 'practical' concept of controllability: structural controllability for switched linear systems has been investigated. Combining the knowledge of the literature of switched linear systems and graph theory, several graphic necessary and graphic sufficient conditions for the structurally controllability of switched linear systems have been proposed. It was shown that switched linear system is structurally controllable if the union graph $\mathcal{G}$ is spanned by a cacti (or no nonaccessible node and no 'dilation') and the colored union graph should have no nonaccessible node and no S-$dilation$ if the switched linear system is structurally controllable. These graphic interpretations provide us better understanding on how the graphic topologies of switched linear systems will influence or determine the structural controllability of switched linear systems and therefore, would be of great practical significance for different kinds of physical systems or processes.

Although we get several graphic interpretations for the structural controllability of switched linear systems, a good graphic necessary and sufficient condition still needs our further study. This shows us a great perspective than we can design the switching algorithm to make the switched linear system structurally controllable conveniently just making sure some properties of the corresponding graph (union or colored union graph) are kept during the switching process.

References

[1] H. Lin and P. J. Antsakis, "Switching stabilizability for continuous-time uncertain switched linear systems," *IEEE Transaction on Automatic Control,* vol. 52, no. 4, pp. 633-646, April 2007.

[2] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic Control,* vol. 54, no. 2, pp. 308-322, Feb. 2009.

[3] Z. Sun, S. S. Ge, and T. H. Lee, "Controllability and reachability criteria for switched linear systems," *Automatica,* vol. 38, no. 5, pp. 775-786, May 2002.

[4] Z. Sun and S. S. Ge, "Analysis and synthesis of switched linear control systems," *Automatica,* vol. 41, no. 2, pp. 181-195, Feb. 2005.

[5] Z. J. Ji, H. Lin, and T. H. Lee, "A new perspective on criteria and algorithms for reachability of discrete-time switched linear systems, " *Automatica,* vol. 45, no. 6, pp. 1584-1587, Jun. 2009.

[6] Z. J. Ji, L. Wang, and X. X. Guo, "On controllability of switched linear systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 796-801, Apr. 2008.

[7] Z. J. Ji, L. Wang, and X. X. Guo, "Design of switching sequences for controllability realization of switched linear systems," *Automatica,* vol.43, no.4, pp. 662-668, Apr. 2007.

[8] Z. J. Ji, G. Feng, and X. X. Guo, "A constructive approach to reachability realization of discrete-time switched linear systems," *Systems & Control Letters,* vol.56, no.11-12, pp. 669-677, Nov.-Dec. 2007.

[9] A. Leonessa, W. M. Haddad, and V. S. Chellaboina, "Nonlinear system stabilization via hierarchical switching control," *IEEE Transactions on Automatic Control,* vol. 46, no. 1, pp. 17-28, Jan. 2001.

[10] A. S. Morse, "Supervisory control of families of linear set-point controllers Part I. Exact matching," *IEEE Transactions on Automatic Control,* vol. 41, no. 10, pp. 1413-1431, Oct. 1996.

[11] K. S. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Transactions on Automatic Control,* vol. 42, no. 2, pp. 171-187, Feb. 1997.

[12] K. A. Loparo, J. T. Aslanis, and O. Hajek, "Analysis of switching linear systems in the plain, part 2, golbal behavior of trajectories, controllability and attainablility," *Journal of Optimization Theory and Applications,* vol. 52, no. 3, pp. 395-427, Mar. 1987.

[13] X. Xu and P. J. Antsaklis, "On the reachability of a class of second-order switched systems," *Technical Report. ISIS-99-003, University of Notre Dame*, 1999.

[14] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *International Journal of Control,* vol. 49, no. 6, pp. 2045-2055, Jun. 1989.

[15] F. Szigeti, "A differential-algebraic condition for controllability and observability of time varying linear systems, " *in Proc. of the 31st IEEE Conference on Decision and Control,* vol. 4, Dec.16-Dec.18, 1992, pp. 3088-3090.

[16] G. Xie and L. Wang, "Controllability and stabilizability of switched linear systems, " *Systems & Control Letters* vol. 48, no. 2, pp. 135-155, Feb 2003.

[17] D. Cheng, "Controllability of switched bilinear systems, " *IEEE Transactions on Automatic Control,* vol. 50, no.4 pp. 511-515, Apr. 2005.

[18] C. T. Lin, "Structural controllability," *IEEE Transactions on Automatic Control,* vol. 19, no. 3, pp. 201-208, Jun. 1974.

[19] H. Mayeda, "On structural controllability theorem," *IEEE Transactions on Automatic Control,* vol. 26, no. 3, pp.795-798, Jun. 1981.

[20] R. W. Shields and J. B. Pearson, "Structural controllability of multi-input linear systems," *IEEE Transactions on Automatic Control,* vol. 21, no. 2, pp. 203-212, Apr. 1976.

[21] K. Glover and L. M. Silverman, "Characterization of structural controllability," *IEEE Transactions on Automatic Control,* vol. 21, no. 4, pp. 534-537, Aug. 1976.

[22] J. M. Dion, C. Commault, and J. van der Woude, "Generic properties and control of linear structured systems" *Automatica,* vol. 39, no. 7, pp. 1125-1144, Jul. 2003.

[23] X. M. Liu, H. Lin, and B. M. Chen, "A graph-theoretic characterization of structural controllability for multi-agent system with switching topology," to appear *in Proc. of the Combined 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference,* 2009.