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Controllability of multi-agent systems with time-delay in state and switching topology

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In this article, the controllability issue is addressed for an interconnected system of multiple agents. The network associated with the system is of the leader–follower structure with some agents taking leader role and others being followers interconnected via the neighbour-based rule. Sufficient conditions are derived for the controllability of multi-agent systems with time-delay in state, as well as a graph-based uncontrollability topology structure is revealed. Both single and double integrator dynamics are considered. For switching topology, two algebraic necessary and sufficient conditions are derived for the controllability of multi-agent systems. Several examples are also presented to illustrate how to control the system to shape into the desired configurations.

Keywords: controllability; multi-agent system; single-integrator; double-integrator; switching topology; state delay

1. Introduction

In recent years, the decentralised coordinated control of multi-agent systems has received considerable attention. This is partly due to the communication of the technological advances and broad applications of multi-agent systems in the area of unmanned air and underwater vehicles, formation control of satellite clusters and so on. Also, studies in this direction have been inspired by the cooperative behaviour of biological swarms, such as ant colonies and bird flocks, where collective motions may emerge from groups of simple individuals through limited interactions. To understand the mechanism inherent in the motion coordination of agents, the information flow and interaction topologies among multiple dynamic agents call for an intensive study. Many researchers have devoted themselves to modelling and understanding the cooperative principles of such collective behaviours, as well as their potential engineering applications (Kozyreff, Vladimirov, and Mandel 2000; Earl and Strogatz 2003; Jadbabaie, Lin, and Morse 2003; Moreau 2004; Tanner 2004; Olfti-Saber and Murray 2004; Amano, Luo, and Hosoe 2005; Ren and Beard 2005; Liu, Xie, Chu, and Wang 2006b; Wang and Xiao 2006; Ji, Muhammad, and Egerstedt 2006; Papachristodoulou and Jadbabaie 2006; Xu and Pei 2006; Sun, Wang, and Xie 2006; Liu, Chu, Wang, and Xie 2006a; Ghabcheloo, Aguiar, Pascoal, and Silvestre 2007; Ji and Egerstedt 2007; Hu and Hong 2007; Ji, Lin, and Lee 2008b; Liu, Zou, Zhang, Chu, and Wang 2008b; Gazi 2008; Liu, Chu, Wang, and Xie 2008a; Bliman and Ferrari-Trecate 2008; Ji, Lin, and Lee 2008a; Rahmani, Ji, Mesbahi, and Egerstedt 2009).

The controllability of a multi-agent system means that the system can be steered from any one state to any other one through certain regulations. The investigation of formation control in terms of controllability has been proved to be advisable for multi-agent systems (see e.g. Ji et al. (2006), Ji and Egerstedt (2007), Liu et al. (2006b), Liu et al. (2008), Tanner (2004), Liu et al. (2006)). The controllability problem was put forward for the first time by Tanner (2004) for multi-agent systems, where necessary and sufficient conditions were derived with respect to fixed topology. The idea is to transform the formation control into a classical controllability problem for fixed topology as well as a switched controllability problem for switching topology. After that, the controllability was characterised from a graphical point of view (Ji et al. 2006; Ji and Egerstedt 2007; Rahmani et al. 2009). Inspired by the above work, graph-based properties were investigated for the controllability of multi-agent systems with respect to fixed topology in Ji et al. (2008a) as well as algebraic conditions were derived for switching

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topology in Ji et al. (2008b). The controllability problem was also studied under fixed and switching topologies for continuous-time case in Liu et al. (2006b, 2008a), and in Liu et al. (2006a) for discrete-time case. In spite of this progress, much work remains to be done to cope with the controllability problem in the presence of e.g. information transmission time-delays. This motivates the present study.

It is well recognised that time-delay phenomenon is ubiquitous in nature and engineering, including mechanical engineering, aeronautics and astronautics, ecology, biology, information technology, economics, etc. (Xu and Pei 2006). Time-delay effect may occur naturally because of the physical characteristics of information transmitting, diversity of signals, as well as the bandwidth of communication channels. In particular, communication delays occur frequently in networks. Many results on multi-agent systems with communication delays have been obtained. For example, consensus problems with communication delays were studied in Moreau (2004), Olfti-Saber and Murray (2004), Sun et al. (2006), Wang and Xiao (2006), Hu and Hong (2007), Bliman and Ferrari-Trecate (2008). The stability with time delays was analysed in Kozyreff et al. (2000), Earl and Strogatz (2003), Amano et al. (2005), Papachristodoulou and Jadbabaie (2006), Ghabcheloo et al. (2007), Gazi (2008), and two sufficient conditions were recently reported in Liu et al. (2008b) with respect to the controllability of multi-agent systems with single time-delay. In this article, we consider a multi-agent system in the leader–follower framework, with both single and double integrator dynamics. The leaders are unidirectional, unaffected by the followers whereas the followers are influenced by leaders directly or indirectly. The leaders play the role of external input to control the subsystem consisting of the followers. Each follower updates its state based on the current information available from its neighbouring agents and the leaders. We introduce time-delay models for the multi-agent system, and derive sufficient conditions for the system to be controllable. Also, a graph-based necessary characterisation is presented for the controllability of delayed multi-agent systems with respect to fixed topology, and necessary and sufficient conditions are derived for the system to be controllable under switching topology.

This article is organised as follows. Section 2 is a brief review of graph theoretic terminologies. Section 3 follows with the system model with delay in state for both single and double integrator dynamics. In Section 4, we analyse the controllability for delayed system and switching topology. Numerical examples are included in Section 5. Finally, the results are briefly summarised in Section 6.

2. Preliminaries

In this section, we briefly recall some basic concepts and notations in graph theory which will be used in this article. The reader is referred to Godsil and Royle (2001) for details.

A directed graph (digraph) G consists of a vertex set $V(G)=\{v_1, v_2, \dots, v_n\}$ and an arc set $E(G)=\{e_{ij}=(v_i, v_j): v_i, v_j \in V(G)\}$, where an arc is an ordered pair of distinct vertices in $V(G)$. An arc (v_i, v_j) in a digraph denotes that agent j can obtain information from agent i, but not necessarily vice versa. In contrast, the pairs of vertices in an undirected graph are unordered, where an edge (v_i, v_j) denotes that agents i and j can obtain information from one another. An undirected graph can be considered a special case of a directed graph, where an edge (v_i, v_j) in the undirected graph corresponds to arcs (v_i, v_j) and (v_j, v_i) in the digraph. If there is an arc from v_i to v_j, v_i is defined as the parent vertex and v_j is defined as the child vertex. The set of neighbours of v_i in G is denoted by $N_i=\{v_j: (v_i, v_j) \in E(G)\}$. The number of elements in the set N_i is called out-degree of vertex v_i. Similarly, the number of elements in the set $\tilde{N}_i=\{v_j: (v_j, v_i) \in E(G)\}$ is called in-degree of vertex v_i. A path from v_i to v_j is meant that there is a sequence of distinct arcs in $E(G)$, (v_i, v_1), $(v_1, v_2), \cdots, (v_r, v_j)$. Here we exclude self-loops and multiple arcs between a pair of distinct vertices. A directed graph is called to be strongly connected if there exists a path between any two distinct vertices of the graph. For undirected graph, the strongly connected property is usually called connected. Let $G=(V, E)$ and $G_s=(V_s, E_s)$ be two directed graphs. A subgraph G_s of a directed graph G is a digraph such that the vertex set $V(G_s) \subset V(G)$ and the arc set $E(G)_s \subset E(G)$. If $V(G_s)=V(G)$, we call G_s a spanning subgraph of G. For any $v_i, v_j \in V(G_s)$, if $(v_i, v_j) \in E(G_s)$ if and only if $(v_i, v_j) \in E(G)$, we call G_s an induced subgraph of G. In this case, we also say that G_s is induced by $V(G_s)$. An induced subgraph of an undirected graph G, which is maximal and connected, is said to be a connected component of the undirected graph.

A weighted directed graph $G(A)$ is a digraph G plus a nonnegative weight matrix $A=[a_{ij}] \in \mathbb{R}^{n\times n}$ such that for any $i \neq j$, $(v_i, v_j) \in E(G) \Leftrightarrow a_{ji} > 0$. Here A is called the weighted adjacency matrix of the directed graph G, and a_{ji} is said to be the weight of arc (v_i, v_j). Particularly, if A is a 0–1 matrix (entries of which are 0 or 1), then we say $G(A)$ is an unweighed directed graph, or say $G(A)$ depicts the directed graph structure $G(V, E)$. If G is an undirected graph, the associated weighted adjacency matrix A is symmetric, i.e. $A^T=A$. Throughout this article, we always use G to represent a graph structure and $G(A)$ to represent a graph with unweighed adjacency matrix A. The Laplacian matrix $L(G)=(l_{ij})\in\mathbb{R}^{n\times n}$ of an unweighed directed graph $G(A)$, abbreviated as L, is defined as: $l_{ij}=-a_{ij}$, if $i\neq j$; otherwise, $l_{ij}=\sum_{v_j\in Ni}a_{ij}$. For an undirected graph, the Laplacian matrix L is always symmetric and positive semidefinite. However, the matrix L for a directed graph does not have this property. And the Laplacian matrix could also be defined as: $L=D-A$, in which $D=\mathrm{diag}(A\cdot\mathbf{1})$ is the in-degree matrix of G with diagonal elements $d_i=\sum_{v_j\in Ni}a_{ij}$. For undirected graph, the in-degree matrix is usually called degree matrix. In the case of undirected graph, all of the eigenvalues of L are nonnegative. In the case of directed graph, all of the eigenvalues of L have nonnegative real parts. In cases of both directed graph and undirected graph, 0 is an eigenvalue of L with an associated eigenvector $\mathbf{1}$, where $\mathbf{1}$ is an $n\times 1$ column vector of all ones.

3. Problem formulation and models

3.1 *Single-integrator dynamics agents*

The multi-agent system consists of $N+l$ dynamic agents, in which the agents indexed by $N+i$, $i=1,\dots,l$, are assigned as leaders; the others indexed by $1,\dots,N$ are referred to as followers. A continuous-time system model, with single-integrator dynamic, is described by

$$\begin{cases}\dot{x}_i=u_i, & i=1,\dots,N\\ \dot{x}_{N+j}=u_{N+j}, & j=1,\dots,l\end{cases}\tag{1}$$

where $x_i\in\mathbb{R}^n$ is the state of agent i and $u_i\in\mathbb{R}^n$ is the input, $i=1,\dots,N+l$. The directed graph $G=(V,E)$ is employed to depict the communication relations among agents of such a system. We begin with defining the directed interconnection graph to describe the multi-agent system.

Definition 1 (Directed interconnection graph (Tanner 2004))**:** The directed interconnection graph, $G=(V,E)$, is being defined as a directed graph consisting of

- a set of vertices, $V=\{v_1,\dots,v_N,v_{N+1},\dots,v_{N+l}\}$, indexed by the agents in the system, and
- a set of arcs, $E=\{e_{ij}=(v_i,v_j):v_i,v_j\in V\}$, containing ordered pairs of vertices that correspond to interconnected agents.

Denote by $N_i=\{v_j:(v_i,v_j)\in E(G)\}$ the neighbouring set of agent i. The topology of an interconnection graph is said to be fixed if each vertex of the graph has a fixed neighbour set. If agent j is not a neighbour of agent i, we denote this by $v_j\nrightarrow v_i$. The communication among agents is realised through the control input u_i, defined by

$$u_i=-\sum_{v_j\in N_i}w_{ij}(x_i-x_j),\quad i\in\{1,\dots,N+l\},$$

where $W=(w_{ij})$, $w_{ij}\geq 0$, is the interactive matrix. Here $w_{ij}=w_{ji}$ is not required, i.e. the system is general anisotropic. With $x=[x_1,\dots,x_{N+1}]^T$ being the stack vector of all the agent states, the interconnected system (1) can be written in a matrix form

$$\dot{x}=-Hx,\tag{2}$$

where the matrix $H=[h_{ij}]_{(N+l)\times(N+l)}$ is defined by

$$h_{ij}=\begin{cases}-w_{ij}, & \text{if } i\neq j \text{ and } v_j\in N_i,\\ \sum_{v_j\in N_i}w_{ij}, & \text{if } i=j,\\ 0 & \text{otherwise.}\end{cases}$$

It can be readily seen that in association with the matrix H, its off-diagonal elements are all negative or zero, and its row sums are all equal to zero. If the directed or undirected graph is unweighed, i.e. $w_{ij}=1$ for any $j\in N_i$, then H degenerates into the Laplacian matrix L of the directed or undirected interconnection graph, and accordingly, (2) becomes

$$\dot{x}=-Lx.\tag{3}$$

If x is n-dimensional, (2) would be changed into the form $\dot{x}=-(H\otimes I_n)x$, where I_n is the $n\times n$ identity matrix, and the symbol $\otimes$ denotes the Kronecker product of matrices. If for any $e_{ij}\in E$, the arc $e_{ji}\in E$ as well, the communication is said to be bidirectional. That is, if agent i can receive information from agent j, agent j can receive information from agent i as well. In this case, we use an edge (represented by —) to depict it; otherwise, the communication is said to be unidirectional and we use an arc (represented by →) to depict it.

Assumption 1: *The communication between leaders and followers is unidirectional, i.e. the leaders are unaffected by the followers whereas a follower may be influenced by leaders as well as other followers, and the leaders are governed by some exogenous control inputs which can drive the states of leaders to be arbitrary values.*

Rewrite the agents as

$$\begin{cases}y_i\triangleq x_i, & i=1,\dots,N,\\ z_j\triangleq x_{N+j}, & j=1,\dots,l.\end{cases}$$

With y and z being the stack vectors of all followers y_i and leaders z_i, respectively, we can rewrite system (3) in the form

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\begin{bmatrix} \mathscr{F} & \mathscr{R} \\ 0 & \mathscr{L} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix},$$

where $\mathscr{F}$ and $\mathscr{L}$ are $N \times N$ and $l \times l$ matrices corresponding to the indexes of followers and leaders, respectively; $\mathscr{F}$, $\mathscr{L}$ and $\mathscr{R}$ are submatrices inherited from Laplacian matrix L, and $u \in \mathbb{R}^l$ is the collection of exogenous control inputs for leaders. Then the dynamics of the followers corresponding to the y component can be extracted as

$$\dot{y} = -\mathscr{F}y - \mathscr{R}z, \tag{4}$$

with the control inputs being the leaders' states. In what follows, to facilitate presentation, we first state the single time-delayed model. The multiple time-delayed one can be formulated in the same manner. It is assumed that the delay only affects the information transmitted from one agent to another, i.e. $\tau_{ii} = 0$, $i = 1, \ldots, N$; and $\tau_{ij} = \tau$ for any $i \neq j$ $(i, j = 1, \ldots, N)$, where τ is a positive integer. It follows from (4) that

$$\begin{aligned}\dot{y}_i(t) = &-f_{i1}y_1(t) - f_{i2}y_2(t) - \cdots - f_{iN}y_N(t) \\ &- r_{i1}z_1(t) - \cdots - r_{il}z_l(t), \quad i = 1, \ldots, N,\end{aligned}$$

where the coefficients are the elements inherited from L. Accordingly, the dynamics with time-delays reads

$$\begin{aligned}\dot{y}_1(t) = &-f_{11}y_1(t) - f_{12}y_2(t-\tau) - \cdots - f_{1N}y_N(t-\tau) \\ &- r_{11}z_1(t) - \cdots - r_{1l}z_l(t), \\ &\vdots \\ \dot{y}_N(t) = &-f_{N1}y_1(t-\tau) - f_{N2}y_2(t-\tau) - \cdots - f_{NN}y_N(t) \\ &- r_{N1}z_1(t) - \cdots - r_{Nl}z_l(t),\end{aligned} \tag{5}$$

which can be written into the matrix form

$$\begin{aligned}\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_N(t) \end{bmatrix} = &-\begin{bmatrix} f_{11} & & & \\ & f_{22} & & \\ & & \ddots & \\ & & & f_{NN} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{bmatrix} \\ &- \begin{bmatrix} 0 & f_{12} & \cdots & f_{1N} \\ f_{21} & 0 & \cdots & f_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N1} & f_{N2} & \cdots & 0 \end{bmatrix}\begin{bmatrix} y_1(t-\tau) \\ y_2(t-\tau) \\ \vdots \\ y_N(t-\tau) \end{bmatrix} \\ &- \mathscr{R}z(t).\end{aligned}$$

Since $L = D - A$, it can be readily seen that $\mathscr{F} = \mathscr{D} - \mathscr{A}$, where $\mathscr{D}$ and $\mathscr{A}$ are the $N \times N$ matrix obtained from in-degree matrix D and unweighed adjacency matrix A after deleting the last l rows and l columns, respectively. With control inputs being the leaders' states, we have the model of multi-agent system with time-delay in state as below

$$\dot{y}(t) = -\mathscr{D}y(t) + \mathscr{A}y(t-\tau) - \mathscr{R}z(t), \quad t > t_0. \tag{6}$$

To state the problem clearly, we give the following definition for System (6).

Definition 2 (Controllability with delay)**:** System (6) is said to be controllable, if for any initial state $y(t)$, $t \in [-\tau, t_0]$ and any final state y_f, there exist a finite time $t_f > t_0$ and an admissible input $z(t)$ defined on $[t_0, t_f]$ such that $y(t_f) = y_f$.

With respect to multiple time-delays, it is assumed that $\tau_{ii} = 0$, $i = 1, \ldots, N$; and there exists a positive integer $\tau_{\max}$ such that $\tau_{ij} \leq \tau_{\max}$ for any $i \neq j$, $(i, j = 1, \ldots, N)$. In this case, communication delays may be different. It can be seen that all time-delays τ_{ij} belong to the set $\{0, 1, 2, \ldots, \tau_{\max}\}$. Under this setup, the model with multiple time-delays in state reads

$$\begin{aligned}\dot{y}(t) = &(-\mathscr{D} + \mathscr{A}_{\tau_0})y(t) + \mathscr{A}_{\tau_1}y(t-\tau_1) + \cdots \\ &+ \mathscr{A}_{\tau_{\max}}y(t-\tau_{\max}) - \mathscr{R}z(t), \quad t > t_0,\end{aligned} \tag{7}$$

where $\tau_{\max} > \tau_{\max-1} > \cdots > \tau_1 > \tau_0 = 0$; the initial function $y(t) = \varphi(t)$ is given for $t \in [-\tau_{\max}, t_0]$; $\mathscr{A}_k$ corresponds to the part of dynamics with time-delay k, and $\sum_{\tau_i=0}^{\tau_{\max}} \mathscr{A}_{\tau_i} = \mathscr{A}$.

Definition 3 (Controllability with multiple delays)**:** System (1) is said to be controllable on $[t_0, t_f]$ if for any specified initial state $y(t)$, $t \in [-\tau_{\max}, t_0]$ and any final state y_f, there exist a finite time $t_f > t_0$ and an admissible input $z(t)$ defined on $[t_0, t_f]$ such that $y(t_f) = y_f$.

3.2 *Double-integrator dynamics agents*

In this subsection, we consider the following continuous-time system of $N + l$ agents with double-integrator dynamics

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad i = 1, \ldots, N + l, \tag{8}$$

where $x_i \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n$ are, respectively, the position and velocity of agent i, and $u_i \in \mathbb{R}^n$ is the control input. We study the controllability problem under the following two protocols (Jiang, Wang, Xie, Ji, and Jia, 2009): one is with the feedbacks of relative velocities

$$u_i = -\sum_{j \in N_i}(x_i - x_j) - k\sum_{j \in N_i}(v_i - v_j), \quad i \in \{1, \ldots, N + l\}, \tag{9}$$

and the other is with the feedbacks of absolute velocity

$$u_i = -\sum_{j\in N_i}(x_i - x_j) + kv_i, \quad i \in \{1,\dots,N+l\}, \tag{10}$$

where $k \neq 0$ is a feedback gain.

The terminologies and notations appearing in this section have the same meaning as those in Section 3.1. So the System (8) with double-integrator dynamics under protocol (9) can be written in the form of

$$\begin{bmatrix}\dot{y}\\ \dot{z}\\ \dot{v}_y\\ \dot{v}_z\end{bmatrix} = \begin{bmatrix}0 & 0 & I_N & 0\\ 0 & 0 & 0 & I_l\\ -\mathscr{F} & -\mathscr{R} & -k\mathscr{F} & -k\mathscr{R}\\ 0 & -\mathscr{L} & 0 & -k\mathscr{L}\end{bmatrix}\begin{bmatrix}y\\ z\\ v_y\\ v_z\end{bmatrix} + \begin{bmatrix}0\\0\\0\\u\end{bmatrix},$$

where $y=[x_1^T,\dots,x_N^T]^T$ is the stacked vector of followers' positions, $z=[x_{N+1}^T,\dots,x_{N+l}^T]^T$ is the stacked vector of leaders' positions, $v_y=[v_1^T,\dots,v_N^T]^T$ and $v_z=[v_{N+1}^T,\dots,v_{N+l}^T]^T$ are the corresponding velocity vector and $u\in\mathbb{R}^l$ is the collection of exogenous control inputs for leaders. Consequently, we can induce the dynamics of followers into the following LTI system:

$$\begin{bmatrix}\dot{y}\\ \dot{v}_y\end{bmatrix} = \begin{bmatrix}0 & I_N\\ -\mathscr{F} & -k\mathscr{F}\end{bmatrix}\begin{bmatrix}y\\ v_y\end{bmatrix} + \begin{bmatrix}0 & 0\\ -\mathscr{R} & -k\mathscr{R}\end{bmatrix}\begin{bmatrix}z\\ v_z\end{bmatrix}, \tag{11}$$

with the control inputs being the leaders' states (positions and velocities). By repeating the same arguments as those in Section 3.1, the double-integrator dynamics (11) with single time-delay in state reads

$$\begin{aligned}\begin{bmatrix}\dot{y}(t)\\ \dot{v}_y(t)\end{bmatrix} &= \begin{bmatrix}0 & I_N\\ -\mathscr{D} & -k\mathscr{D}\end{bmatrix}\begin{bmatrix}y(t)\\ v_y(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ \mathscr{A} & k\mathscr{A}\end{bmatrix}\begin{bmatrix}y(t-\tau)\\ v_y(t-\tau)\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ -\mathscr{R} & -k\mathscr{R}\end{bmatrix}\begin{bmatrix}z(t)\\ v_z(t)\end{bmatrix}, \quad t > t_0.\end{aligned} \tag{12}$$

If protocol (10) is employed, the closed-loop system of multi-agent system (8) is

$$\begin{bmatrix}\dot{y}\\ \dot{z}\\ \dot{v}_y\\ \dot{v}_z\end{bmatrix} = \begin{bmatrix}0 & 0 & I_N & 0\\ 0 & 0 & 0 & I_l\\ -\mathscr{F} & -\mathscr{R} & kI_N & 0\\ 0 & -\mathscr{L} & 0 & kI_l\end{bmatrix}\begin{bmatrix}y\\ z\\ v_y\\ v_z\end{bmatrix} + \begin{bmatrix}0\\0\\0\\u\end{bmatrix},$$

where y, z, v_y, v_z, u are defined as above. This yields the following dynamics of followers:

$$\begin{bmatrix}\dot{y}\\ \dot{v}_y\end{bmatrix} = \begin{bmatrix}0 & I_N\\ -\mathscr{F} & kI_N\end{bmatrix}\begin{bmatrix}y\\ v_y\end{bmatrix} + \begin{bmatrix}0 & 0\\ -\mathscr{R} & 0\end{bmatrix}\begin{bmatrix}z\\ v_z\end{bmatrix}, \tag{13}$$

with the control inputs being the leaders' states (positions). Then the double-integrator dynamics (13) with delay in state can be written as

$$\begin{aligned}\begin{bmatrix}\dot{y}(t)\\ \dot{v}_y(t)\end{bmatrix} &= \begin{bmatrix}0 & I_N\\ -\mathscr{D} & kI_N\end{bmatrix}\begin{bmatrix}y(t)\\ v_y(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ \mathscr{A} & 0\end{bmatrix}\begin{bmatrix}y(t-\tau)\\ v_y(t-\tau)\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ -\mathscr{R} & 0\end{bmatrix}\begin{bmatrix}z(t)\\ v_z(t)\end{bmatrix}, \quad t > t_0.\end{aligned} \tag{14}$$

Definition 4: System (12) or (14) is said to be controllable if for any initial state $[y^T(t), v_y^T(t)]^T$, $t\in[-\tau, t_0]$ and any final state $[y^T, v_y^T]_f^T$, there exist a finite time $t_f > t_0$ and an admissible input $[z^T(t), v_z^T(t)]^T$ defined on $[t_0, t_f]$ such that $[y^T(t_f), v_y^T(t_f)]^T = [y^T, v_y^T]_f^T$.

Under protocol (9), the model with multiple delays reads

$$\begin{aligned}\begin{bmatrix}\dot{y}(t)\\ \dot{v}_y(t)\end{bmatrix} &= \begin{bmatrix}0 & I_N\\ -\mathscr{D}+\mathscr{A}_{\tau_0} & -k\mathscr{D}+k\mathscr{A}_{\tau_0}\end{bmatrix}\begin{bmatrix}y(t)\\ v_y(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ \mathscr{A}_{\tau_1} & k\mathscr{A}_{\tau_1}\end{bmatrix}\begin{bmatrix}y(t-\tau_1)\\ v_y(t-\tau_1)\end{bmatrix}+\cdots\\ &\quad+\begin{bmatrix}0 & 0\\ \mathscr{A}_{\tau_{\max}} & k\mathscr{A}_{\tau_{\max}}\end{bmatrix}\begin{bmatrix}y(t-\tau_{\max})\\ v_y(t-\tau_{\max})\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ -\mathscr{R} & -k\mathscr{R}\end{bmatrix}\begin{bmatrix}z(t)\\ v_z(t)\end{bmatrix}, \quad t > t_0,\end{aligned} \tag{15}$$

and under protocol (10), the model reads

$$\begin{aligned}\begin{bmatrix}\dot{y}(t)\\ \dot{v}_y(t)\end{bmatrix} &= \begin{bmatrix}0 & I_N\\ -\mathscr{D}+\mathscr{A}_{\tau_0} & kI_N\end{bmatrix}\begin{bmatrix}y(t)\\ v_y(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ \mathscr{A}_{\tau_1} & 0\end{bmatrix}\begin{bmatrix}y(t-\tau_1)\\ v_y(t-\tau_1)\end{bmatrix}+\cdots\\ &\quad+\begin{bmatrix}0 & 0\\ \mathscr{A}_{\tau_{\max}} & 0\end{bmatrix}\begin{bmatrix}y(t-\tau_{\max})\\ v_y(t-\tau_{\max})\end{bmatrix}\\ &\quad+\begin{bmatrix}0 & 0\\ -\mathscr{R} & 0\end{bmatrix}\begin{bmatrix}z(t)\\ v_z(t)\end{bmatrix}, \quad t > t_0,\end{aligned} \tag{16}$$

where $\tau_{\max} > \tau_{\max-1} > \cdots > \tau_1 > \tau_0 = 0$, the initial function $y(t)=\phi(t)$ is given for $t\in[-\tau_{\max}, t_0]$, $\mathscr{A}_k$ corresponds to the part of dynamics with time-delay k and $\sum_{\tau_i=0}^{\tau_{\max}}\mathscr{A}_{\tau_i} = \mathscr{A}$.

Definition 5 (Controllability with multiple delays)**:** System (15) or (16) is said to be controllable on $[t_0, t_f]$ if for any specified initial state $[y^T(t), v_y^T(t)]^T$, $t\in[-\tau_{\max}, t_0]$ and any final state $[y^T, v_y^T]_f^T$, there exist a finite time $t_f > t_0$ and an admissible input $[z^T(t), v_z^T(t)]^T$ defined on $[t_0, t_f]$ such that $[y^T(t_f), v_y^T(t_f)]^T = [y^T, v_y^T]_f^T$.

4. Main results

4.1 *Controllability of the delayed system with single integrator dynamics*

It has been shown in Bellman and Cooke (1963) that the solution of (6) can be represented as

$$y(t,t_0,\varphi,z)=y(t,t_0,\varphi,0)+\int_{t_0}^{t}K(t,s)(-\mathscr{R})z(s)\mathrm{d}s,$$

where $y(t,t_0,\varphi,z)$ denotes the solution to (6) at time t corresponding to the initial time $t_0, y(t,t_0,\varphi,0)$ denotes the free solution and $K(t,s)$ is the $N\times N$ fundamental matrix of (6) which satisfies the following equations:

(1) $\partial K(t,s)/\partial s=K(t,s)\mathscr{D}-K(t,s+\tau)\mathscr{A},\ s\in[t_0,t-\tau]$
(2) $K(t,t)=I$
(3) $K(t,s)=0$, for all $s>t$.

As presented in Chyung and Lee (1966), system (6) is controllable to the origin from time t_0 if there exists a finite value of time $t_1>t_0$ such that rank $\int_{t_0}^{t_1}K(t_1,s)\mathscr{R}\mathscr{R}'K'(t_1,s)\mathrm{d}s=N$, which is the controllable Gramian for the delayed system (6). Due to the difficulty of computing $K(t,s)$, some algebraic criteria on controllability of linear systems with time-delay have been established in Chyung and Lee (1966), Kirillova and Curakova (1967), Weiss (1970), where a matrix only consisted with the coefficients of differential equations is constructed for the delayed system, and the delayed system is controllable if the matrix has full rank. So the controllability problem of the delayed system is transformed to the one without time-delay through judging the rank of the matrix. To formulate the problem clearly, we denominate the matrix as the controllable matrix. The following controllability matrix Q is constructed for the multi-agent system with state delay described by (6).

$$Q=[Q_1^1,Q_1^2,\ldots,Q_1^N,Q_2^2,\ldots,Q_2^N,\ldots,Q_N^N],\tag{17}$$

where $Q_1^1=-\mathscr{R},\ Q_j^k=0$ for $j>k$ or $j=0$, and $Q_j^{k+1}=-\mathscr{D}Q_j^k+\mathscr{A}Q_{j-1}^k, j=1,2,\ldots,k, k=1,2,\ldots,N$.

The controllability of delayed system (6) relies on the rank of controllability matrix Q. If Q has full row rank, i.e. $\mathrm{rank}(Q)=N$, system (6) is controllable. To simplify the computation for the rank of Q with $l\times N(N+1)/2$ columns, we will give sufficient conditions for the delayed system to be controllable. The following lemma will be used in the derivation of the result.

Lemma 1 (Hewer 1972): *The identity*

$$(-\mathscr{D}+\mathscr{A})^N(-\mathscr{R})=Q_1^{N+1}+Q_2^{N+1}+\cdots+Q_N^{N+1}+Q_{N+1}^{N+1}$$

is valid.

If the communications between followers are bidirectional, they will be depicted by edges except those between leaders and followers. In this case, the interconnection graph is undirected and the following theorem can be derived to guarantee the delayed system to be controllable. Here the associated unweighed adjacent matrix A, in-degree matrix D and Laplacian matrix L are denoted by adding the inverse-orientation links of unidirectional arcs between the leaders and followers of the original graph G. So, A, D and L are all symmetric.

Theorem 1: *The delayed multi-agent system* (6), *with undirected interconnection graph G, l leaders and N followers, is controllable for $\tau>0$ if any of the following five conditions is fulfilled*:

(I) (i) *The eigenvalues of $\mathscr{A}$ are all distinct.*
 (ii) *The eigenvectors of $\mathscr{A}$ are not orthogonal to at least one column of $\mathscr{R}$.*
(II) *A and $\mathscr{A}$ share no common eigenvalues.*
(III) (i) *The eigenvalues, i.e. the diagonal elements of $\mathscr{D}$ are all distinct.*
 (ii) *At least all the elements in one column of $\mathscr{R}$ are nonzero.*
(IV) (i) *The eigenvalues of $\mathscr{F}$ are all distinct.*
 (ii) *The eigenvectors of $\mathscr{F}$ are not orthogonal to at least one column of $\mathscr{R}$.*
(V) *L and $\mathscr{F}$ share no common eigenvalues.*

Proof: We shall show that *rank*$(Q)=N$ under each of Conditions (I)–(V). Accordingly, system (6) is controllable since the controllability matrix Q is full row rank.

Conditions (I) and (III): In view of the expression (17) for the controllability matrix Q, calculations show that

$$Q_1^1=-\mathscr{R},\ \ Q_2^2=-\mathscr{A}\mathscr{R},\ldots,\ Q_N^N=-\mathscr{A}^{N-1}\mathscr{R},$$

and

$$Q_1^1=-\mathscr{R},\ \ Q_1^2=\mathscr{D}\mathscr{R},\ldots,\ Q_1^N=(-1)^N\mathscr{D}^{N-1}\mathscr{R}.$$

Set $Q_1\triangleq[Q_1^1,Q_2^2,\ldots,Q_N^N]$, $Q_2\triangleq[Q_1^1,Q_1^2,\ldots,Q_1^N]$, it follows that Q_1 and Q_2 are the controllability matrices of the following non-delayed systems, respectively,

$$\dot{y}(t)=\mathscr{A}y(t)-\mathscr{R}z(t),\tag{18}$$

$$\dot{y}(t)=-\mathscr{D}y(t)-\mathscr{R}z(t).\tag{19}$$

The idea is that since Q consists of Q_1 (or Q_2) and other matrices, the controllability matrix Q is full row rank if so is for matrix Q_1 (or Q_2). The remaining argument in this part of proof is then to show that the non-delayed system (18) and (19) are controllable under Conditions (I) and (III), respectively, as follows.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of $\mathscr{A}$. Since $\mathscr{A}$ is symmetric, there exists an orthogonal matrix U such that $\mathscr{A} = U\Lambda U^T$, where U consists of the orthogonal eigenvectors of $\mathscr{A}$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Denote $U^T\mathscr{R} \triangleq [r_1, \ldots, r_l]$ with $r_i \triangleq [r_{1i}, \ldots, r_{Ni}]^T$, $i = 1, \ldots, l$. Then

$$\begin{aligned} Q_1 &= -U\big[U^T\mathscr{R}, \Lambda U^T\mathscr{R}, \ldots, \Lambda^{N-1}U^T\mathscr{R}\big] \\ &= -U\big[[r_1, \ldots, r_l], \Lambda[r_1, \ldots, r_l], \ldots, \Lambda^{N-1}[r_1, \ldots, r_l]\big]. \end{aligned}$$

Consider the following matrix:

$$\begin{aligned} \widetilde{Q}_1 &= \big[[r_1, \Lambda r_1, \ldots, \Lambda^{N-1}r_1], \ldots, [r_l, \Lambda r_l, \ldots, \Lambda^{N-1}r_l]\big] \\ &= [\mathrm{diag}\{r_{11}, \ldots, r_{N1}\}\Xi, \ldots, \mathrm{diag}\{r_{1l}, \ldots, r_{Nl}\}\Xi], \end{aligned} \tag{20}$$

where

$$\Xi = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^{N-1} \end{bmatrix}$$

is a Vandermonde matrix. Since $\widetilde{Q}_1$ is obtained by rearranging the columns of Q_1 and U is an orthogonal matrix, we have

$$\mathrm{rank}(Q_1) = \mathrm{rank}(\widetilde{Q}_1). \tag{21}$$

By (i) of Condition (I), $\lambda_1, \ldots, \lambda_N$ are all distinct. As a consequence, Ξ is nonsingular. By (ii) of Condition (I), it can be assumed, without loss of generality, that r_1 is not orthogonal to the eigenvectors of $\mathscr{A}$. Accordingly, $r_{11}, \ldots, r_{N1}$ are nonzero and then $\mathrm{diag}\{r_{11}, \ldots, r_{N1}\}$ is nonsingular. Therefore, it follows from (20) and (21) that $\widetilde{Q}_1$ and then Q_1 is full row rank if Condition (I) is fulfilled. So, under Condition (I), the controllability matrix Q is full row rank, that is, the state delayed system (6) is controllable.

With respect to the Condition (III), we denote $\mathscr{R} = [\gamma_1, \ldots, \gamma_l]$. Then Q_2 can be written as

$$Q_2 = \big[-[\gamma_1, \ldots, \gamma_l], \mathscr{D}[\gamma_1, \ldots, \gamma_l], \ldots, (-1)^N\mathscr{D}^{N-1}[\gamma_1, \ldots, \gamma_l]\big],$$

which has the same rank as $\widetilde{Q}_2$, where

$$\begin{aligned} \widetilde{Q}_2 &= \big[[-\gamma_1, \mathscr{D}\gamma_1, \ldots, (-1)^N\mathscr{D}^{N-1}\gamma_1], \ldots, \\ &\quad [-\gamma_l, \mathscr{D}\gamma_l, \ldots, (-1)^N\mathscr{D}^{N-1}\gamma_l]\big] \\ &= \big[\mathrm{diag}\{-\gamma_{11}, \ldots, (-1)^N\gamma_{N1}\}\widetilde{\Xi}, \ldots, \\ &\quad \mathrm{diag}\{-\gamma_{1l}, \ldots, (-1)^N\gamma_{Nl}\}\widetilde{\Xi}\big], \end{aligned}$$

$\gamma_i \triangleq [\gamma_{1i}, \ldots, \gamma_{Ni}]^T$, $i = 1, \ldots, l$, and

$$\widetilde{\Xi} = \begin{bmatrix} 1 & d_1 & \cdots & d_1^{N-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & d_N & \cdots & d_N^{N-1} \end{bmatrix}.$$

By repeating the same lines of arguments as those for the system (18) in the proof concerning Condition (I), it can be seen that system (19), and then system (6), is controllable if Condition (III) is fulfilled.

Condition (II): Since $\mathscr{A}$ is obtained by deleting the last l rows and l columns of A, the adjacency matrix A can be partitioned as

$$A = \begin{bmatrix} \mathscr{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

With respect to the nondelayed system (18), the control input matrix $\mathscr{R}$ is obtained by deleting the last l rows and the first N columns of the Laplacian matrix L. This, together with the observation that $L = D - A$ and D is a diagonal matrix, yields that $A_{12} = \mathscr{R}$. Noticing that A is symmetric since it is the adjacency matrix of the undirected interconnection graph G, we have $A_{21} = \mathscr{R}^T$.

Next, we shall prove that the non-delayed system (18) is controllable under Condition (II). We are to show this by contradiction. Suppose that system (18) is uncontrollable. It follows from the controllability PBH criteria that there exists a vector $p \in \mathbb{R}^N$ such that $\mathscr{A}\,p = \mu p$ for some $\mu \in \mathbb{R}$, with $\mathscr{R}^T p = 0$. Accordingly,

$$A\begin{bmatrix} q \\ 0 \end{bmatrix} = \begin{bmatrix} \mathscr{A} & \mathscr{R} \\ \mathscr{R}^T & A_{22} \end{bmatrix}\begin{bmatrix} q \\ 0 \end{bmatrix} = \mu\begin{bmatrix} q \\ 0 \end{bmatrix},$$

implying that μ is a common eigenvalue of A and $\mathscr{A}$. Hence, the fulfilment of Condition (II) leads to the controllability of system (18). By following the arguments in the first part of proof for Condition (I), the original system (6) is then controllable.

Condition (IV): Consider the following matrix $\hat{Q}_3$ derived from the controllability matrix Q of the delayed system (6) by elementary column operations:

$$\begin{aligned} \hat{Q}_3 = \Big[&Q_1^1, Q_1^2 + Q_2^2, \ldots, Q_1^N + Q_2^N + \cdots + Q_{N-1}^N \\ &+ Q_N^N, Q_2^2, \ldots, Q_2^N, \ldots, Q_N^N\Big]. \end{aligned}$$

Since elementary column operations do not affect the rank of a matrix, $\hat{Q}_3$ has the same row rank with Q. By Lemma 1, we have

$$\begin{aligned} \hat{Q}_3 &= \big[(-\mathscr{R}), (-\mathscr{D} + \mathscr{A})(-\mathscr{R}), \ldots, (-\mathscr{D} + \mathscr{A})^{N-1} \\ &\quad \times (-\mathscr{R}), Q_2^2, \ldots, Q_2^N, \ldots, Q_N^N\big] \\ &= \big[-\mathscr{R}, \mathscr{F}\mathscr{R}, -\mathscr{F}^2\mathscr{R}, \ldots, (-1)^N \\ &\quad \times \mathscr{F}^{N-1}\mathscr{R}, Q_2^2, \ldots, Q_2^N, \ldots, Q_N^N\big]. \end{aligned}$$

Note that $Q_3 \triangleq [-\mathscr{R}, \mathscr{F}\mathscr{R}, \ldots, (-1)^N\mathscr{F}^{N-1}\mathscr{R}]$ is the controllability matrix with respect to the following nondelayed system:

$$\dot{y}(t) = -\mathscr{F}y(t) - \mathscr{R}z(t).$$

The controllability matrix Q is full row rank if so is matrix Q_3. The remaining argument is a repetition of the same part in the proof of Condition (I), with the only difference being to replace $\mathscr{A}$ by $\mathscr{F}$.

Condition (V): The proof in this part can be carried on in the same way as that for Condition (II) by replacing $\mathscr{A}$ with $\mathscr{F}$.

The proof is completed. □

The advantage of Conditions (I)–(III) in Theorem 1 consists in the independence of matrices $\mathscr{D}$ and $\mathscr{A}$.

Corollary 1: *The nondelayed system* (4) *with multiple leaders is controllable if the eigenvalues of $\mathscr{F}$ are all distinct and the eigenvectors of $\mathscr{F}$ are not orthogonal to at least one column of $\mathscr{R}$.*

Proof: The assertion is a direct consequence of proof with respect to Condition (I) in Theorem 1. □

In practice, the information channels among agents are influenced by environment, obstacles along the trajectories of agents, etc. So, not all the communications between followers are bidirectional. Some agents could only send information to others or receive information from others. Accordingly, there only exist unidirectional communications between some followers. In this case, we use directed graph to describe the system. The following result is on the controllability under directed interconnection graph.

Corollary 2: *The delayed multi-agent system* (6), *with undirected interconnection graph G, l leaders and N followers, is controllable for $\tau > 0$ if any of the two matrices Q_1, Q_3 has full row rank or Condition* (III) *in Theorem* 1 *is fulfilled, where*

$$Q_1 = [-\mathscr{R},\ -\mathscr{A}\mathscr{R},\ -\mathscr{A}^2\mathscr{R}, \ldots,\ -\mathscr{A}^{N-1}\mathscr{R}],$$
$$Q_3 = [-\mathscr{R}, \mathscr{F}\mathscr{R},\ -\mathscr{F}^2\mathscr{R}, \ldots, (-1)^N\mathscr{F}^{N-1}\mathscr{R}].$$

Proof: Set $Q_2 = [-\mathscr{R}, \mathscr{D}\mathscr{R}, -\mathscr{D}^2\mathscr{R}, \ldots, (-1)^N\mathscr{D}^{N-1}\mathscr{R}]$, it follows from the proof of Theorem 1 that the controllability matrix Q of the state delayed system (6) is full row rank if so is any of the matrices Q_1, Q_2 and Q_3. Furthermore, if Condition (III) in Theorem 1 is fulfilled, Q_2 is full row rank. Hence, the assertion holds. □

Since the solution to (7) can be written in the form of $y(t, t_0, \varphi, z) = y(t, t_0, \varphi, 0) + \int_{t_0}^{t} K(t, s)(-\mathscr{R})z(s)\mathrm{d}s$, the results in Buckalo (1968) can be employed to cope with the controllability problem with multiple time-delays.

Theorem 2: *The delayed multi-agent system* (7), *with undirected interconnection graph G, l leaders and N followers, is controllable for $\tau_1 > 0, \ldots, \tau_{\max} > 0$ if the following conditions are fulfilled:*

(I) (i) *The eigenvalues, i.e. the diagonal elements of $-\mathscr{D} + \mathscr{A}_0$ are all distinct.*
(ii) *At least all the elements in one column of $\mathscr{R}$ are nonzero.*

(II) $\mathscr{A}_{\tau_{\max}} y(t - \tau_{\max}) - \mathscr{R}z(t) \equiv 0$ *admits to a piecewise continuous solution $z(t)$ for $t \in [t_1, t_1 + \tau_{\max}]$.*

Proof: The matrix $[\mathscr{R}, (-\mathscr{D} + \mathscr{A}_0)\mathscr{R}, \ldots, (-\mathscr{D} + \mathscr{A}_0)^{N-1}\mathscr{R}]$ has full row rank of N if (i) and (ii) of Condition (I) are fulfilled. The rest of the proof is a direct consequence of the result in Buckalo (1968). □

4.2 *Controllability of the delayed system with double integrator dynamics*

The terminologies and notations in this section have the same meanings as those in Section 4.1. The controllable matrices for the delayed systems (12) and (14) have the same structure as those for system (6). The only difference consists in the following specific forms of the associated matrices in the controllability matrix:

$$Q_1^1 = \begin{bmatrix} 0 & 0 \\ -\mathscr{R} & -k\mathscr{R} \end{bmatrix}, \quad Q_j^{k+1} = \begin{bmatrix} 0 & I_N \\ -\mathscr{D} & -k\mathscr{D} \end{bmatrix} Q_j^k + \begin{bmatrix} 0 & 0 \\ \mathscr{A} & k\mathscr{A} \end{bmatrix} Q_{j-1}^k;$$

$$Q_1^1 = \begin{bmatrix} 0 & 0 \\ -\mathscr{R} & 0 \end{bmatrix}, \quad Q_j^{k+1} = \begin{bmatrix} 0 & I_N \\ -\mathscr{D} & kI_N \end{bmatrix} Q_j^k + \begin{bmatrix} 0 & 0 \\ \mathscr{A} & 0 \end{bmatrix} Q_{j-1}^k.$$

Theorem 3: *For the state-delayed system described by* (12) *or* (14), *with undirected interconnection graph G, l leaders and N followers, the dimension of the controllable subspace of system* (12) *or* (14) *is not less than N if the following conditions are both fulfilled:*

(i) *The eigenvalues of $\mathscr{A}$ are all distinct.*
(ii) *The eigenvectors of $\mathscr{A}$ are not orthogonal to at least one column of $\mathscr{R}$.*

Proof: Consider the following nondelayed system:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v}_y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathscr{A} & k\mathscr{A} \end{bmatrix} \begin{bmatrix} y(t) \\ v_y(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\mathscr{R} & -k\mathscr{R} \end{bmatrix} \begin{bmatrix} z(t) \\ v_z(t) \end{bmatrix}, \quad t > t_0. \tag{22}$$

By following the same reasonings as those in the proof of Condition (I) in Theorem 1, we see that the controllable matrix of the nondelayed system (22) constitutes part of that of the state-delayed system (12). So the dimension of the controllable subspace associated with system (12) is always greater than that of system (22). Computations show that the controllability matrix $\mathscr{C}_1$ of system (22) is

$$\mathscr{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathscr{R} & -k\mathscr{R} & -k\mathscr{A}\mathscr{R} & -k^2\mathscr{A}\mathscr{R} & \cdots & -k^{2N-1}\mathscr{A}^{2N-1}\mathscr{R} & -k^{2N}\mathscr{A}^{2N-1}\mathscr{R} \end{bmatrix}.$$

Accordingly, the rank of $\mathscr{C}_1$ equals that of matrix $\widetilde{\mathscr{C}}_1$ with

$$\widetilde{\mathscr{C}}_1 = [\widetilde{\mathscr{C}}_{11}, k\widetilde{\mathscr{C}}_{11}],$$

where

$$\widetilde{\mathscr{C}}_{11} = [-\mathscr{R},\ -k\mathscr{A}\mathscr{R},\ -k^2\mathscr{A}^2\mathscr{R}, \ldots,\ -k^{2N-1}\mathscr{A}^{2N-1}\mathscr{R}].$$

Since $\mathscr{A}$ is symmetric, it can be assumed that $\mathscr{A} = U\Lambda U^T$, with U being an orthogonal matrix. Denote $U^T\mathscr{R} \triangleq [r_1, \ldots, r_l]$ with $r_i \triangleq [r_{1i}, \ldots, r_{Ni}]^T$, $i = 1, \ldots, l$. There exists a permutation matrix P such that

$$\begin{aligned}
\widetilde{\mathscr{C}}_{11} &= -U\big[[r_1, \ldots, r_l], k\Lambda[r_1, \ldots, r_l], k^2\Lambda^2[r_1, \ldots, r_l], \ldots, \\
&\qquad k^{2N-1}\Lambda^{2N-1}[r_1, \ldots, r_l]\big] \\
&= -U\big[[r_1, k\Lambda r_1, \ldots, k^{2N-1}\Lambda^{2N-1}r_1], \ldots, \\
&\qquad [r_l, k\Lambda r_l, \ldots, k^{2N-1}\Lambda^{2N-1}r_l]\big] \\
&= -U\Big[\mathrm{diag}\{r_{11}, \ldots, r_{N1}\}\widetilde{\Xi}, \ldots, \mathrm{diag}\{r_{1l}, \ldots, r_{Nl}\}\widetilde{\Xi}\Big]P,
\end{aligned}$$

where

$$\widetilde{\Xi} \triangleq \begin{bmatrix} 1 & k\lambda_1 & \cdots & k^{2N-1}\lambda_1^{2N-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & k\lambda_N & \cdots & k^{2N-1}\lambda_N^{2N-1} \end{bmatrix}.$$

It can be seen from the preceding arguments that under Condition (I), $\widetilde{\mathscr{C}}_{11}$ and then $\widetilde{\mathscr{C}}_1$ is full row rank, which leads to rank $\mathscr{C}_1 = N$. Consequently, the dimension of the state-delayed system (12) is not less than N under the two conditions. □

It can be seen from the proof of Theorem 3 that the velocities of the followers are controllable states.

Theorem 4: *The state-delayed system described by* (12) *or* (14) *with undirected interconnection graph G, l leaders and N followers, is controllable for any* $\tau > 0$ *if* G_1 *has nonzero eigenvalues and L and* $\mathscr{F}$ *share no common eigenvalues, where* $G_1 = \begin{bmatrix} 0 & I_N \\ -\mathscr{F} & -k\mathscr{F} \end{bmatrix}$.

Proof: Denote $H_1 = \left[\begin{smallmatrix} 0 & 0 \\ -\mathscr{R} & -k\mathscr{R} \end{smallmatrix}\right]$, and consider the system

$$\begin{bmatrix} \dot{y} \\ \dot{v}_y \end{bmatrix} = G_1 \begin{bmatrix} y \\ v_y \end{bmatrix} + H_1 \begin{bmatrix} z \\ v_z \end{bmatrix}. \tag{23}$$

By repeating the same lines of proof for Condition (IV) in Theorem 1, we see that system (12) is controllable if so is system (23).

Next, we are to show that if G_1 has nonzero eigenvalues and L and $\mathscr{F}$ share no common eigenvalues, system (23) is controllable. We show this by contradiction. Suppose system (23) is uncontrollable. By the controllability Popov–Belevitch–Hautus (PBH) criteria, there exist an eigenvalue λ of G_1 and an associated left eigenvector $[\xi^T, \eta^T]$ such that

$$[\xi^T, \eta^T]G_1 = \lambda[\xi^T, \eta^T] \quad \text{and} \quad [\xi^T, \eta^T]H_1 = 0. \tag{24}$$

The first equality of (24) gives rise to

$$\begin{cases} -\eta^T\mathscr{F} = \lambda\xi^T, \\ \xi^T I_N - k\eta^T\mathscr{F} = \lambda\eta^T. \end{cases} \tag{25}$$

Then

$$\lambda^2\eta^T = \eta^T(-1 - k\lambda)\mathscr{F}. \tag{26}$$

We claim that $1 + k\lambda \neq 0$. Otherwise, if $1 + k\lambda = 0$, it follows from $\lambda \neq 0$ and (26) that $\eta^T = 0$. This, together with the second equality of (25) results in $\xi^T = 0$, which contradicts to the fact that the eigenvector $[\xi^T,\ \eta^T]$ is nonzero. Accordingly, one has $\eta^T\mathscr{F} = \mu\eta^T$ with $\mu = -\frac{\lambda^2}{1+k\lambda}$. So, μ is an eigenvalue of $\mathscr{F}$. On the other hand, the second equality of (24) implies $\eta^T\mathscr{R} = 0$. We have

$$[\eta^T, 0^T]L = [\eta^T, 0^T]\begin{bmatrix} \mathscr{F} & \mathscr{R} \\ \mathscr{R}^T & L_{22} \end{bmatrix} = \mu[\eta^T, 0].$$

That is, μ is also an eigenvalue of L. This contradicts to the assumption that L and $\mathscr{F}$ share no common eigenvalues. The above analysis shows that if G_1 has nonzero eigenvalues and L and $\mathscr{F}$ share no common eigenvalues, system (23) and then system (12) is controllable. □

Remark 1: The latter part of proof in this theorem is inspired by the proof of Theorem 3 in Jiang et al. (2009).

Next, we present a result on multiple time-delays.

Theorem 5: *The state delayed system described by* (12) *or* (14), *with undirected interconnection graph G, l leaders and N followers, is controllable for* $\tau_1 > 0, \ldots, \tau_{\max} > 0$ *if the following conditions are fulfilled*:

(1) *The matrix* $\tilde{C} = \left[\Gamma, \Gamma\Theta, \cdots \Gamma^{N-1}\Theta\right]$ *has full row rank.*

(2) $\Psi\left[\begin{smallmatrix} y(t-\tau_{\max}) \\ v_y(t-\tau_{\max}) \end{smallmatrix}\right] - \Theta\left[\begin{smallmatrix} z(t) \\ v_z(t) \end{smallmatrix}\right] \equiv 0$ *admits to a piecewise continuous solution* $\left[\begin{smallmatrix} z(t) \\ v_z(t) \end{smallmatrix}\right]$ *for* $t \in [t_1,\ t_1 + \tau_{\max}]$,

where for system (15),

$$\Gamma = \begin{bmatrix} 0 & I_N \\ -\mathscr{D} + \mathscr{A}_{\tau_0} & -k\mathscr{D} + k\mathscr{A}_{\tau_0} \end{bmatrix},$$
$$\Theta = \begin{bmatrix} 0 & 0 \\ -\mathscr{R} & -k\mathscr{R} \end{bmatrix}$$
$$\Psi = \begin{bmatrix} 0 & 0 \\ \mathscr{A}_{\tau_k} & k\mathscr{A}_{\tau_k} \end{bmatrix};$$

and for (16),

$$\Gamma = \begin{bmatrix} 0 & I_N \\ -\mathscr{D} + \mathscr{A}_{\tau_0} & kI_N \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 0 \\ -\mathscr{R} & 0 \end{bmatrix},$$
$$\Psi = \begin{bmatrix} 0 & 0 \\ \mathscr{A}_{\tau_k} & 0 \end{bmatrix}.$$

Proof: The proof can be conducted in the same way as that in Theorem 2. □

The controllability problem has been studied hereinbefore for the delayed multi-agent system (12). The same idea can be employed in the investigation of controllability for system (14) with the establishment of similar results. In the sequel, we focus on a graph-based perspective for the controllability of system (4) and (6). To this end, we first introduce the following definition.

Definition 6: For a multi-agent system with directed or undirected interconnection graph G, any two agents are said to have the same direct neighbour set of parent vertices if the two parent vertex sets associated with these two agents are identical; and are said to have the same indirect neighbour set of parent vertices if the two sets obtained by adding themselves to each set of their parent vertices are identical.

For example, in Figure 1, the direct parent vertex sets of agents 3 and 4 are, respectively, $S_3 = \{2, 4\}$ and $S_4 = \{2, 3\}$. By adding agent 3 to its direct parent vertex set S_3 and 4 to its direct parent vertex set S_4, one can get the *indirect* parent vertex of agents 3 and 4, which are, respectively, $\widetilde{S}_3 = \{2, 3, 4\}$ and $\widetilde{S}_4 = \{2, 3, 4\}$. So, with respect to Figure 1, agents 3 and 4 have the same indirect neighbour set of parent vertices. At the same time, it can be readily seen that agents 1 and 2 have the same direct neighbour set of parent vertices, which is $S_1 = S_2 = \{5, 6\}$.

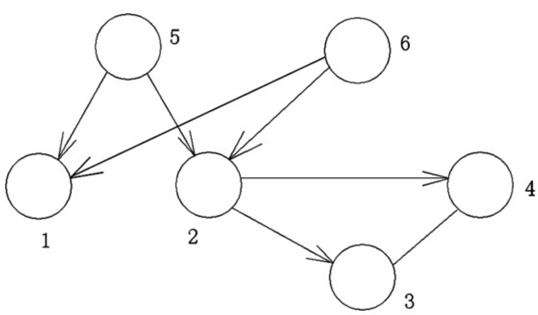

Figure 1. Agents 3 and 4 have the indirect same set of parent vertices.

Theorem 6: *The multi-agent system with or without delay in state, described by* (4) *or* (6), *is uncontrollable if at least two followers have the same direct or indirect neighbour set of parent vertices.*

Proof: For the convenience of presentation, we assume that there are two followers, say the i-th and j-th followers, having the same direct or indirect neighbour set of parent vertices. Since $\mathscr{F}$ is obtained by deleting the last l rows and l columns of the Laplacian matrix L and $\mathscr{R}$ is obtained by deleting the last l rows and the first N columns of L, it follows that

$$\mathscr{F} = \begin{array}{cl} \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & \delta & \cdots & \varepsilon & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & \varepsilon & \cdots & \delta & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix} & \begin{array}{l} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \\ \begin{array}{ccccccc} & & i & & j & & \end{array} & \end{array},$$

$$\mathscr{R} = \begin{array}{cl} \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix} & \begin{array}{l} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \end{array},$$

where $\varepsilon = 0$ if followers i and j have the same direct neighbour set of parent vertices and $\varepsilon = -1$ if they have the same indirect neighbour set of parent vertices. Furthermore, the i-th row in $\mathscr{F}$ is identical with the j-th row if the i-th and j-th elements are exchanged from each other. Since the i-th and j-th followers have the same direct or indirect neighbour set of parent vertices, the i-th and j-th rows in $\mathscr{R}$ are also identical. Recall that the controllability matrix of system (4) is $\mathscr{C} = [-\mathscr{R}, \mathscr{F}\mathscr{R}, \ldots, (-1)^N \mathscr{F}^{N-1}\mathscr{R}]$. It can be seen from the specific structure of $\mathscr{F}$ and $\mathscr{R}$ that the i-th and j-th rows of the controllability matrix are identical. As a consequence, the controllability matrix is rank deficient, i.e. the system is uncontrollable. □

The above theorem presents a method to decide the uncontrollability directly from the interconnection graph itself by searching for some followers with the same direct or indirect neighbour set of parent vertices.

4.3 *Controllability under switching topology*

Since the interconnection graph G is time-variant, the dynamic (4) can be viewed more reasonably as a system in switching networks, which can be written in the form

$$\dot{y} = -\mathscr{F}_{\sigma(t)}y - \mathscr{R}_{\sigma(t)}z, \tag{27}$$

where $\sigma(t) : \mathbb{R}^+ \to \mathscr{M} \triangleq \{1, 2, \ldots, M\}$ is the switching signal/sequence to be designed. Given a switching signal $\sigma(t) : [t_0, t_f] \to \mathscr{M}$, we refer to $t_0, t_1, \ldots, t_{s-1}$ with $t_0 < t_1 < \cdots < t_{s-1}$ as the switching time sequence, and $\sigma(t_0) = i_0,\ \sigma(t_1) = i_1, \ldots, \sigma(t_{s-1}) = i_{s-1}$ as the switching index sequence. Let $h_i \triangleq t_{i+1} - t_i,\ i = 0, 1, \ldots, s-1$, and $t_s \triangleq t_f$. We denote by $\pi \triangleq \{(i_0, h_0)\ldots(i_{s-1}, h_{s-1})\}$ a switching signal. $(\mathscr{F}_{\sigma(t)}, \mathscr{R}_{\sigma(t)})$ is said to be the system and control input matrix pair (in short matrix pair) of the multi-agent system (1) with switching topology. In particular, $(\mathscr{F}, \mathscr{R})$ is said to be the matrix pair of system (1) with fixed topology.

Definition 7: The multi-agent system (1) is said to be controllable under leaders $x_{N+j}, j = 1, \ldots, l$ and switching topology if system (27) is controllable.

The system (27) is controllable if for any nonzero state $y \in \mathbb{R}^N$, there exist a switching sequence π and input z such that $y(0) = y$, and $y(t_f) = 0$. We denote by $\mathscr{C}$ the controllable state set of system (27). To derive conditions for the controllability of multi-agent systems under switched dynamic networks, results on the controllability of switched linear systems (27) are first recalled in the sequel. Given a matrix $A \in \mathbb{R}^{N\times N}$, and a linear subspace $\mathscr{W} \subseteq \mathbb{R}^N$, we denote $\langle A|\mathscr{W}\rangle \triangleq \sum_{i=1}^{N} A^{i-1}\mathscr{W}$. It follows that $\langle A|\mathscr{W}\rangle$ is a minimum A-invariant subspace that contains $\mathscr{W}$. Given $B \in \mathbb{R}^{N\times p}$, let $\operatorname{Im} B$ denote the image space of B. For notational simplicity, we denote by $\langle A|B\rangle$ the $\langle A|\operatorname{Im} B\rangle$. For system (27), we denote by $\mathscr{C}$ the set of all its controllable states and consider the nested subspace sequence defined by

$$\mathscr{W}_1 = \sum_{k=1}^{M} \langle -F_k| - R_k\rangle, \quad \mathscr{W}_{s+1} = \sum_{k=1}^{M} \langle -F_k|\mathscr{W}_s\rangle, \tag{28}$$

with $s = 1, 2, \ldots$. Let $\mathscr{W} = \lim_{s\to\infty} \mathscr{W}_s$, and $\mu = \min\{s|\ \mathscr{W}_s = \mathscr{W}_{s+1},\ s = 1, 2, \ldots\}$. It follows that $\mu \le N - \dim \mathscr{W}_1 + 1$. The following result is on the controllability of system (27).

Lemma 2: *For system* (27), $\mathscr{C} = \mathscr{W}_\mu = \mathscr{W}_N = \mathscr{W}$.

Proof: It is a direct consequence of Theorem 1 in Sun, Ge, and Lee (2002) and Ji, Wang, and Guo (2008d). □

To calculate the controllable subspace $\mathscr{C}$, the following subspace sequence $\mathscr{E}_0, \mathscr{E}_1, \ldots$ is defined:

$$\mathscr{E}_0 = \sum_{k=1}^{M} \operatorname{Im}(-R_k),\ \mathscr{E}_{s+1} = \mathscr{E}_0 + \sum_{k=1}^{M} (-F_k)\mathscr{E}_s,\quad s = 0, 1, 2, \ldots. \tag{29}$$

Let $\gamma = \min\{s|\mathscr{E}_s = \mathscr{E}_{s+1},\ s = 0, 1, 2, \ldots\}$, and

$$\begin{aligned}\Gamma \triangleq \big[&B_1, \ldots, B_M, H_1B_1, \ldots, H_1B_M, \ldots, H_MB_1, \ldots,\\ &H_MB_M, \ldots, H_1^{\gamma}B_1, \ldots, H_1^{\gamma}B_M, H_1^{\gamma-1}H_2B_1, \ldots,\\ &H_1^{\gamma-1}H_2B_M, \ldots, H_M^{\gamma}B_1, \ldots, H_M^{\gamma}B_M\big],\end{aligned} \tag{30}$$

where $H_i \triangleq -F_i,\ B_i \triangleq -R_i,\ i = 1, \ldots, M$. Below is a Kalman-type rank criteria.

Theorem 7: *The interconnected system with n_l leaders and switching networks described by* (27) *is controllable if and only if matrix Γ is full row rank, i.e.*

$$\operatorname{rank}\Gamma = N.$$

Proof: By the proof of Proposition 1 in Ji, Lin, and Lee (2008c), the subspace $\mathscr{W}$ coincides with the image space of Γ, i.e. $\mathscr{W} = \operatorname{Im}\Gamma$. The result then follows from Lemma 2. □

Since $\gamma \le N - \dim \mathscr{E}_0$, the theorem still holds if γ is replaced by N. In the sequel, we derive directly from the nested subspace sequence (28) rather than (29). To facilitate statement, we begin with the situation $n_l = 2$, $\mu = N = 3$ and $M = 2$. The remainder is to calculate $\mathscr{W}_\mu$. By (28),

$$\begin{aligned}\mathscr{W}_\mu &= \mathscr{W}_2 + H_1\mathscr{W}_2 + H_1^2\mathscr{W}_2 + H_2\mathscr{W}_2 + H_2^2\mathscr{W}_2\\ &= \mathscr{W}_{\mu 1} + \mathscr{W}_{\mu 2},\end{aligned}$$

where

$$\begin{aligned}\mathscr{W}_{\mu 1} \triangleq &\langle H_1|B_1\rangle + H_2\langle H_1|B_1\rangle + H_2^2\langle H_1|B_1\rangle\\ &+ H_1H_2\langle H_1|B_1\rangle + H_1H_2^2\langle H_1|B_1\rangle\\ &+ H_1^2H_2\langle H_1|B_1\rangle + H_1^2H_2^2\langle H_1|B_1\rangle,\end{aligned}$$

$$\begin{aligned}\mathscr{W}_{!\mu 2} \triangleq &\langle H_2|B_2\rangle + H_1\langle H_2|B_2\rangle + H_1^2\langle H_2|B_2\rangle\\ &+ H_2H_1\langle H_2|B_2\rangle + H_2H_1^2\langle H_2|B_2\rangle\\ &+ H_2^2H_1\langle H_2|B_2\rangle + H_2^2H_1^2\langle H_2|B_2\rangle.\end{aligned}$$

Since F_i is symmetric, it can be expressed as

$$-F_i = -U_i D_i U_i^T = U_i \widehat{D}_i U_i^T \triangleq H_i, \quad i = 1, \ldots, M, \tag{31}$$

where $\widehat{D}_i \triangleq -D_i$, U_i is an orthogonal matrix. Denote $(-F_i, -R_i) \triangleq (H_i, B_i)$, one has

$$\langle H_i | B_i \rangle = \sum_{j=1}^{N} H_i^{j-1} \operatorname{Im} B_i = U_i \sum_{j=1}^{N} \widehat{D}_i^{j-1} \operatorname{Im} \widehat{B}_i,$$

where $\widehat{B}_i \triangleq U_i^T B_i$. Set $\widehat{D}_i \triangleq \operatorname{diag}\{\hat{d}_{1i}, \ldots, \hat{d}_{Ni}\}$, $\widehat{B}_i \triangleq [\widehat{b}_{i1}, \widehat{b}_{i2}]$, and $\widehat{b}_{ik} \triangleq [\hat{b}_{ik}^{(1)}, \ldots, \hat{b}_{ik}^{(N)}]^T$, $i = 1, \ldots, M$; $k = 1, \ldots, n_l$. Some calculations show that

$$\langle H_i | B_i \rangle = \operatorname{Im} \Gamma_i, \tag{32}$$

where

$$\Gamma_i \triangleq U_i \left[\begin{bmatrix} \hat{b}_{i1}^{(1)} & \hat{d}_{1i} \hat{b}_{i1}^{(1)} & \cdots & \hat{d}_{1i}^{N-1} \hat{b}_{i1}^{(1)} \\ \hat{b}_{i1}^{(2)} & \hat{d}_{2i} \hat{b}_{i1}^{(2)} & \cdots & \hat{d}_{2i}^{N-1} \hat{b}_{i1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{i1}^{(N)} & \hat{d}_{Ni} \hat{b}_{i1}^{(N)} & \cdots & \hat{d}_{Ni}^{N-1} \hat{b}_{i1}^{(N)} \end{bmatrix}, \begin{bmatrix} \hat{b}_{i2}^{(1)} & \hat{d}_{1i} \hat{b}_{i2}^{(1)} & \cdots & \hat{d}_{1i}^{N-1} \hat{b}_{i2}^{(1)} \\ \hat{b}_{i2}^{(2)} & \hat{d}_{2i} \hat{b}_{i2}^{(2)} & \cdots & \hat{d}_{2i}^{N-1} \hat{b}_{i2}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}_{i2}^{(N)} & \hat{d}_{Ni} \hat{b}_{i2}^{(N)} & \cdots & \hat{d}_{Ni}^{N-1} \hat{b}_{i2}^{(N)} \end{bmatrix} \right]$$

$$= [U_i, U_i] \begin{bmatrix} \Lambda_{i1} & \\ & \Lambda_{i2} \end{bmatrix} \begin{bmatrix} \Xi_i & \\ & \Xi_i \end{bmatrix},$$

$$\Lambda_{ik} = \begin{bmatrix} \hat{b}_{ik}^{(1)} & & \\ & \ddots & \\ & & \hat{b}_{ik}^{(N)} \end{bmatrix},$$

$$\Xi_i = \begin{bmatrix} 1 & \hat{d}_{1i} & \cdots & \hat{d}_{1i}^{N-1} \\ 1 & \hat{d}_{2i} & \cdots & \hat{d}_{2i}^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \hat{d}_{Ni} & \cdots & \hat{d}_{Ni}^{N-1} \end{bmatrix},$$

$$i = 1, \ldots, M; \quad k = 1, \ldots, n_l. \tag{33}$$

Let $U_{ij}^T \triangleq U_i^T U_j$. Then U_{ij} is an orthogonal matrix since U_i, U_j are orthogonal. To express $\mathscr{W}_\mu$ further, the following two matrices are introduced:

$$\Theta_{\mu 1} = \Theta_{\mu 1}^{(1)} \Theta_{\mu 1}^{(2)} \Theta_{\mu 1}^{(3)},$$

where

$$\begin{aligned} \Theta_{\mu 1}^{(1)} = \Big[& U_1, U_1, U_2 \widehat{D}_2 U_{21}^T, U_2 \widehat{D}_2 U_{21}^T, U_2 \widehat{D}_2^2 U_{21}^T, \\ & U_2 \widehat{D}_2^2 U_{21}^T U_1 \widehat{D}_1 U_{21} \widehat{D}_2 U_{21}^T, U_1 \widehat{D}_1 U_{21} \widehat{D}_2 U_{21}^T, \\ & U_1 \widehat{D}_1 U_{21} \widehat{D}_2^2 U_{21}^T, U_1 \widehat{D}_1 U_{21} \widehat{D}_2^2 U_{21}^T U_1 \widehat{D}_1^2 U_{21} \widehat{D}_2 U_{21}^T, \\ & U_1 \widehat{D}_1^2 U_{21} \widehat{D}_2 U_{21}^T, U_1 \widehat{D}_1^2 U_{21} \widehat{D}_2^2 U_{21}^T, U_1 \widehat{D}_1^2 U_{21} \widehat{D}_2^2 U_{21}^T \Big], \end{aligned}$$

$$\begin{aligned} \Theta_{\mu 1}^{(2)} = \operatorname{diag}\{ & \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \\ & \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \Lambda_{11}, \Lambda_{12}, \}, \end{aligned}$$

$$\begin{aligned} \Theta_{\mu 1}^{(3)} = \operatorname{diag}\{ & \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \\ & \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1, \Xi_1 \}, \end{aligned}$$

and

$$\Theta_{\mu 2} = \Theta_{\mu 2}^{(1)} \Theta_{\mu 2}^{(2)} \Theta_{\mu 2}^{(3)},$$

where

$$\begin{aligned} \Theta_{\mu 2}^{(1)} = \Big[& U_2, U_2, U_1 \widehat{D}_1 U_{12}^T, U_1 \widehat{D}_1 U_{12}^T, U_1 \widehat{D}_1^2 U_{12}^T, U_1 \widehat{D}_1^2 U_{12}^T, \\ & U_2 \widehat{D}_2 U_{12} \widehat{D}_1 U_{12}^T, U_2 \widehat{D}_2 U_{12} \widehat{D}_1 U_{12}^T, U_2 \widehat{D}_2 U_{12} \widehat{D}_1^2 U_{12}^T, \\ & U_2 \widehat{D}_2 U_{12} \widehat{D}_1^2 U_{12}^T, U_2 \widehat{D}_2^2 U_{12} \widehat{D}_1 U_{12}^T, U_2 \widehat{D}_2^2 U_{12} \widehat{D}_1 U_{12}^T, \\ & U_2 \widehat{D}_2^2 U_{12} \widehat{D}_1^2 U_{12}^T, U_2 \widehat{D}_2^2 U_{12} \widehat{D}_1^2 U_{12}^T \Big] \end{aligned}$$

$$\begin{aligned} \Theta_{\mu 2}^{(2)} = \times \operatorname{diag}\Big\{ & \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \\ & \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22}, \Lambda_{21}, \Lambda_{22} \Big\} \end{aligned}$$

$$\begin{aligned} \Theta_{\mu 2}^{(3)} = \times \operatorname{diag}\Big\{ & \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \\ & \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2, \Xi_2 \Big\}. \end{aligned}$$

Using (32), it can be verified that

$$\mathscr{W}_{\mu 1} = \operatorname{Im} \Theta_{\mu 1}, \quad \mathscr{W}_{\mu 2} = \operatorname{Im} \Theta_{\mu 2}.$$

Accordingly,

$$\mathscr{W}_\mu = \operatorname{Im} \Theta_\mu,$$

with

$$\Theta_\mu \triangleq \Big[\Theta_{\mu 1}^{(1)}, \Theta_{\mu 2}^{(1)} \Big] \times \operatorname{diag}\Big\{ \Theta_{\mu 1}^{(2)}, \Theta_{\mu 2}^{(2)} \Big\} \times \operatorname{diag}\Big\{ \Theta_{\mu 1}^{(3)}, \Theta_{\mu 2}^{(3)} \Big\}.$$

By Lemma 2, the interconnected system (27) with $n_l = 2$, $N = 3$ and $M = 2$ is controllable if and only if Θ_μ is full row rank, i.e.

$$\operatorname{rank} \Theta_\mu = N.$$

The derivation of general situation can be carried out in the same way. The difference consists in the expression of Θ_μ, which becomes much more complex as the state dimension N and the number of subsystems M increase. In general case, Θ_μ is expressed as

$$\Theta_\mu = \left[\Theta^{(1)}_{\mu 1}, \dots, \Theta^{(1)}_{\mu M}\right] \times \mathrm{diag}\left\{\Theta^{(2)}_{\mu 1}, \dots, \Theta^{(2)}_{\mu M}\right\} \times \mathrm{diag}\left\{\Theta^{(3)}_{\mu 1}, \dots, \Theta^{(3)}_{\mu M}\right\}, \tag{34}$$

where

$$\Theta^{(1)}_{\mu i} = [\underbrace{U_i, \dots, U_i}_{n_l}, \dots, \underbrace{U_{i_1}\widehat{D}^{N-1}_{i_1}U_{i_2 i_1}\widehat{D}^{N-1}_{i_2}\dots\widehat{D}^{N-1}_{i_M}U^T_{i_M i}, \dots, U_{i_1}\widehat{D}^{N-1}_{i_1}U_{i_2 i_1}\widehat{D}^{N-1}_{i_2}\dots\widehat{D}^{N-1}_{i_M}U^T_{i_M i}}_{n_l}],$$

with $i_1, \dots, i_M \in \{1, \dots, M\}$ $i_M \neq i$; U_i, U_{ij} are orthogonal matrices, $\widehat{D}_{i_j}$ are diagonal matrices defined in (31), and

$$\Theta^{(2)}_{\mu i} = \mathrm{diag}\{\Lambda_{i1}, \dots, \Lambda_{in_l}\},\ \Theta^{(3)}_{\mu i} = \mathrm{diag}\{\Xi_i, \dots, \Xi_i\},\quad i = 1, \dots, M,$$

Λ_{ik}, Ξ_i are diagonal and Vandermonde matrices defined, respectively, in (33). The following conclusion is a summary of the previous arguments.

Theorem 8: *The interconnected system with l leaders and switching networks described by* (27) *is controllable if and only if matrix Θ_μ is full row rank, i.e.*

$$\mathrm{rank}\,\Theta_\mu = N,$$

where Θ_μ is defined in (34).

Remark 2: Comparing with Theorem 8, the controllability matrix expression in Theorem 7 is more laconic and the controllability can be therein verified directly via computing Γ. The advantage of Theorem 8 consists in the isolation of eigenvalues and eigenvectors, which is incorporated, respectively, in $\Theta^{(3)}_{\mu i}$ and $\Theta^{(2)}_{\mu i}$ in (34) and accordingly results in a specific expression for controllability matrix Θ_μ. But no such information is extracted in the expression of Γ. In this sense, Theorem 8 provides further information on controllability for a multi-agent system although it seems that Θ_μ is not as simple as Γ. The difference between Theorems 7 and 8 originates from the starting point of derivation. Theorem 7 is deduced from the viewpoint of subspace-based algorithm for the controllability of switching linear systems, while Theorem 8 is deduced directly from the iteratively defined subspace sequence.

5. Illustrative examples

Example 1: We consider a directed interconnection graph consisting of four vertices, with nodes 3 and 4 being leaders. The associated fixed topology is shown in Figure 2. The objective is to move the two leaders in a two-dimensional space so that the followers can be successively steered into the desired configuration, i.e. a straight line in the vertical direction with time delay $\tau = 1\,\mathrm{s}$. The directed interconnection graph associated with the system has the following coefficient matrices:

$$\mathscr{F} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix},\quad \mathscr{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$\mathscr{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\quad \mathscr{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The system moving in a two-dimensional space is described by the following equation:

$$\dot{y}(t) = -\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes I_2\right) y(t) + \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes I_2\right) \times y(t-\tau) - \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I_2\right) z(t).$$

Figure 3 shows the trajectories of the two followers with the initial state $y(t) = [1, 4, 0, 3]^T$, $t \in [-1, 0]$. At the time instant $t_f = 11\,\mathrm{s}$, the two followers are steered into a straight line in a vertical direction.

Example 2: Consider a multi-agent system with the interconnection graph depicted by Figure 1, where nodes 5 and 6 are chosen to take the leaders role. Calculations show that system matrices are

$$\mathscr{F} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix},\quad \mathscr{R} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

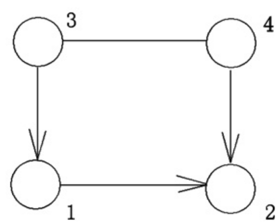

Figure 2. The directed interconnection graph of four agents, where agents 3 and 4 are leaders.

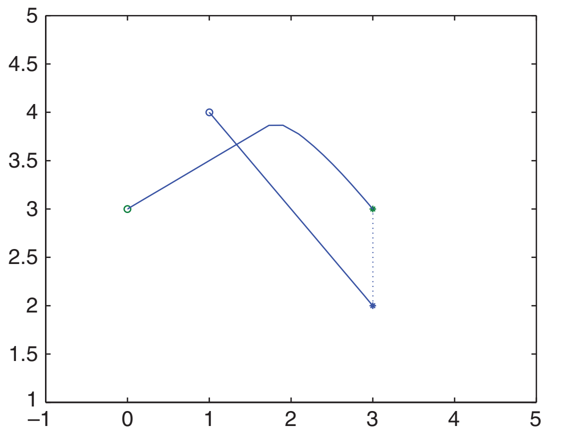

Figure 3. A straight line in the vertical direction with $\tau = 1$s.

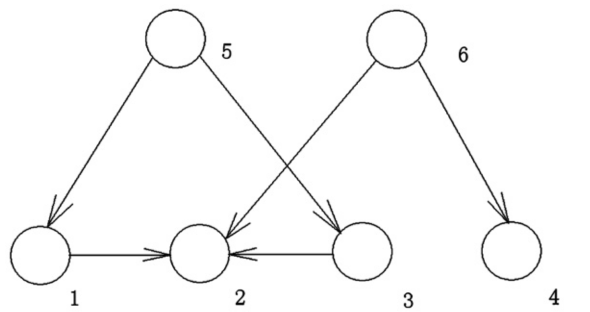

Figure 4. The directed interconnection graph of six agents, where agents 1 and 3 have the same topology.

It can be verified that the controllability matrix $\mathscr{C}$ has the row rank of 2, i.e. the system is uncontrollable, which coincides with Theorem 1.

In the preceding sections, we discussed the controllability problem with respect to unweighed directed or undirected graph. It should be noted that communication weights distinct from 1 are ubiquitous in practice. In what follows, we will show by an example that it is possible to turn an uncontrollable system with fix topology into a controllable one by selecting appropriate weights for the communication links between leaders and followers.

Example 3: For the multi-agent system with directed interconnection graph depicted by Figure 4, agents 5 and 6 are assigned to take the leader roles and agents 1 and 3 have the same input channel. If the digraph is unweighed, we have

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathscr{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

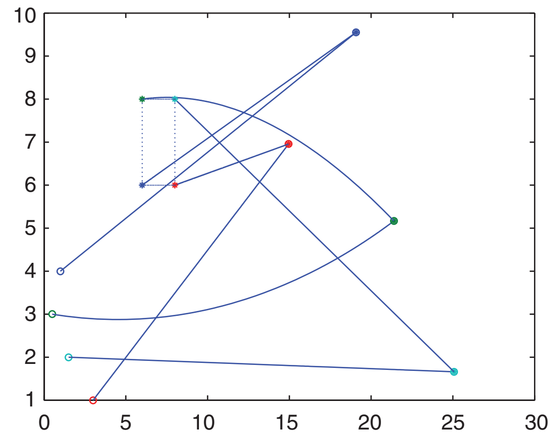

Figure 5. An rectangle configuration.

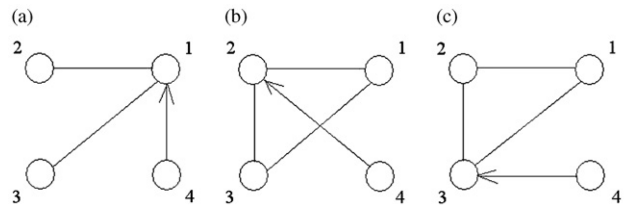

Figure 6. Three interconnection graphs associated with three multi-agent systems.

The controllability matrix has the rank 3. The system is uncontrollable. If different weight, say weight value 2, is imposed between the leader 5 and the follower 1 (or between 5 and 3), then

$$\mathscr{F} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathscr{R} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The rank of the controllability matrix is 4, implying that the uncontrollable system is turned to be controllable. Figure 5 shows the motion trajectories of the followers 1–4. The four followers start from arbitrary initial positions in the plane represented by $\circ$, and reach a desired rectangular configuration depicted by *'s.

Example 4: The example is employed to show that under switching topologies, the desired configuration can be achieved for a multi-agent system with time-delay in state. Consider a multi-agent system with switching topology described by Figure 6, where agent 4 is the leader and the others are followers. Our objective is to move the leader in a two-dimensional space so that the followers are successively steered into a straight line with time delay $\tau = 1$ s under switching $(a) \to (b) \to (c)$. Starting from a random initial position

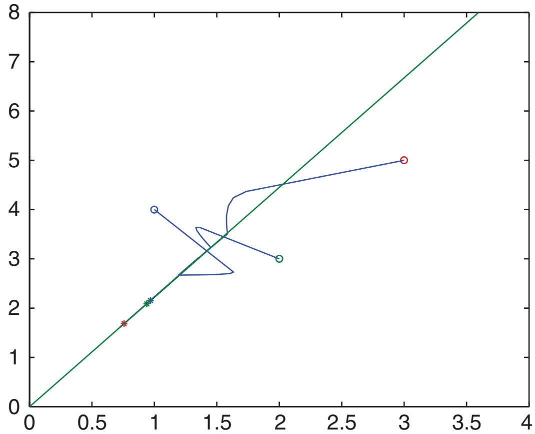

Figure 7. The trajectories of the three followers in the plane with time-delay $\tau = 1$s under switching topology.

(1, 4), (2, 3) and (3, 5), Figure 7 shows the evolution of the three followers, which forms a configuration of straight line at time $t = 6$ s.

6. Conclusion

As an attractive issue of research, the controllability of networks of dynamic agents has recently aroused increasing attention. In this article, the controllability problem is formulated and studied for continuous-time multi-agent systems with time-delay in state and switching topology. Sufficient algebraic conditions are derived for the interconnected system to be controllable, as well as a graph-based uncontrollability topology structure is constructed. The results are analysed for both single and double integrator dynamics. For switching topology, two necessary and sufficient algebraic conditions are derived for the controllability of networked multi-agent system. The work provides insight into the effect of the interaction pattern on self-organised motion in a multi-agent system.

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