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Necessary and sufficient conditions for regional stabilisability of generic switched linear systems with a pair of planar subsystems

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In this article, the regional stabilisability issues of a pair of planar LTI systems are investigated through the geometrical approach, and easily verifiable necessary and sufficient conditions are derived. The main idea of the article is to characterise the best case switching signals based upon the variations of the constants of the integration of the subsystems. The conditions are generic as all possible combinations of the subsystem dynamics are considered.

Keywords: switched linear systems; stabilisability; geometrical approach; best case analysis

1. Introduction

A switched system is a type of hybrid system which comprises a collection of discrete-time or continuous-time dynamic systems described by difference or differential equations and a switching rule that specifies the switching between the subsystems. Many real-world processes and systems, for example, chemical, power systems and communication networks, can be modelled as switched systems. The last decade has witnessed increasing research activities in this area due to its success in application and importance in theory. Among the various research topics, stability and stabilisation issues have attracted most of the attention, for example, Morse (1996), Narendra and Balakrishnan (1997), Dayawansa and Martin (1999), Hespanha and Morse (1999), Liberzon, Hespanha, and Morse (1999), Shorten and Narendra (1999), Decarlo, Braanicky, Pettersson, and Lennartson (2000), Narendra and Xiang (2000), Pettersson (2003) and Cheng (2004). For more references, the reader may refer to the survey papers by Liberzon and Morse (1999), Lin and Antsaklis (2005) and the recent books by Liberzon (2003), Sun and Ge (2005).

There are two categories of stabilisation strategies for switched systems. The first one is feedback stabilisation, where the switching signals are assumed to be given or restricted. The problem is to design appropriate feedback control laws, in the form of state or output feedback, to make the closed-loop systems stable under these given switching signals. Several classes of switching signals are considered in the literature, for example arbitrary switching (Daafouz, Reidinger, and Iung 2002), slow switching (Cheng, Guo, Lin, and Wang 2005) and restricted switching induced by partitions of the state space (Cuzzola and Morari 2002; Rodrigues, Hassibi, and How 2003; Arapostathis and Broucke 2007, etc.).

Besides the feedback stabilisation described above, switching stabilisation has also been investigated. It is known that even when all the subsystems are unstable, it is still possible to stabilise the switched system by designing the switching signals carefully. It leads to a very interesting question: how 'unstable' these subsystems are while there still exist switching signals to stabilise them. This is usually referred to in the literature as the switching stabilisability problem, which is the focus of this article.

Early efforts on this issue focused on quadratic stabilisation using a common quadratic Lyapunov function. For example, a quadratic stabilisation switching law between two LTI systems was considered by Wicks, Peleties, and DeCarlo (1998), in which it was shown that the existence of a stable convex combination of the two subsystem matrices implies the existence of a state-dependent switching rule that stabilises the switched system along with a quadratic Lyapunov function. The stable convex combination condition was also shown to be necessary for the quadratic stabilisability of two-mode switched LTI system by Feron (1996). However, it is only sufficient for switched LTI systems with more than two modes. A necessary and sufficient condition for quadratic

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stabilisability of more general switched systems was derived by Skafidas, Evans, Savkin, and Petersen (1999). There are other extensions of Wicks et al. (1998) to output-dependent switching by Liberzon and Morse (1999) and to the discrete-time switched systems with polytopic uncertainty based on linear matrix inequalities by Zhai, Lin, and Antsaklis (2003). However, all of these methods which guarantee quadratic stabilisation, are conservative in the sense that there are switched systems that can be asymptotically stabilised without using a common quadratic Lyapunov function (Hespanha, Liberzon, Angeli, and Sontag 2005). More recent efforts were based on multiple Lyapunov functions (Branicky 1998), especially piecewise Lyapunov functions (Wicks and DeCarlo 1997; Ishii, Basar, and Tempo 2003; Pettersson 2003), to construct stabilising switching signals.

Note that the existing stabilisability conditions, which may be expressed as the feasibility of certain linear/bilinear matrix inequalities, are sufficient only except for certain cases of quadratic stabilisation. An algebraic necessary and sufficient condition for asymptotic stabilisability of second-order switched LTI systems was derived by Xu and Antsaklis (2000) using detailed vector field analysis. Similar idea was also applied in recent works (Zhang, Chen and Cui 2005; Bacciotti and Ceragioli 2006). However, all these conditions are not generic as not all the possible combinations of subsystem dynamics were considered. Recently, another necessary and sufficient algebraic condition was proposed by Lin and Antsaklis (2007) for the global stabilisability of switched linear system affected by parameter variations. However, the checking of the necessity is not easy in general.

This article aims to derive easily verifiable, necessary and sufficient conditions for the switching stabilisability of switched linear systems. In particular, we consider the switched systems with a pair of second-order continuous-time LTI subsystems:

$$S_{ij} : \dot{x} = \sigma x \quad \sigma = \{A_i, B_j\} \tag{1}$$

where $A_i, B_j \in \mathbb{R}^{2\times 2}$ are not asymptotically stable, and $i, j \in \{1, 2, 3\}$ denote the types of A and B, respectively. A matrix $A \in \mathbb{R}^{2\times 2}$ is classified into three types according to its eigenvalue and eigenstructure. Type 1: A has real eigenvalues and diagonalisable; Type 2: A has real eigenvalues but undiagonalisable; Type 3: A has two complex eigenvalues.

For the convenience of discussion and presentation, two types of asymptotic stabilisability are defined as follows.

Definition 1: The switched system (1) is said to be *globally asymptotically stabilisable* (GAS), if for any non-zero initial state, there exists a switching signal under which the trajectory will asymptotically converge to zero.

Definition 2: The switched system (1) is said to be *regionally asymptotically stabilisable* (RAS), if there exists at least one region (non-empty, open set) such that for any initial state in that region, there exists a switching signal under which the trajectory will asymptotically converge to zero.

In addition to the global asymptotic stabilisability, which is the focus of the most of the research in the literature, regional asymptotic stabilisability will also be considered in this article. It is due to the fact that there exists a class of switched systems which are not GAS, but still can be stabilised if the initial state is within certain regions. Those switched systems are not hopeless compared to the ones which cannot be stabilised for any initial state. In practice, it is quite possible that the initial state is fortunately within the stabilisable region.

The main technique for stabilisability analysis throughout the whole article is based on the characterisation of the 'best' case switching signal (BCSS) for the given switched system. The logic is very simple: if the switched system cannot be stabilised under the most 'stable' switching signal, then the switched system is not stabilisable. The similar approach has been used to study the stability of switched second-order LTI systems under arbitrary switching by Boscain (2002) and Huang, Xiang, Lin, and Lee (2007) and absolute stability of second-order linear systems by Margaliot and Langholz (2003) using the 'worst' case switching signal. In this article, the BCSS is identified based upon the variation of the constants of integration of individual subsystems.

The article is organised as follows. In Section 2, polar coordinates are utilised to analyse the switched system and to construct functions to describe the variation of the constants of integration. In Section 3, the core concept of the BCSS is introduced and characterised. In Section 4, the main result regarding an easily verifiable, necessary and sufficient condition for stabilisability of the switched system

$$\begin{aligned} S_{ij} : \dot{x} = \sigma x, \quad \sigma = \{A_i, B_j\}, A_i, B_j \in \mathbb{R}^{2\times 2}, \\ Re\{\lambda_{A_i}\} > 0, \ Re\{\lambda_{B_j}\} > 0 \end{aligned} \tag{2}$$

is derived, where $Re\{\lambda_{A_i}\}$ denotes the real parts of the eigenvalues of A_i. In Section 5, the result is extended to the switched system

$$\begin{aligned} S_{ij} : \dot{x} = \sigma x, \quad \sigma = \{A_i, B_j\}, A_i, B_j \in \mathbb{R}^{2\times 2}, \\ Re\{\lambda_{A_i}\} \geq 0, \ Re\{\lambda_{B_j}\} \geq 0, \end{aligned} \tag{3}$$

and the switched system consisting of at least one subsystem (assumed to be A_1) has a negative real eigenvalue

$$S_{ij}: \dot{x} = \sigma x, \quad \sigma = \{A_1, B_j\}, \ A_1, B_j \in \mathbb{R}^{2\times 2}, \tag{4}$$

where $\lambda_{1A}\lambda_{2A} \le 0$ and B_j is not asymptotically stable. When $\lambda_{1A}\lambda_{2A} < 0$, A_1 is a saddle point. When $\lambda_{1A}\lambda_{2A} = 0$, A_1 is marginally stable but not asymptotically stable.

In Section 6, we discuss the connections between the stabilisability conditions derived in this article and the ones in the literature. Finally, Section 7 concludes the article.

2. Mathematical preliminaries

2.1 *Solution of a second-order LTI system in polar coordinates*

Consider a second-order LTI system

$$\dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x \tag{5}$$

and define $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, it follows that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r[a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta] \tag{6}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta \tag{7}$$

When $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$, it corresponds to the real eigenvector of A. The solutions on the real eigenvectors are $r = r_0 e^{\lambda_A t}$, where r_0 is the magnitude of the initial state and λ_A is the corresponding eigenvalue of the real eigenvector.

Since the BCSS is straightforward on the eigenvectors, we focus on the trajectories not on the eigenvectors.

When $\frac{\mathrm{d}\theta}{\mathrm{d}t} \neq 0$,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r\frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta}. \tag{8}$$

Denote

$$f(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta} \tag{9}$$

we have

$$\frac{1}{r}\mathrm{d}r = f(\theta)\mathrm{d}\theta. \tag{10}$$

Lemma 1: *The trajectories of the LTI system* (5) *in r–θ coordinates, except the ones along the eigenvectors, can be expressed as*

$$r(\theta) = Cg(\theta), \tag{11}$$

where $g(\theta(t)) = e^{\int_{\theta^*}^{\theta(t)} f(\tau)\mathrm{d}\tau}$ is positive and C is a positive constant depending on the initial state (r_0, θ_0), the so-called constant of integration. Note that θ^* can be chosen as any value except the angle of any real eigenvector of A.

Proof: By integrating both sides of (10), we have

$$\int_{r_0}^{r} \frac{1}{r}\mathrm{d}r = \int_{\theta_0}^{\theta} f(\tau)\mathrm{d}\tau \Longrightarrow \ln r = \int_{\theta_0}^{\theta} f(\tau)\mathrm{d}\tau + \ln r_0$$
$$\Longrightarrow r(\theta) = r_0 e^{\int_{\theta_0}^{\theta} f(\tau)\mathrm{d}\tau}. \tag{12}$$

Equation (12) can be rewritten as (13) by splitting the integral interval,

$$r(\theta) = r_0 e^{\int_{\theta_0}^{\theta} f(\tau)\mathrm{d}\tau} = r_0 e^{\int_{\theta_0}^{\theta^*} f(\tau)\mathrm{d}\tau} e^{\int_{\theta^*}^{\theta} f(\tau)\mathrm{d}\tau}. \tag{13}$$

Denote the angle of the eigenvector of A as θ_e. Since $\theta^* \neq \theta_e$, $\theta \neq \theta_e$, the integrals $\int_{\theta_0}^{\theta^*} f(\tau)\mathrm{d}\tau$ and $\int_{\theta^*}^{\theta} f(\tau)\mathrm{d}\tau$ are bounded[1] and (13) can be further reduced to (11). It can be readily seen that $C = r_0 e^{\int_{\theta_0}^{\theta^*} f(\tau)\mathrm{d}\tau}$ is a constant determined by the initial state (r_0, θ_0).

Typical phase trajectories of planar LTI systems in polar coordinates are shown in Figure 1. It follows from (12) that

$$\frac{r(\theta+\pi)}{r(\theta)} = \frac{r_0 e^{\int_{\theta_0}^{\theta+\pi} f(\tau)\mathrm{d}\tau}}{r_0 e^{\int_{\theta_0}^{\theta} f(\tau)\mathrm{d}\tau}} = e^{\int_0^{\pi} f(\tau)\mathrm{d}\tau} \tag{14}$$

which is a constant since $f(\theta)$ is a periodical function with a period of π. Therefore, it is sufficient to analyse the stability of systems (5), regardless of the types of A, in an interval of θ with the length of π. Without loss of generality, this interval is chosen to be $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark 1: It was shown that C is the constant of integration that depends on the initial state. It remains the same along the trajectories of A. Geometrically, a larger C indicates an outer layer trajectory as shown in Figure 1, where $C_1 < C_2 < C_3 \cdots < C_n$. Note that when A has real eigenvalues, Equation (11) does not only represent a single trajectory, but an assembly of trajectories. More precisely, for each trajectory corresponding to some constant of integration lying to the right of the eigenvector direction, there exists one and only one trajectory corresponding to the same constant of integration and lying to the left of the eigenvector direction such that (11) holds. It is also worth noting that $r(t)$ will go to infinity because $g(\theta)$ will go to infinity as θ approaches the asymptote of an unstable A.

Definition 3: The line $\theta = \theta_a$ is said to be asymptote of A in $r-\theta$ coordinates if the angle of the trajectory of $\dot{x} = Ax$ approaches θ_a as the time $t \to +\infty$.

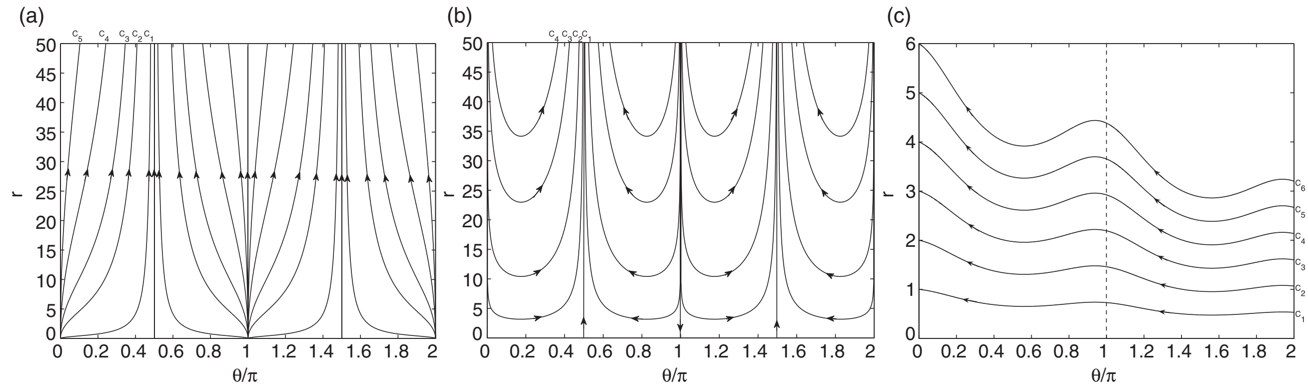

Figure 1. The phase diagrams of second-order LTI systems in polar coordinates: (a) node, (b) saddle point and (c) focus.

Similarly, the line $\theta = \theta_{na}$ is said to be non-asymptote of A if the angle of the trajectory of $\dot{x} = Ax$ approaches θ_{na} as the time $t \to -\infty$.

For a given $A \in \mathbb{R}^{2\times 2}$ with real eigenvalues, the asymptote θ_a is the angle of the real eigenvector corresponding to the larger eigenvalue of A. This definition is applicable to all matrices $A \in \mathbb{R}^{2\times 2}$ with real eigenvalues regardless of the dynamics of A (stable/unstable node, saddle point). If A is a degenerate node (has only one eigenvector with an angle θ_r), θ_a and θ_{na} are chosen from θ_r^+ or θ_r^- based on the trajectory direction of A.

Note that the asymptote of A_3 in $r-\theta$ coordinates is actually $\theta_a = +\infty$ if the trajectories of A_3 are counter clockwise and $\theta_a = -\infty$ if the trajectories of A_3 are clockwise.

2.2 *Solution of the switched system (1) in polar coordinates*

In this subsection, we proceed to analyse the switched system (1) with two unstable subsystems. Using the variation of the subsystems' constants of integration, we reveal how a convergent trajectory can be constructed by switching between two unstable subsystems.

First of all, the two subsystems are defined in terms of their entries.

$$\Sigma_A : \dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x \tag{15}$$

$$\Sigma_B : \dot{x} = Bx = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} x \tag{16}$$

Assumption 1: $A \neq cB$, $c \in \mathbb{R}$.

Assumption 2: *A_i and B_j do not share any real eigenvector*.

The special cases that Assumption 2 is violated will be discussed in Appendix A, just for the completeness of the results.

Following the definition of $f(\theta)$ in Equation (8), we define $f_A(\theta)$ and $f_B(\theta)$ for subsystems A and B respectively.

$$f_A(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta} \tag{17}$$

$$f_B(\theta) = \frac{b_{11}\cos^2\theta + b_{22}\sin^2\theta + (b_{12}+b_{21})\sin\theta\cos\theta}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22}-b_{11})\sin\theta\cos\theta}. \tag{18}$$

It follows from Lemma 1 that for subsystem A,

$$r = C_A e^{\int f_A(\theta)\mathrm{d}\theta} = C_A g_A(\theta) \tag{19}$$

and for subsystem B,

$$r = C_B e^{\int f_B(\theta)\mathrm{d}\theta} = C_B g_B(\theta). \tag{20}$$

By combining (19) and (20), the trajectories of the switched system, except the ones along the eigenvectors, can be described as

$$r(t) = h_A(\theta(t))\, g_A(\theta(t)), \tag{21}$$

where

$$h_A(\theta(t)) = \begin{cases} C_A(t), & \sigma(t) = A \\ C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}, & \sigma(t) = B \end{cases} \tag{22}$$

or similarly

$$r(t) = h_B(\theta(t))\, g_B(\theta(t)), \tag{23}$$

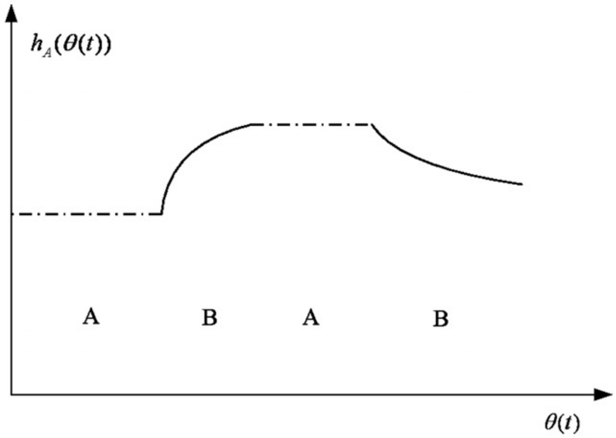

Figure 2. The variation of h_A under switching.

where

$$h_B(\theta(t)) = \begin{cases} C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}, & \sigma(t) = A \\ C_B(t), & \sigma(t) = B \end{cases}. \tag{24}$$

For convenience, we denote

$$H_A(\theta(t)) \triangleq \frac{\mathrm{d}h_A(\theta(t))}{\mathrm{d}t}\bigg|_{\sigma(t)=B}, \quad H_B(\theta(t)) \triangleq \frac{\mathrm{d}h_B(\theta(t))}{\mathrm{d}t}\bigg|_{\sigma(t)=A}. \tag{25}$$

Equation (21) indicates that even when the actual trajectory follows Σ_B, it can still be described by the same form as that of the solution of Σ_A with a varying h_A. Thus, we can use the variation of h_A to describe the behaviour of the switched system (1), as shown in Figure 2.

Geometrically, a negative $H_A(\theta)$, or equivalently a decrease in $h_A(\theta)$, means that the vector field of Σ_B points inwards relative to Σ_A. Intuitively, if the decrease in h_A can compensate the divergence of g_A for a long term, or in a period of $\theta(t)$, then it is possible to stabilise the switched system (1). Although the existence of negative $H_A(\theta)$ or $H_B(\theta)$ is considered to be necessary, it is not sufficient for stabilisability. Therefore, a comprehensive analysis is needed.

3. Characterisation of the BCSS

As mentioned before, if we are able to find the BCSS for a given switched system, then the stabilisability problem can be transformed to the stability problem under the BCSS. To find the BCSS, we need to know which subsystem is more 'stable' for every θ and how θ varies with the time t. The former is determined through the signs of $H_A(\theta)$ and $H_B(\theta)$ (25), while the latter is based on the signs of $Q_A(\theta)$ and $Q_B(\theta)$ which are defined as

$$Q_A(\theta(t)) \triangleq \frac{\mathrm{d}\theta}{\mathrm{d}t}\bigg|_{\sigma=A}, \quad Q_B(\theta(t)) \triangleq \frac{\mathrm{d}\theta}{\mathrm{d}t}\bigg|_{\sigma=B}. \tag{26}$$

It follows from Equations (22) and (25) that

$$\begin{aligned} H_A(\theta(t)) &= \frac{\mathrm{d}h_A(t)}{\mathrm{d}t}\bigg|_{\sigma(t)=B} = C_B(t)\left(\frac{g_B(\theta(t))}{g_A(\theta(t))}\right)' \\ &= -C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}[f_A(\theta(t)) - f_B(\theta(t))]\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\bigg|_{\sigma(t)=B}, \end{aligned} \tag{27}$$

where $C_B(t)$ is a constant since $\sigma(t)=B$ in (27). Similarly, we have

$$H_B(\theta(t)) = C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}[f_A(\theta(t)) - f_B(\theta(t))]\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\bigg|_{\sigma(t)=A}. \tag{28}$$

Equations (27) and (28) can be rewritten as

$$H_A(\theta(t)) = -K_B(\theta(t))G(\theta(t))Q_B(\theta(t)) \tag{29}$$

$$H_B(\theta(t)) = K_A(\theta(t))G(\theta(t))Q_A(\theta(t)), \tag{30}$$

where $K_A(\theta(t)) = C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}$, $K_B(\theta(t)) = C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}$ and

$$G(\theta) = f_A(\theta) - f_B(\theta). \tag{31}$$

Remark 2: In (29) and (30), both $K_A(\theta)$ and $K_B(\theta)$ are positive, and $G(\theta)$ is the common part. It can be readily shown that

- If the signs of $Q_A(\theta)$ and $Q_B(\theta)$ are the same, then the signs of $H_A(\theta)$ and $H_B(\theta)$ are opposite.
- If the signs of $Q_A(\theta)$ and $Q_B(\theta)$ are opposite, then the signs of $H_A(\theta)$ and $H_B(\theta)$ are the same.

The geometrical meaning of the signs of $Q_A(\theta)$ and $Q_B(\theta)$ is the trajectory direction. A positive $Q_A(\theta)$ implies a counter clockwise trajectory of Σ_A in $x-y$ coordinates.

Since the interval of interest of θ is $[-\frac{\pi}{2}, \frac{\pi}{2})$, all the functions of θ could be transformed to the functions of k by denoting $k = \tan\theta$. Straightforward algebraic manipulation yields

$$H_A(k) = K_B(k)\frac{N(k)}{D_B(k)} \tag{32}$$

$$H_B(k) = -K_A(k)\frac{N(k)}{D_A(k)} \tag{33}$$

$$Q_A(k) = -\frac{1}{k^2+1}D_A(k) \tag{34}$$

$$Q_B(k) = -\frac{1}{k^2+1}D_B(k), \tag{35}$$

where

$$D_A(k) = a_{12}k^2 + (a_{11} - a_{22})k - a_{21} \quad (36)$$

$$D_B(k) = b_{12}k^2 + (b_{11} - b_{22})k - b_{21} \quad (37)$$

and

$$N(k) = p_2k^2 + p_1k + p_0, \quad (38)$$

where $p_2 = a_{12}b_{22} - a_{22}b_{12}$, $p_1 = a_{12}b_{21} + a_{11}b_{22} - a_{21}b_{12} - a_{22}b_{11}$ and $p_0 = a_{11}b_{21} - a_{21}b_{11}$.

Denote the two distinct real roots of $N(k)$, if exist, by k_1 and k_2, and assume $k_2 < k_1$. Notice that the signs of Equations (32)–(35) depend on the signs of $D_A(k)$, $D_B(k)$ and $N(k)$.

Lemma 2: *If A and B do not share any real eigenvector, which was guaranteed by Assumption 2, then the real roots of $N(k)$ do not overlap with the real roots of $D_A(k)$ or $D_B(k)$ for non-singular A and B.*

The proof of Lemma 2 is presented in Appendix B.

Definition 4: A region of k is a continuous interval where the signs of (32)–(35) are preserved for all k in this interval.

Remark 3: The boundaries of the regions of k, if exist, are the lines whose angles satisfy $D_A(k) = 0$, $D_B(k) = 0$ or $N(k) = 0$.

- If $D_A(k) = 0$, then $Q_A(\theta) = 0$, the lines are the real eigenvectors of A.
- If $D_B(k) = 0$, then $Q_B(\theta) = 0$, the lines are the real eigenvectors of B.
- Since the real eigenvectors are only located on the boundaries, the solution expressions of (21) and (23) are always valid inside the regions of k.
- If $N(k) = 0$, they are the lines where the trajectories of the two subsystems are tangent to each other. It can be readily shown that $N(k) = 0$ is equivalent to the collinear condition $\det(Ax, Bx) = 0$, where k represents the slope of vector x.
- If $N(k) = (k - k_m)^2$, in the two regions that share the boundary $k = k_m$, the signs of (32)–(35) do not change. Hence the BCSS in these two regions are the same. In addition, trajectories of the two subsystems are tangent to each other on $k = k_m$ and both of them can cross this boundary, then we can choose any subsystem as the BCSS on the vector $k = k_m$. It follows that the BCSS and the stabilisability of the switched system will not be affected by ignoring $k = k_m$.
- With reference to Equations (32)–(35), when trajectories cross the boundary k_1 or k_2, the trajectory directions remain unchanged while the signs of $H_A(k)$ and $H_B(k)$ change simultaneously.

These boundaries divide the $x{-}y$ plane to several conic sectors, i.e. regions of k. Now we proceed to establish criteria to determine the BCSS for every θ, or k equivalently, based on the signs of H_A and H_B.

3.1 *Both H_A and H_B are negative*

Lemma 3: *The switched system (1) is RAS if there is a region of k, $[k_l, k_u]$, where both $H_A(k)$ and $H_B(k)$ are negative.*

With reference to Figure 3, a stable trajectory can be easily constructed by switching inside this region. The proof of Lemma 3 is shown in Appendix C.

3.2 *H_A is positive and H_B is negative*

The BCSS is Σ_A. In this case, the trajectories of two subsystems have the same direction based on Remark 2. With reference to Figure 4, consider an initial state with an angle θ_0 at t_0. Let $r_B(\theta)$ be the trajectory along Σ_B and let $r_A(\theta)$ be the trajectory along Σ_A. Comparing the magnitudes of the trajectories along different subsystems, we have

$$\begin{aligned} r_B(\theta) - r_A(\theta) &= h_A(\theta)\, g_A(\theta) - C_A g_A(\theta) \\ &= g_A(\theta) \int_{t_0}^{t} H_A(\theta(t))\mathrm{d}t > 0, \end{aligned} \quad (39)$$

which shows that the trajectory of Σ_A always has a smaller magnitude than the corresponding one of Σ_B for all θ in this region.

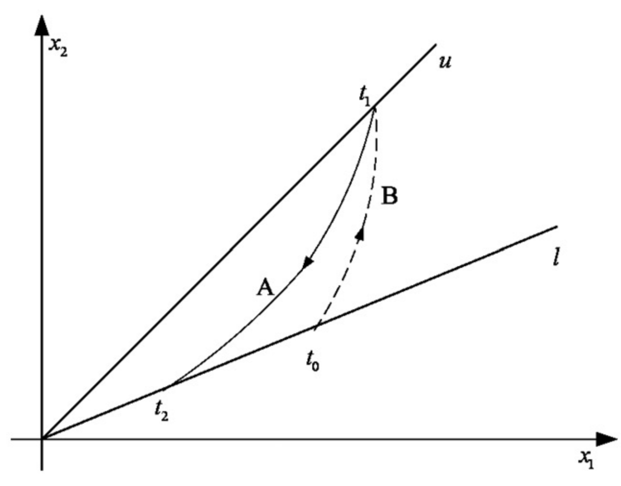

Figure 3. The region where both H_A and H_B are negative.

3.3 H_A *is negative and* H_B *is positive*

Similarly, the BCSS is Σ_B.

3.4 *Both* H_A *and* H_B *are positive*

First, we will show that the switched system cannot be stabilised in this region if its trajectory does not move out. It follows from Assumption 2 that at least one of $g_A(\theta)$ and $g_B(\theta)$ is lower-bounded for any given θ. Since both H_A and H_B are positive, we have $h_A(t) \geq h_A(t_0)$ and $h_B(t) \geq h_B(t_0)$. With reference to (21) and (23), the magnitude of trajectories r is lower-bounded in this region. Hence the stabilisability of the switched system is determined by other regions.

Next we will discuss the scenarios when the trajectory may move out.

(1) If only the trajectory of one subsystem, say Σ_A, can go out of this region, then the BCSS in this region is Σ_A. Let r_{σ^*} be the trajectory along Σ_A and let r_σ be the trajectory under any other switching signal. Comparing the magnitudes of the trajectories under different switching on the boundary ($\theta = \theta_{bn}$) where the trajectories move out, it can be shown that any switching other than Σ_A in this region will make the switched system more unstable since

$$r_{\sigma^*}(\theta_{bn}) = h_A(t_0)\, g_A(\theta_{bn}) < r_\sigma = h_A(t)\, g_A(\theta_{bn}). \qquad (40)$$

(2) If the trajectories of both subsystems can go out and neither can come back, then no matter which subsystem is chosen, the trajectory will leave this region and the stabilisability of the switched system is determined by other regions.

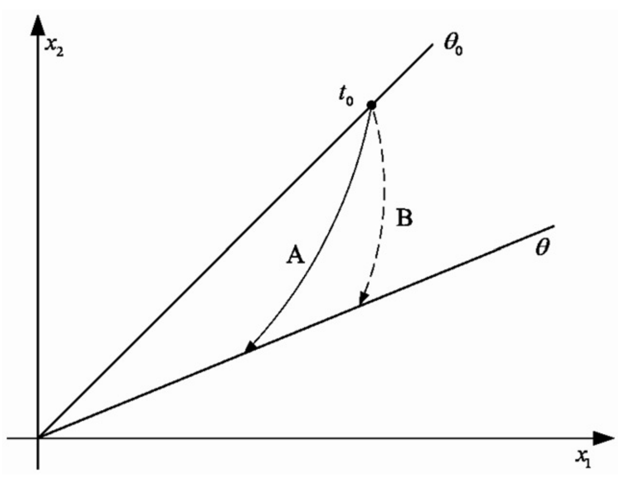

Figure 4. The region where H_A is negative and H_B is positive.

(3) If the trajectories of both subsystems can go out and at least one of them can come back, then at least one of the boundaries of this region is k_1 or k_2, the root of $N(k)$. It was mentioned in Remark 3 that $H_A(k)$ and $H_B(k)$ change their signs simultaneously when trajectories cross the boundary k_1 or k_2, then there must exist a stabilisable region, where both H_A and H_B are negative, next to this region. Therefore, the switched system (1) is RAS based on Lemma 3.

3.5 *One of* H_A *and* H_B *is zero*

If one of $H_A(k)$ and $H_B(k)$ is zero, it implies $N(k) = 0$, then both of them are zero at the line k.

(1) If the trajectories of the subsystems cross the line with the same direction, we can choose either subsystem as the BCSS since the trajectories are tangent to each other on this line.
(2) If the trajectories of the subsystems cross the line with opposite direction, it follows from Remark 3 that there exists a stabilisable region near the line where $N(k) = 0$. Hence the switched system is RAS from Lemma 3.

3.6 *On real eigenvectors*

It can be readily shown that the BCSS is Σ_A on the eigenvectors of B, and vice versa.

3.7 *Procedure*

In this section, we have characterised the BCSS based on the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$. Then one is able to determine the stabilisability of switched systems (1) by the following procedure.

(1) Determine all the boundaries: the real eigenvectors of two subsystems and the distinct real roots of $N(k)$. All the boundaries are known since all the entries of the subsystems are known.
(2) Determine the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$ for every region of k.
(3) Determine the BCSS for every region based on (2) and then obtain the BCSS for the whole phase plane.
(4) Determine the stabilisability of the switched system based on the BCSS for the whole phase plane.

4. A necessary and sufficient condition for stabilisability of switched system (2)

In this section, we are going to apply the best case analysis to derive an easily verifiable, necessary and sufficient condition for the switching stabilisability of switched systems (2). In order to reduce the degrees of freedom, standard forms and standard transformation matrices for different types of second-order matrices are defined and some assumptions are made.

4.1 *Standard forms*

To reduce the degree of freedom, the standard forms for different types of second-order matrices are defined as follows.

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda & 0 \\ -1 & \lambda \end{bmatrix}, \quad J_3 = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \quad (41)$$

Since the switched system (2) is considered, we have

$$\lambda_2 \geq \lambda_1 > 0; \quad \lambda > 0; \quad \mu > 0, \ \omega < 0. \quad (42)$$

4.2 *Standard transformation matrices*

It is assumed that one of the subsystems is in its standard form, i.e. $A_i = J_i$, then the other one can be expressed as $B_j = P_j J_j P_j^{-1}$ with $i \leq j$, where J_j is the standard form of B_j and P_j is the transformation matrix, which are defined for different types of B_j as follows.

$$P_1 = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}. \quad (43)$$

For any given B_j with its standard form J_j, P_j can be derived from the eigenvectors of B_j.

(1) α and β in P_1 can be obtained by calculating the real eigenvectors of B_1. Make sure that the eigenvector $[1, \alpha]^T$ corresponds to λ_1.
(2) α in P_2 can be derived by calculating the eigenvector of B_2. And then β can be uniquely determined by the equation $B_2 = P_2 J_2 P_2^{-1}$.
(3) α and β in P_3 can be derived from the eigenvector of B_3. If the eigenvector corresponding to the eigenvalue $\mu - j\omega$, is $v = \begin{bmatrix} p_{11} + p_{12} i \\ p_{21} + p_{22} i \end{bmatrix}$, then $P_3 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. It is always possible to ensure $p_{11} = 0$ and $p_{12} = 1$ by multiplying v with a factor of $(p_{11} - p_{12} i) i / (p_{11}^2 + p_{12}^2)$.

4.3 *Assumptions*

In order to further reduce the degrees of freedom such that the final result can be presented in a compact form, certain assumptions have to be made concerning the various parameters. These are listed below.

Assumption 3:

(I) *if* $S_{ij} = S_{11}$, $\beta < 0$;
(II) *if* $S_{ij} = S_{12}$, $\alpha < 0$;
(III) *if* $S_{ij} = S_{13}$, $k_2 < 0$, *where* k_2 *is the smaller root of* $N(k)$;
(IV) *if* $S_{ij} = S_{33}$, $p_2 \neq 0$, *where* p_2 *is the leading coefficient of* $N(k)$;
(V) *if* $S_{ij} = S_{33}$, $p_2 < 0$ (*if* $N(k)$ *has two distinct real roots*).

Please note that those assumptions do not impose any constraint on the subsystems A_i and B_j as shown by following lemma.

Lemma 4: *Any given switched linear system* (2) *subjected to Assumptions* 1 *and* 2 *can be transformed to satisfy Assumption* 3 *by similarity transformations.*

The proof of Lemma 4 is provided in Appendix D.

4.4 *A necessary and sufficient stabilisability condition*

Now we are ready to state the principal result of this article as follows.

Theorem 1: *The switched system* (2), *subject to Assumptions* 1–3, *is RAS if and only if there exist two independent real-valued vectors* w_1, w_2, *satisfying the collinear condition*

$$\det([A_i w \ B_j w]) = 0, \quad (44)$$

and the slopes of w_1 *and* w_2, *denoted as* k_1 *and* k_2 *with* $k_2 < k_1$, *satisfy the following inequality*:

$$\begin{cases} L < k_2 < k_1 < M & \text{if } \det(P_j) < 0 \\ \|\exp(B_j T_B)\exp(A_i T_A) w_1\|_2 < \|w_1\|_2 & \text{if } \det(P_j) > 0, \end{cases} \quad (45)$$

where M *and* L *correspond to the slopes of the non-asymptotes of* A_i *and* B_j *respectively such that*

$$M = \begin{cases} 0, & i = 1 \\ +\infty, & i = 2 \\ +\infty, & i = 3 \end{cases}, \quad L = \begin{cases} \alpha, & j = 1 \\ \alpha, & j = 2 \\ -\infty & j = 3 \end{cases} \quad (46)$$

and

$$T_A = \int_{\theta_2}^{\theta_1} \frac{1}{Q_A(\theta)} \mathrm{d}\theta = \int_{\theta_2}^{\theta_1} \frac{1}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta} \mathrm{d}\theta \tag{47}$$

$$T_B = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{Q_B(\theta)} \mathrm{d}\theta = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22} - b_{11})\sin\theta\cos\theta} \mathrm{d}\theta, \tag{48}$$

where $\theta_1 = \tan^{-1} k_1$, $\theta_2 = \tan^{-1} k_2$.

Theorem 1 shows that the existence of two independent vectors w_1, w_2, along which the trajectories of the two subsystems are collinear, is a necessary condition for the switched system (2) to be stabilisable.

Theorem 1 also indicates that there are two classes of switched systems (2) categorised by the sign of $\det(P_j)$, which implies the relative trajectory direction of two subsystems in certain regions. For example, when both A_i and B_j are with complex eigenvalues, the positive/negative $\det(P_j)$ implies that the trajectory directions of the two subsystems are the same/opposite for the whole phase plane.

The possible stabilisation mechanisms corresponding to the two classes mentioned above are totally different as detailed below.

Class I ($\det(P_j) < 0$): *stable chattering* (sliding or sliding-like motion), i.e. when system trajectories can be driven into a conic region where both $H_A(k)$ and $H_B(k)$ are negative, there exists a switching sequence to stabilise the system inside this region. In Class I, the switched systems are only RAS in the region (L, M), but not GAS unless one of the subsystem is with spiral, which can bring any initial state into the stabilisable region.

Class II ($\det(P_j) > 0$): *stable spiralling*, i.e. when the system trajectory is a spiral around the origin and there exists a switching action to make the magnitude decrease after one or half circle. In Class II, if the condition (45) is satisfied, the switched systems are not only RAS, but also GAS.

Remark 4: The existence of two distinct stabilisation mechanisms was also discussed by Xu and Antsaklis (2000). However, no simple algebraic index has been reported in the literature to classify given switched system (2) into those two classes. It was shown above that this can be readily done by checking the sign of $\det(P_j)$.

The condition in Theorem 1 is easily verifiable, by the following procedure.

(1) Calculate the eigenvalues and the eigenvectors of two subsystems, and check the following:
   (a) If one of the subsystems is asymptotically stable, then the switched system (2) is RAS (GAS).
   (b) If either Assumption 1 or 2 is violated, the switched system (2) is not RAS.
(2) Determine S_{ij} with $i \le j$, where the subscripts i and j denote the types of A_i and B_j respectively.
(3) Check whether A_i is in its standard form J_i. Do a similarity transformation for the two subsystems simultaneously to guarantee $A_i = J_i$ if necessary.
(4) Calculate P_j, k_1, k_2 and check Assumption 3.
   (a) If Assumption 3 is satisfied, go to step 5.
   (b) Otherwise, do a similarity transformation, as stated previously, for two subsystems simultaneously such that Assumption 3 is satisfied. Recalculate P_j, k_1 and k_2.
(5) If the real roots $k_1 \neq k_2$, go to the next step, otherwise the switched system is not RAS.
(6) Calculate $\det(P_j)$.
   (a) If $\det(P_j) < 0$, determine the values of L and M with reference to (46), and check the first inequality of Theorem 1.
   (b) If $\det(P_j) > 0$, calculate the values of T_A and T_B using Equations (47) and (48), which are easily integrable by changing variable, and check the second inequality of Theorem 1.

Theorem 1 is proved in the following fashion. For every possible combination of the subsystems S_{ij}, it will be shown that if the condition (45) is satisfied, then there exists a switching signal to stabilise the switched system (2) for initial states in some regions of k, which constitutes the proof for the sufficiency. It will also be demonstrated that for all the cases when this condition is violated, the switched system cannot be stabilised for all non-zero initial states by all possible switching, which would establish the necessity.

We prove Theorem 1 for the case $S_{ij} = S_{11}$ in the following as an example to show the main idea and process of the proof of Theorem 1. The rest of the proof is presented in Appendix E.

Proof: In the case of $S_{ij} = S_{11}$,

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, \quad B_1 = P_2 J_2 P_2^{-1}$$
$$= \frac{1}{\beta - \alpha} \begin{bmatrix} \beta\lambda_{1b} - \alpha\lambda_{2b} & \lambda_{2b} - \lambda_{1b} \\ \alpha\beta(\lambda_{1b} - \lambda_{2b}) & \beta\lambda_{2b} - \alpha\lambda_{1b} \end{bmatrix}. \tag{49}$$

Denote

$$\lambda_{1a} = k_A \lambda_{2a}, \lambda_{1b} = k_B \lambda_{2b}, \tag{50}$$

we have $0 < k_A$, $k_B < 1$,[2] $\alpha \neq 0$ by Assumption 2 and $\beta < 0$ by Assumption 3.1. Substituting A_1 and B_1 into (32)–(38), it follows that

$$N(k) = \frac{\lambda_{2a}\lambda_{2b}(k_A - 1)}{\beta - \alpha} \bar{N}(k), \tag{51}$$

where

$$\bar{N}(k) = k^2 + \frac{(k_A - k_B)\beta + (1 - k_A k_B)\alpha}{k_B - 1} k + \alpha\beta k_A \tag{52}$$

is a monic polynomial with the same roots as $N(k)$ and

$$H_A(k) = K_B(k)\lambda_{2b} \frac{-\bar{N}(k)}{(\alpha - \beta)k} \tag{53}$$

$$H_B(k) = K_A(k) \frac{\lambda_{2a}(1 - k_A)\bar{N}(k)}{(1 - k_B)(k - \alpha)(k - \beta)} \tag{54}$$

$$Q_A(k) = -\frac{1}{1 + k^2} \lambda_{2a}(k_A - 1)k \tag{55}$$

$$Q_B(k) = \frac{\lambda_{2b}(1 - k_B)}{1 + k^2} \frac{(k - \alpha)(k - \beta)}{\alpha - \beta}. \tag{56}$$

It can be readily shown that

$$\mathrm{sgn}(H_A(k)) = -\mathrm{sgn}(\alpha - \beta)\mathrm{sgn}(\bar{N}(k))\mathrm{sgn}(k) \tag{57}$$

$$\mathrm{sgn}(H_B(k)) = \mathrm{sgn}(\bar{N}(k))\mathrm{sgn}(k - \alpha)\mathrm{sgn}(k - \beta) \tag{58}$$

$$\mathrm{sgn}(Q_A(k)) = \mathrm{sgn}(k) \tag{59}$$

$$\mathrm{sgn}(Q_B(k)) = \mathrm{sgn}(\alpha - \beta)\mathrm{sgn}(k - \alpha)\mathrm{sgn}(k - \beta). \tag{60}$$

In order to determine the signs of Equations (57)–(60) in every region of k, the relative position of the boundaries including two eigenvectors of A_1 which are $k = 0$ and $k = \infty$ in S_{11}, two eigenvectors of B_1 which are $k = \alpha$ and $k = \beta$, the two distinct real roots of $N(k)$ which were defined as k_1 and k_2, are required. We go through all possible sequences of these boundaries with respect to the following three exclusive and exhaustive cases. Note that the root condition of $\bar{N}(k)$, or $N(k)$, is essentially the same as the one for $\det(Aw, Bw)$ by denoting k as the slope of w. For simplicity, we use the root condition of $\bar{N}(k)$ in the following analysis.

Case 1: $\bar{N}(k)$ does not have two distinct real roots.

There are three possibilities: (1) two complex roots; (2) two identical real roots; (3) one root, which are discussed as follows.

(1.1) $\bar{N}(k)$ has two complex roots. Since the complex roots of $N(k)$, denoted as c_1 and c_2, are conjugate, Equation (61) below should be positive for any α.

$$(\alpha - c_1)(\alpha - c_2) = \frac{(1 - k_A)k_B\alpha(\alpha - \beta)}{k_B - 1}. \tag{61}$$

As a result, the only possible sequence of these boundaries is $\beta < \alpha < 0$. Then the signs of (57)–(60) could be determined for every region of k, as shown in Figure 5.

Figure 5 is the main tool for us to determine the stability of switched systems (2), as well as switched systems (1). It shows the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$ versus k ranged from $-\infty$ to $+\infty$, which corresponds to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. The dashed vertical lines are the boundaries of the regions of k. The horizontal lines represent the signs of $H_A(k)$ (solid) and $H_B(k)$ (dashed) while the arrows represent the signs of $Q_A(k)$ and $Q_B(k)$ in different regions. If $H_A(k)$ is positive, then the solid line is above the horizontal axis. If $Q_A(k)$ is positive, the arrow on the dashed line points to the right (counter clockwise in $x{-}y$ plane).

With reference to Figure 5, regions I and III are unstabilisable since both $H_A(k)$ and $H_B(k)$ are positive in these regions. Region I is a special region, where none of the trajectories can go out. Consider all possible initial states in different regions as follows.

- If the initial state is in region I, it cannot go out of this region.
- If the initial state is in region II or IV, it will be brought into region I by the BCSS, which is Σ_A in region II (H_A is positive and H_B is negative) and Σ_B in region IV.
- If the initial state is in region III, it must be brought out because region III is unstabilisable. Then the trajectory will go to region II or region IV, and goes to region I eventually.

Therefore, when $\bar{N}(k)$ has two complex roots, the switched system (2) is not RAS.

(1.2) $\bar{N}(k)$ has two identical real roots. Based on Remark 3, the best case analysis for this case is similar to the one for Figure 5 regardless of the position of the multiple roots. Since this is true for all S_{ij}, the analysis for the case that $\bar{N}(k)$ has two identical real roots will be omitted in all other cases.

(1.3) $\bar{N}(k)$ has only one root. In this case, the leading coefficient of $N(k)$, $p_2 = a_{12}b_{22} - a_{22}b_{12} = 0$ from (38). With reference to (49), we have $a_{12} = 0$ and $a_{22} \neq 0$.

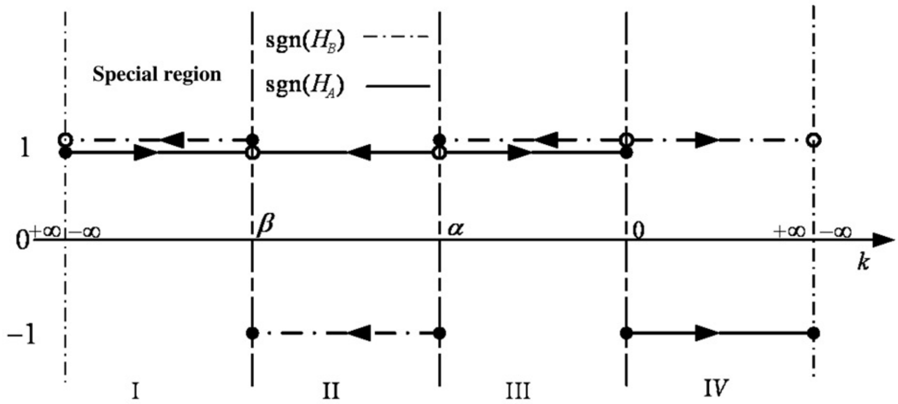

Figure 5. S_{11}: $N(k)$ has two complex real roots, the switched system is not RAS.

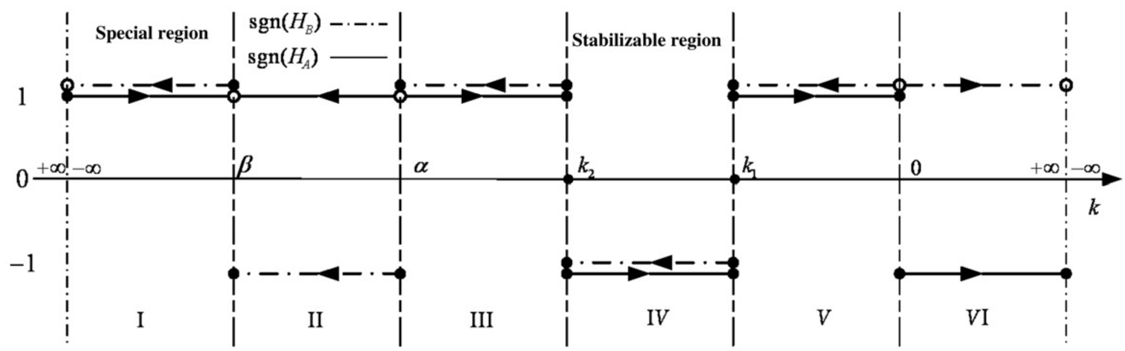

Figure 6. S_{11}: $\det(P_1)<0$, $\beta<\alpha<k_2<k_1<0$, the switched system is RAS.

So $p_2=0$ results in $b_{12}=0$, which implies that B_1 shares a real eigenvector (the y axis) with A_1, which violates Assumption 2. Therefore, this case cannot happen for S_{11}. It can be readily shown that this is true for all other cases of S_{1j} and S_{2j}. In S_{33}, $p_2=0$ was excluded by Assumption 3.4. Hence, we will omit the case that $N(k)$ has only one root in the rest of the proof of Theorem 1.

Case 2: $\bar{N}(k)$ has two distinct real roots and $\det(P_1)<0$

$\alpha>\beta$, with reference to (52) and (61), there are totally four possibilities:

(2.1) $\beta<\alpha<k_2<k_1<0$

With reference to Figure 6, if the initial state is in the region of $k\in(-\infty,\alpha]$ or $k\in[0,\infty)$, the trajectory will be driven into the unstabilisable region I and cannot move out no matter which subsystem is chosen. However, if the initial state is in (α, 0), the trajectory can be brought into region IV, where both $H_A(k)$ and $H_B(k)$ are negative, then the system can be stabilised by switching inside the stabilisable region IV. Therefore, in this case, the switched system is RAS. The stabilisable region is $(\alpha, 0)$.

(2.2) $\beta<k_2<k_1<\alpha<0$

The switched system is not RAS with reference to Figure 7.

(2.3) $\beta<\alpha<0<k_2<k_1$

The switched system is not RAS with reference to Figure 8.

(2.4) $\beta<k_2<0<\alpha<k_1$

The switched system is not RAS with reference to Figure 9.

Case 3: $\bar{N}(k)$ has two distinct real roots and $\det(P_1)>0$.

$\alpha<\beta$, it follows from (52) and (61) that $k_2<\alpha<\beta<k_1<0$.

With reference to Figure 10, it is straightforward that the BCSS is Σ_B in regions I and V because H_A is negative and H_B are positive. Similarly, the BCSS is Σ_A in regions II and IV because H_A is positive and H_B are negative. In region III, both H_A and H_B are positive, but Σ_A is the only subsystem whose trajectory can go out of region III because the boundaries of region III are α and β that correspond to the eigenvectors of B. Similarly, the BCSS is Σ_B in region VI. On k_1 and k_2, without loss of generality, we can choose Σ_A and Σ_B

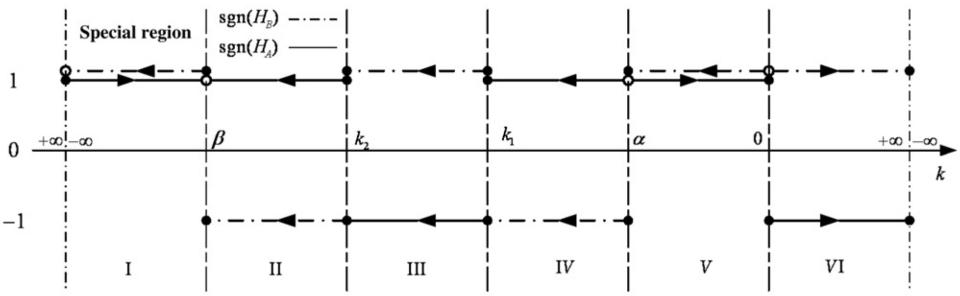

Figure 7. S_{11}: $\det(P_1)<0$, $\beta<k_2<k_1<\alpha<0$, the switched system is unstabilisable.

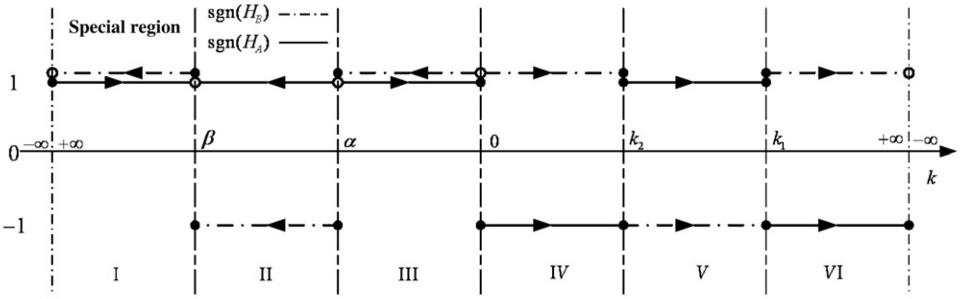

Figure 8. S_{11}: $\det(P_1)<0$, $\beta<\alpha<0<k_2<k_1$, the switched system is unstabilisable.

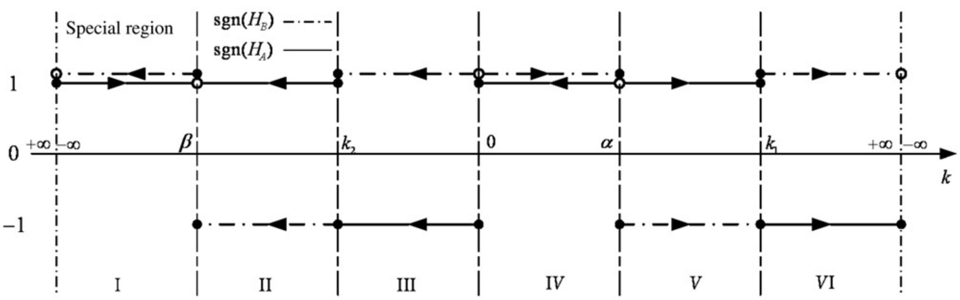

Figure 9. S_{11}: $\det(P_1)<0$, $\beta<k_2<0<\alpha<k_1$, the switched system is unstabilisable.

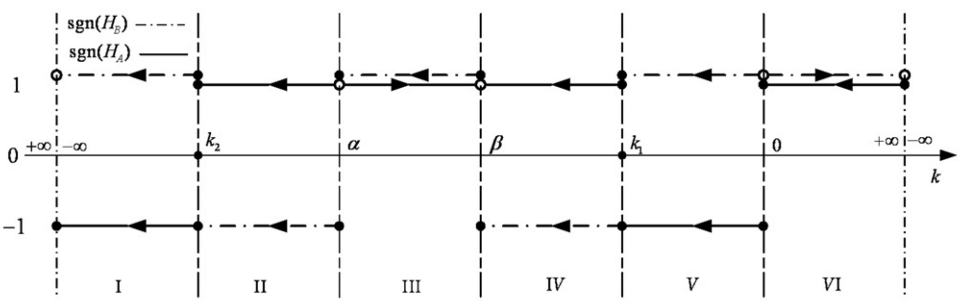

Figure 10. S_{11}: $\det(P_1)>0$, the trajectory under the BCSS rotates around the origin.

respectively as the BCSS since both H_A and H_B are zero. It is concluded that the BCSS in the whole interval of k is

$$\begin{cases} \sigma = A & k_2 < k \leq k_1, \\ \sigma = B & \text{otherwise}. \end{cases} \tag{62}$$

In this case, the trajectory under the BCSS rotates around the origin clockwise. The simplest way to determine stabilisability of the system is to follow a trajectory under the BCSS originating from a line l until it returns to l again and evaluate its expansion or contraction in the radial direction. Without loss of generality, let $w_1=[1,k_1]$, the system is GAS if and only if $\|\exp(B_1T_B)\exp(A_1T_A)w_1\|_2<\|w_1\|_2$. T_A and T_B are the time on Σ_A and Σ_B respectively, which could be calculated by

$$T_A=\int_{\theta_2}^{\theta_1}\frac{\mathrm{d}t}{\mathrm{d}\theta}\bigg|_{\sigma=A}\mathrm{d}\theta=\int_{\theta_2}^{\theta_1}\frac{1}{Q_A(\theta)}\mathrm{d}\theta \quad (63)$$

$$T_B=\int_{\theta_1}^{\theta_2+\pi}\frac{\mathrm{d}t}{\mathrm{d}\theta}\bigg|_{\sigma=B}\mathrm{d}\theta=\int_{\theta_1}^{\theta_2+\pi}\frac{1}{Q_B(\theta)}\mathrm{d}\theta, \quad (64)$$

where $\theta_1=\tan^{-1}k_1$ and $\theta_2=\tan^{-1}k_2$.

Example 1:

$$A=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix},\quad B=\begin{bmatrix}-9 & 5\\ -20 & 11\end{bmatrix}. \quad (65)$$

(1) Simple check shows that A has two distinct real eigenvalues: $\lambda_{1a}=1$ and $\lambda_{2a}=3$ with corresponding eigenvectors: $[1,0]^T$ and $[0,1]^T$, respectively. B has two multiple eigenvalues $\lambda_b=1$ with a single eigenvector $[1,2]^T$, which is undiagonalisable. It is the case S_{12}. And it follows that Assumptions 1 and 2 are satisfied.

(2) A is already in its standard form J_1.

(3) $P_2=\begin{bmatrix}0 & 1\\ -0.2 & 2\end{bmatrix}$ is derived from $B=P_2J_2P_2^{-1}$. It follows that $\alpha=2$, which violates Assumption 3.2. Therefore, we need to transform A and B simultaneously. By denoting $\bar{x}_1=-x_1$, we obtain a new switched system

$$\bar{A}=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix},\quad \bar{B}=\begin{bmatrix}-9 & -5\\ 20 & 11\end{bmatrix}, \quad (66)$$

which has the same stabilisability property as the switched system (65). Recalculate $\bar{P}_2=\begin{bmatrix}0 & 1\\ 0.2 & -2\end{bmatrix}$, where $\alpha=-1$ satisfies Assumption 3.2. And we have $k_1=-0.7460$, $k_2=-1.7873$.

(4) The first inequality of Theorem 1 should be checked because $\det(\bar{P}_2)=-0.2<0$. With reference to (46), we have $L=\alpha=-2$ and $M=0$ for S_{12}, hence the inequality $L<k_2<k_1<M$ is satisfied.

It can be concluded that the switched system (66), or equivalently the switched system (65), is RAS. A typical stabilising trajectory of the switched system (66) is shown in Figure 11.

Note that this example corresponds to a class of switched systems, which was not considered by Xu and Antsaklis (2000), Zhang et al. (2005) or Bacciotti and Ceragioli (2006).

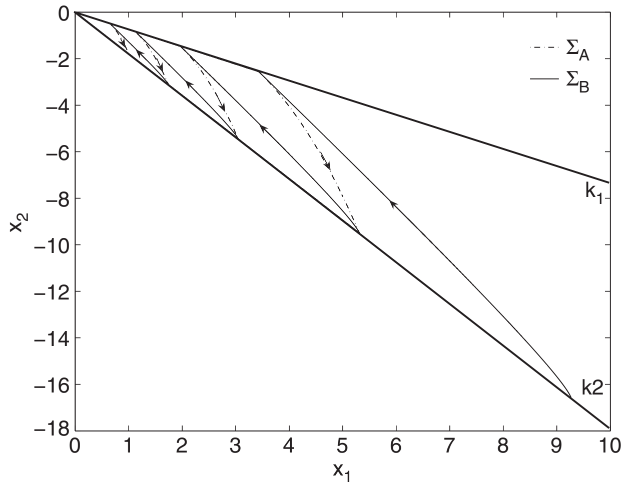

Figure 11. A typical stabilising trajectory of the switched system (66).

5. Extensions

5.1 *Stabilisability conditions for the switched systems (3)*

For the switched systems (3), the standard forms and standard transformation matrices are the same as those for the switched systems (2) in (41) and (43) except Equation (42) is revised as

$$\lambda_2\geq\lambda_1\geq 0;\quad \lambda\geq 0;\quad \mu\geq 0,\omega<0. \quad (67)$$

Theorem 2: *The switched system* (3), *subject to Assumptions* 1–3, *is RAS if and only if there exist two independent real-valued vectors* w_1 *and* w_2, *satisfying the collinear condition*

$$\det([A_iw\ B_jw])=0, \quad (68)$$

and the slopes of w_1 *and* w_2, *denoted as* k_1 *and* k_2 *with* $k_2<k_1$, *satisfy the following inequality*:

$$\begin{cases} L\leq k_2<k_1\leq M & \text{if } \det(P_j)<0\\ \left\|\exp(B_jT_B)\exp(A_iT_A)w_1\right\|_2<\|w_1\|_2 & \text{if } \det(P_j)>0, \end{cases} \quad (69)$$

where M, L, T_A *and* T_B *are the same as those defined in Theorem* 1.

Theorem 2 is an extension of Theorem 1 by including the case when the eigenvalue of the subsystems has zero real part. The proof for Theorem 2 is very similar to that of Theorem 1 by considering the special cases when $k_A=0$, $k_B=0$ (50) or $\mu_A=0$ (42), hence is omitted in this article. The reader is referred to Huang (2008) for the detailed proof.

5.2 *A stabilisability condition for the switched system (4)*

In this subsection, we are going to analyse the stabilisability of the switched system (4), where at least one of the subsystems has a negative eigenvalue.

It is worth noting that the trajectory staying on the eigenvector with a negative eigenvalue will not be considered as a valid stabilising trajectory, because it is not possible to bring the trajectory to this eigenvector exactly in practice. Furthermore a small disturbance will divert the trajectory from the eigenvector even if the initial state is on the eigenvector.

Theorem 3: *The switched system* (4), *subject to Assumptions* 1 *and* 2, *is always RAS.*

The proof for Theorem 3 is very similar to that of Theorem 1 by analysing the cases when k_A and/or k_B are negative, hence is omitted in this article. The reader is referred to Huang (2008) for the detailed proof.

Remark 5: The switched system (4) which is RAS can also be said to be GAS

- if $S_{ij} = S_{13}$. In this case, there exists a subsystem along which the trajectories can be driven into the stabilisable region regardless of the initial state.
- if the switched system (4) subjected to Assumptions 1–3 satisfies the condition $\det(P_j) > 0$. It is the spiralling case. There always exists a trajectory that can rotate around the origin regardless of the type of subsystems.

6. Discussion

In this section, we discuss the connections between the stabilisability conditions in this article and the ones in the literature. We refer in particular to the articles by Xu and Antsaklis (2000), Boscain (2002), Bacciotti and Ceragioli (2006) and Balde and Boscain (2008).

In the article by Xu and Antsaklis (2000), necessary and sufficient stabilisability conditions for the switched systems (1) are firstly found in following cases:

(1) both A and B are unstable nodes;
(2) both A and B are unstable spirals;
(3) both A and B are saddle points.

All of above cases are considered in Theorems 1 and 3.

In the article by Bacciotti and Ceragioli (2006), the authors analyse switched systems in the cases when A has complex conjugates eigenvalues with null real part and any B (stable/unstable node, spiral or saddle), and derive necessary and sufficient conditions for the switching stabilisability. These conditions are mathematically elegant and are easy to verify. In our article, these cases are included in Theorem 2 (when B is node or spiral) and Theorem 3 (when B is saddle). The equivalence between these conditions can be proved by following the proofs of Theorem 2 for $S_{ij} = S_{i3}$, $i = 1, 2, 3$ and considering the special case when the real part of the complex eigenvalue is zero. Due to the limitation of the space, we take $S_{ij} = S_{33}$ as an example to show the equivalence.

With reference to Appendix E5, we assume

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \mu_b & -\omega_b \\ \omega_b & \mu_b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}^{-1}$$
$$= \frac{\omega_b}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix},$$

where $\omega_b < 0$ and $\xi = \frac{\mu_b}{\omega_b} < 0$. With reference to Theorem 1 in the paper by Bacciotti and Ceragioli (2006), where ω_a is chosen to be -1 (it is feasible by scaling time t or do linear transformation $x_1 = -x_1$), the switched system is stabilisable if and only if there exists $x \in \mathbb{R}^2$ such that $\det(Ax : Bx) < 0$. By substituting A and B and denoting $x = [1, k]^T$, the stabilisability condition by Bacciotti and Ceragioli (2006) can be written as:

$$\det(Ax : Bx) = \frac{\omega_b}{\beta} \{(\beta\xi + \alpha)k^2 + [1 - (\alpha^2 + \beta^2)]k + (\beta\xi - \alpha)\} < 0. \tag{70}$$

Case (1) $\det(Ax : Bx)$ does not have two distinct real roots. From Theorem 2, the switched system is not stabilisable. To prove the equivalence, we need to show $\det(Ax : Bx)$ is non-negative for all x, or equivalently, the leading coefficient of (70), denoted as p_2, is positive. It follows from $\det(Ax : Bx)$ does not have two distinct real roots that $|\beta\xi| > |\alpha|$. If $\beta > 0$, then $\beta\xi + \alpha < 0$, we have $p_2 = \frac{\omega_b}{\beta}(\beta\xi + \alpha) > 0$. Similarly, if $\beta < 0$, then $\beta\xi + \alpha > 0$, we also have $p_2 > 0$. So the equivalence is proved for this case.

Case (2) $\det(Ax : Bx)$ has two distinct real roots and $\beta > 0$. In this case, Equation (70) is always true regardless of the sign of p_2. As a result, the switched system is stabilisable. The similar result can be obtained from Theorem 2 by checking the first inequality of (69), which is always satisfied since $L = -\infty$ and $M = +\infty$.

Case (3) $\det(Ax : Bx)$ has two distinct real roots and $\beta < 0$. In this case, Equation (70) is always true regardless of the sign of p_2. As a result, the switched system is stabilisable. The similar result can be obtained from Theorem 2 by checking the second inequality of (69), which is always satisfied due to the property of A.

In this article, we relax the constraint that A has complex eigenvalues with null real part, and extend the stabilisability conditions proposed in Bacciotti and Ceragioli (2006) to more general combinations of subsystems A_i and B_j.

In the articles by Boscain (2002), Balde and Boscain (2008), the authors analyse the stability of switched systems with two asymptotically stable planar LTI subsystems under arbitrary switching and derive necessary and sufficient conditions by finding the worst case switching signals. In Theorem 1, we deal with the regional asymptotical stabilisability of switched systems (2) and derive necessary and sufficient stabilisability condition by finding the BCSS. There are some similarities on the approaches applied in the papers by Balde and Boscain and our article such as analysing worst/best switching signal in different conic sections, finding the vectors where the trajectories of two subsystems are parallel, using some parameters to denote the relative trajectory directions of two subsystems and so on. By reversing time, Theorem 1 is essentially equivalent to the conditions in Boscain (2002), Balde and Boscain (2008), although they are formulated in different forms. Simply speaking, if a switched system (2) with a pair of A_i and B_j is not RAS, then the corresponding switched system with $-A_i$ and $-B_j$ is stable under arbitrary switching. Similarly, if a switched system (2) with A_i and B_j is RAS, then the corresponding switched system with $-A_i$ and $-B_j$ is not stable under arbitrary switching. The equivalence is shown by the following example:

Example 2:

$$A=\begin{bmatrix}-1 & 0\\ 0 & -6\end{bmatrix},\quad B=\begin{bmatrix}12 & 14\\ -21 & -23\end{bmatrix}. \tag{71}$$

With reference to Theorem 2.3 in the paper by Boscain (2002), we have $\rho_A=i\frac{7}{5}$, $\rho_B=i\frac{11}{7}$, $\mathcal{K}=5$, $\mathcal{D}=6.4294$, $\mathcal{K}+\rho_A\rho_B=2.8>0$. So the switched system (71) is not stable under arbitrary switching because it belongs to Case (RR.2.1). By reversing time, we have

$$-A=\begin{bmatrix}1 & 0\\ 0 & 6\end{bmatrix},\ -B=\begin{bmatrix}-12 & -14\\ 21 & 23\end{bmatrix}. \tag{72}$$

With reference to Theorem 1 in this article, we have $k_1=-0.3013$, $k_2=-0.8296$, $\det(P_2)=-0.5<0$, $M=0$, $L=-1.5$. It follows from $L<k_2<k_1<M$ that the switched system (72) is RAS.

It has to be pointed out that the study on the regional asymptotical stabilisability in this article is not trivial although there exists an equivalence between Theorem 1 and the conditions proposed in the papers by Balde and Boscain by reversing time. The reasons are listed below:

(1) When the stabilisability problem is considered, we need to know (i) when a switched system is GAS and (ii) where the stabilisable region is if a switched system is only RAS. In example 2, the initial state has to be inside the region of k bounded by (L, M) such that its trajectory can go into the stabilisable region (k_2, k_1), where $H_A(k)$ and $H_B(k)$ are negative. The situation is different for the problem of the stability under arbitrary switching: if there exists an unstable region, then the trajectory can be driven into this region regardless of its initial state.

(2) The formulation of Theorem 1 is different with the conditions in Boscain and Balde–Boscain's papers: the latter are mathematically elegant by presenting the results for difference cases separately while the former is given in a compact form for all of combinations of dynamics of subsystems by assumptions, which is able to provide more geometrical insights.

(3) In Theorems 2 and 3, the cases when subsystems have eigenvalues with null real part or a negative eigenvalue are considered. No corresponding result is found in the papers by Boscain (2002) and Balde and Boscain (2008).

7. Conclusion

This article deals with the long-standing open problem of deriving easily verifiable, necessary and sufficient conditions for the regional asymptotical stabilisability of switched system with a pair of planar LTI unstable systems. The conditions derived in this article are the extensions to the one proposed by Xu and Antsaklis (2000), and are demonstrated to be (i) more generic in the sense that all the possible combinations of subsystem dynamics (node, saddle point and focus) and marginally unstable subsystems were considered; and (ii) easily verifiable since the checking algorithm, shown in Section 4.4, is easy to follow and all the calculations can be done by hand and (iii) in a compact form which is possible to provide more geometric insights.

In contrast to the Lyapunov function approach commonly adopted by many researchers, a geometric approach was utilised in this article. In order to facilitate the best case analysis, a tool of using the variations of the constants of the integration of subsystems, namely $H_A(k)$ and $H_B(k)$, as the indictors of the 'goodness' or 'badness' of the trajectory, was developed in this article. With this powerful tool, the best case trajectory can be easily identified, which showed that the existence of two independent vectors, where the trajectories of two subsystems are collinear, is a necessary condition for the switched systems (3) to be stabilisable, and these two vectors play a key role on switching strategies. It was also found that the sign of $\det(P_j)$ (43), associating with the relative trajectory directions of the two subsystems in certain regions, can be used to classify any given switched system (3) into two classes, which correspond to the two possible stabilisation mechanisms: stable chattering and stable spiralling.

It is also believed that the idea of using $H_A(k)$ and $H_B(k)$ to characterise the best case switching and the proposed geometrical insights, i.e. the existence of collinear vectors, relative trajectory directions, can be extended to cope with third-order linear systems and some special classes of nonlinear systems.

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Notes

1. If $\theta_e \in (\theta^*, \theta)$, the Cauchy principal value of the improper integral is introduced as $P.V. \int_{\theta^*}^{\theta} f(\tau)\mathrm{d}\tau = \lim_{\varepsilon \to 0^+} (\int_{\theta^*}^{\theta_e - \varepsilon} f(\tau)\mathrm{d}\tau + \int_{\theta_e + \varepsilon}^{\theta} f(\tau)\mathrm{d}\tau)$, which is also bounded because $\lim_{\varepsilon \to 0^+} \int_{\theta_e - \varepsilon}^{\theta_e + \varepsilon} f(\tau)\mathrm{d}\tau = 0$.
2. If $k_A = 1$, any vector in the phase plane is the eigenvector of A, which contradicts Assumption 2 since B have two real eigenvectors.
3. Note that $\beta = 0$ in S_{11} and $\alpha = 0$ in S_{12} have been excluded by Assumption 2.

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Appendix A. Analysis of the special cases when Assumption 2 is violated

Case (1) A and B have only one common eigenvector. Without loss of generality, we assume that the eigenvalues of A and B corresponding to the common eigenvector are λ_{2A} and λ_{2B}, and the common eigenvector is $[0, 1]^T$, then we have

$$A = \begin{bmatrix} \lambda_{1A} & 0 \\ a_{21} & \lambda_{2A} \end{bmatrix}, \quad B = \begin{bmatrix} \lambda_{1B} & 0 \\ b_{21} & \lambda_{2B} \end{bmatrix},$$

where at least one of a_{21} and b_{21} is not zero. Thus the dynamic of the switched system can be described as

$$\dot{x} = \begin{bmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{bmatrix} x,$$

where $\sigma_{11}(t) \in \{\lambda_{1A}, \lambda_{1B}\}$, $\sigma_{21}(t) \in \{a_{21}, b_{21}\}$ and $\sigma_{22}(t) \in \{\lambda_{2A}, \lambda_{2B}\}$.

For the switched systems (2) and (3), $\sigma_{11}(t)$ is non-negative because all the eigenvalues of A and B are non-negative. It follows that $|x_1(t)| = e^{\int_0^t \sigma_{11}(\tau)\mathrm{d}\tau}|x_1(0)|$ is lower-bounded by $|x_1(0)|$, so the switched system (2) or (3) is not RAS in this case.

For the switched system (4), if both λ_{1A} and λ_{1B} are non-negative, similarly $|x_1(t)|$ is lower-bounded by $|x_1(0)|$, the switched system (4) is not RAS. If one of λ_{1A} and λ_{1B} is negative, the switched system (4) is RAS, which is proved as follows.

Consider a periodical switching signal $\sigma_T(t)$ with a period of $T = t_A + t_B$

$$\sigma_T(t) = \begin{cases} A & \text{if } 0 \le t < t_A \\ B & \text{if } t_A \le t < T \end{cases}.$$

It follows that

$$x(T) = e^{Bt_B}e^{At_A}x(0) \triangleq \Gamma x(0) = \begin{bmatrix} \Gamma_{11} & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} x(0),$$

where $\Gamma_{11} = e^{\lambda_{1A}t_A + \lambda_{1B}t_B}$, $\Gamma_{22} = e^{\lambda_{2A}t_A + \lambda_{2B}t_B}$,

$$\Gamma_{21} = \frac{a_{21}}{(\lambda_{1A} - \lambda_{2A})}\left(e^{\lambda_{1A}t_A} - e^{\lambda_{2A}t_A}\right)e^{\lambda_{1B}t_B} + \frac{b_{21}}{(\lambda_{1B} - \lambda_{2B})}\left(e^{\lambda_{1B}t_B} - e^{\lambda_{2B}t_B}\right)e^{\lambda_{2A}t_A}.$$

Let $x(0)$ be on the eigenvector corresponding to the eigenvalue Γ_{11}, i.e.

$$x(0) = \left[1, \quad \frac{\Gamma_{21}}{(\Gamma_{11} - \Gamma_{22})}\right]^T,$$

we have $x(T) = \Gamma_{11}x(0)$. If one of λ_{1A} and λ_{1B} is negative, for every pair (t_A, t_B) satisfying $\lambda_{1A}t_A + \lambda_{1B}t_B < 0$, there exists a corresponding vector such that the trajectory starting from this vector is asymptotically stable under the switching signal $\sigma_T(t)$. Since one of a_{21} and b_{21} is non-zero, the collection of these vectors, corresponding to the different pairs (t_A, t_B) with $0 < \Gamma_{11} < 1$, is a region instead of a single line. Based on Definition 1, the switched system (4) is RAS.

Case (2) A and B have two common eigenvectors. In this case, we have

$$\dot{x} = \begin{bmatrix} \sigma_{11}(t) & 0 \\ 0 & \sigma_{22}(t) \end{bmatrix} x.$$

Similarly, the switched system (2) or (3) is not RAS since both $\sigma_{11}(t)$ and $\sigma_{22}(t)$ are non-negative.

In this case, the switched system (4) is RAS if and only if (a) one of λ_{1A} and λ_{1B} is negative; and (b) one of λ_{2A} and λ_{2B} is negative and (c) the product of the two negative eigenvectors is greater than the product of the other two non-negative eigenvectors. These conditions are equivalent to the existence of a pair (t_A, t_B) such that both $\lambda_{1A}t_A + \lambda_{1B}t_B$ and $\lambda_{2A}t_A + \lambda_{2B}t_B$ are negative.

Note that the special cases that Assumption 2 is violated can also be solved by direct inspection. They are discussed here just for the completeness of the results.

Appendix B. Proof of Lemma 2

It follows from (17) and (18) that

$$
\begin{aligned}
f_A(\theta) - f_B(\theta) &\\
&= \frac{\left\{\begin{array}{l}(\tan^2\theta+1)[(a_{12}b_{22}-a_{22}b_{12})\tan^2\theta \\ \quad +(a_{12}b_{21}+a_{11}b_{22}-b_{12}a_{21}-b_{11}a_{22})\tan\theta \\ \quad +(a_{11}b_{21}-b_{11}a_{21})]\end{array}\right\}}{\left\{\begin{array}{l}[a_{12}\tan^2\theta+(a_{11}-a_{22})\tan\theta-a_{21}] \\ \times[b_{12}\tan^2\theta+(b_{11}-b_{22})\tan\theta-b_{21}]\end{array}\right\}} \\
&= \frac{(k^2+1)N(k)}{D_A(k)D_B(k)} \qquad \text{(B1)}
\end{aligned}
$$

With reference to (8) and (10), we have

$$f_A(\theta) - f_B(\theta) = \frac{1}{r}\left(\left.\frac{\mathrm{d}r}{\mathrm{d}\theta}\right|_{\sigma=A} - \left.\frac{\mathrm{d}r}{\mathrm{d}\theta}\right|_{\sigma=B}\right). \qquad \text{(B2)}$$

Combining (B1) and (B2) yields

$$N(k) = \frac{1}{r(k^2+1)}\left\{\left.\frac{\mathrm{d}r}{\mathrm{d}\theta}\right|_{\sigma=A} D_A(k)D_B(k) - \left.\frac{\mathrm{d}r}{\mathrm{d}\theta}\right|_{\sigma=B} D_A(k)D_B(k)\right\}. \qquad \text{(B3)}$$

It follows from (26), (34) and (35) that

$$N(k) = \frac{1}{r}\left\{\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{\sigma=A}(k)D_B(k) - \left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{\sigma=B}(k)D_A(k)\right\}. \qquad \text{(B4)}$$

Let $\bar{k}$ be a real root of $D_A(k)$, then $\bar{k}$ is an eigenvector of A. It follows from Assumption 2 that $D_B(\bar{k}) \neq 0$. So $N(\bar{k}) = 0$ only if $\frac{\mathrm{d}r}{\mathrm{d}t}\big|_{\sigma=A}(\bar{k}) = 0$, which implies that the eigenvalue, corresponding to the eigenvector $k = \bar{k}$, is zero. It contradicts the condition that A is non-singular.

Appendix C. Proof of Lemma 3

Since $H_A(k)$ and $H_B(k)$ are both negative, the trajectories of the two subsystems have opposite directions in this region. With reference to Figure 3, define u and l as the lines where $x_2 = k_u x_1$ and $x_2 = k_l x_1$. Consider an initial state on l at t_0. Let the trajectory follow Σ_A until it hits u at t_1 and switch to Σ_B until it returns to the line l again at t_2. Define the states at t_0, t_1 and t_2 as (r_0, θ_0), (r_1, θ_1) and (r_2, θ_0) respectively, it yields

$$
\begin{aligned}
&r_0 = C_{A0}g_A(\theta_0) = C_{B0}g_B(\theta_0), \quad r_1 = C_{A0}g_A(\theta_1) = C_{B1}g_B(\theta_1), \\
&r_2 = C_{A1}g_A(\theta_0) = C_{B1}g_B(\theta_0).
\end{aligned} \qquad \text{(C1)}
$$

It follows from (27) that $C_{A1} = C_{A0}(1+\Delta)$, where

$$\Delta = \frac{1}{C_{A0}}\int_{t_1}^{t_2} H_A(\theta(t))\mathrm{d}t = \frac{g_A(\theta_1)}{g_B(\theta_1)}\int_{\theta_1}^{\theta_0}\frac{g_B(\theta)}{g_A(\theta)}[f_B(\theta)-f_A(\theta)]\mathrm{d}\theta$$

is a constant between $(-1, 0)$ depending on the known parameters: k_l, k_u and the entries of A and B. An asymptotically stable trajectory can be easily constructed by repeating the switching from t_0 to t_2.

$$\lim_{n\to\infty} r(t_0 + nT) = \lim_{n\to\infty} C_{A0}(1+\Delta)^n g(\theta_0) \to 0,$$

where $T = t_2 - t_0 = \int_{\theta_0}^{\theta_1}\frac{1}{Q_A(\theta)}\mathrm{d}\theta + \int_{\theta_1}^{\theta_0}\frac{1}{Q_B(\theta)}\mathrm{d}\theta$ and n is the number of switching periods.

Appendix D. Proof of Lemma 4

Assumptions 3.1–3.3 can be satisfied by the transformation $\bar{x}_1 = -x_1$ when necessary. When $S_{ij} = S_{1j}$, A_1 equals J_1, which is invariant under the transformation $\bar{x}_1 = -x_1$. Therefore, it is reasonable to transform A_1 and B_j simultaneously by $\bar{x}_1 = -x_1$ while the stability of the switched systems S_{1j} preserves. It is assumed that one of the eigenvectors of B is in the fourth quadrant in S_{11} and S_{12}.[3] Similarly, it is assumed that the vector $[1, k_2]^T$ is in the fourth quadrant in S_{13}.

Assumptions 3.4 and 3.5 can be satisfied by similarity transformation with a unitary matrix $W = \begin{bmatrix}\gamma & -\eta \\ \eta & \gamma\end{bmatrix}$ when necessary, where $\det(W) = \sqrt{\gamma^2+\eta^2} = 1$. Geometrically, transformation with W is a coordinate rotation. The phase diagram of $A_3 = J_3$ is a spiral that is invariant under the rotation. Therefore, it is possible to rotate the original coordinate to satisfy Assumptions 3.4 and 3.5 while the stability property preserves.

Since W is unitary and real, $W^{-1} = W^T$. In addition, A_3 is in its standard form J_3. It follows that

$$
\begin{aligned}
\bar{A}_3 &= W^{-1}A_3W = W^TA_3W = W^TJ_3W \\
&= \begin{bmatrix}\gamma & \eta \\ -\eta & \gamma\end{bmatrix}\begin{bmatrix}\mu & -\omega \\ \omega & \mu\end{bmatrix}\begin{bmatrix}\gamma & -\eta \\ \eta & \gamma\end{bmatrix} \\
&= \begin{bmatrix}\mu\gamma^2-(-\omega+\omega)\gamma\eta+\mu\eta^2 & -\omega\gamma^2+(\mu-\mu)\gamma\eta-\omega\eta^2 \\ \omega\gamma^2+(\mu-\mu)\gamma\eta-(-\omega)\eta^2 & \mu\gamma^2+(-\omega+\omega)\gamma\eta+\mu\eta^2\end{bmatrix} \\
&= J_3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\bar{B}_3 &\triangleq \begin{bmatrix}\bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22}\end{bmatrix} = W^{-1}B_3W = W^TB_3W \\
&= \begin{bmatrix}\left\{\begin{array}{l}b_{11}\gamma^2-(b_{12}+b_{21})\gamma\eta \\ +b_{22}\eta^2\end{array}\right\} & \left\{\begin{array}{l}b_{12}\gamma^2+(b_{11}-b_{22})\gamma\eta \\ -b_{21}\eta^2\end{array}\right\} \\ \left\{\begin{array}{l}b_{21}\gamma^2+(b_{11}-b_{22})\gamma\eta \\ -b_{12}\eta^2\end{array}\right\} & \left\{\begin{array}{l}b_{22}\gamma^2+(b_{12}+b_{21})\gamma\eta \\ +b_{11}\eta^2\end{array}\right\}\end{bmatrix}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\bar{p}_2 &= \bar{a}_{12}\bar{b}_{22} - \bar{b}_{12}\bar{a}_{22} = a_{12}\bar{b}_{22} - \bar{b}_{12}a_{22} \\
&= \eta^2\left[p_2\left(\frac{\gamma}{\eta}\right)^2 + p_1\frac{\gamma}{\eta} + p_0\right].
\end{aligned} \qquad \text{(D1)}
$$

The polynomial inside the bracket in (D1) has the same coefficients as $N(k)$. If $p_2 > 0$ and $N(k)$ has two roots $k_2 < k_1$, it is always possible to get a negative $\bar{p}_2$ by a pair of (γ, η) satisfying $k_2 < \frac{\gamma}{\eta} < k_1$.

Similarly, if $p_2 = 0$ and $p_2^2 + p_1^2 + p_0^2 \neq 0$ that was guaranteed by Assumption 1, it is always possible to find a pair of (γ, η) to guarantee $\bar{p}_2 \neq 0$.

Appendix E. The proof of Theorem 1 for other cases of S_{ij}

E1: Proof of $S_{ij} = S_{12}$

In this case, the two subsystems are expressed as

$$A_1 = \begin{bmatrix}\lambda_{1a} & 0 \\ 0 & \lambda_{2a}\end{bmatrix}, \quad B_2 = \frac{1}{\beta}\begin{bmatrix}\beta\lambda_b+\alpha & -1 \\ \alpha^2 & \beta\lambda_b-\alpha\end{bmatrix}, \qquad \text{(E1)}$$

where $\alpha<0$ by Assumption 3.2, $\lambda_{2a}>\lambda_{1a}>0$, and $\lambda_b>0$. Denote $\lambda_{1a}=k_A\lambda_{2a}$, then $k_A\in(0,1)$. Substituting the entries of A_1 and B_2 into (32)–(35), it follows that

$$\mathrm{sgn}(H_A(k)=-\mathrm{sgn}(\beta)\mathrm{sgn}(\bar{N}(k))\mathrm{sgn}(k) \tag{E2}$$

$$\mathrm{sgn}(H_B(k)=\mathrm{sgn}(\bar{N}(k)) \tag{E3}$$

$$\mathrm{sgn}(Q_A(k))=\mathrm{sgn}(k) \tag{E4}$$

$$\mathrm{sgn}(Q_B(k))=\mathrm{sgn}(\beta), \tag{E5}$$

where

$$\bar{N}(k)=k^2+[(k_A-1)\beta\lambda_b-(k_A+1)\alpha]k+k_A\alpha^2. \tag{E6}$$

Similar to the case $S_{ij}=S_{11}$, we need to know the locations of k_1, k_2 relative to α, which is based on

$$\mathrm{sgn}((\alpha-k_1)(\alpha-k_2))=\mathrm{sgn}(\beta). \tag{E7}$$

Case 1: $\bar{N}(k)$ does not have two distinct real roots.

(1.1) $\beta<0$: It follows that the discriminant of Equation (E6)

$$\begin{aligned}\Delta_{12}&=\beta^2\lambda_b^2(k_A-1)^2+(k_A+1)^2\alpha^2\\&\quad-2\alpha\beta\lambda_b(k_A-1)(k_A+1)-4k_A\alpha^2\\&=\beta\lambda_b(k_A-1)[\beta\lambda_b(k_A-1)-2\alpha(k_A+1)]+(k_A-1)^2\alpha^2>0,\end{aligned}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta<0$ is impossible in this case.

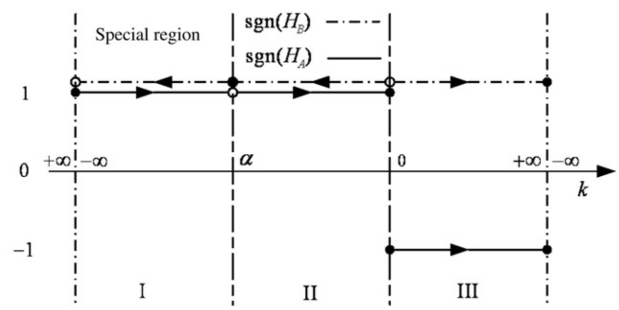

Figure A1. S_{12}: $N(k)$ does not have two distinct real roots, the switched system is unstabilisable.

(1.2) $\beta>0$: With reference to Figure A1 and following the similar argument as that for Figure 5, it can be concluded that the switched system is unstabilisable.

Case 2: $\bar{N}(k)$ has two distinct real roots and $\det(P_2)<0$.

$\det(P_2)=-\beta<0$ leads to $\beta>0$. It follows from $\beta>0$ and $\alpha<0$ (Assumption 3.2) that Equation (E7) is positive. Thus k_1 and k_2 are in the same side of α. In addition, $|k_1k_2|=k_A\alpha^2<\alpha^2$. It results in $\alpha<k_2<k_1<0$ or $\alpha<0<k_2<k_1$.

(2.1) $\alpha<k_2<k_1<0$: Both (E2) and (E3) are negative when $k\in(k_2,k_1)$. Therefore, the switched system is regionally stabilisable based on Lemma 3.

(2.2) $\alpha<0<k_2<k_1$: With reference to Figure A2, the switched system is stable by similar argument as that for Figure 5. It can be concluded that $\alpha<k_2<k_1<0$ is necessary and sufficient for the stabilisability in Case 2.

Case 3: $\bar{N}(k)$ has two distinct real roots and $\det(P_2)>0$.

It follows from $\det(P_2)>0$ that $\beta<0$. With reference to (E6) and (E7), the only possible sequence is $k_2<\alpha<k_1<0$ in this case. With reference to Figure A3, the BCSS σ^* for this case is the same as (62) by similar argument as that for Figure 10.

E2: Proof of $S_{ij}=S_{13}$

$$A_1=\begin{bmatrix}\lambda_{1a}&0\\0&\lambda_{2a}\end{bmatrix},\quad B_3=\frac{\omega}{\beta}\begin{bmatrix}\beta\xi-\alpha&1\\-(\alpha^2+\beta^2)&\beta\xi+\alpha\end{bmatrix}, \tag{E8}$$

where $\mu>0$, $\omega<0$, and $\xi=\frac{\mu}{\omega}<0$. Substituting A_1 and B_3 into (32)–(35), it follows that $\mathrm{sgn}(H_A(k))=-\mathrm{sgn}(\beta)\,\mathrm{sgn}(\bar{N}(k))\mathrm{sgn}(k)$, $\mathrm{sgn}(H_B(k))=\mathrm{sgn}(\bar{N}(k))$, $\mathrm{sgn}(Q_A(k))=\mathrm{sgn}(k)$, $\mathrm{sgn}(Q_B(k))=\mathrm{sgn}(\beta)$, where

$$\bar{N}(k)=k^2-[(k_A-1)\beta\xi+(k_A+1)\alpha]k+k_A(\alpha^2+\beta^2). \tag{E9}$$

Case 1: $\bar{N}(k)$ does not have two distinct real roots.

Figure A4 shows that the BCSS is Σ_B for all k regardless of the sign of $\det(P_3)$. Hence the switched system is unstabilisable.

Case 2: $\bar{N}(k)$ has two distinct real roots and $\det(P_3)<0$.

In this case, $\beta>0$. It follows from $k_2<0$ (Assumption 3.3) and $k_1k_2=k_A(\alpha^2+\beta^2)>0$ that $k_2<k_1<0$. Hence $H_A(k)$ and $H_B(k)$ are negative when $k\in(k_2,k_1)$, the switched system is regionally stabilisable based on Lemma 3.

Case 3: $\bar{N}(k)$ has two distinct real roots and $\det(P_3)>0$.

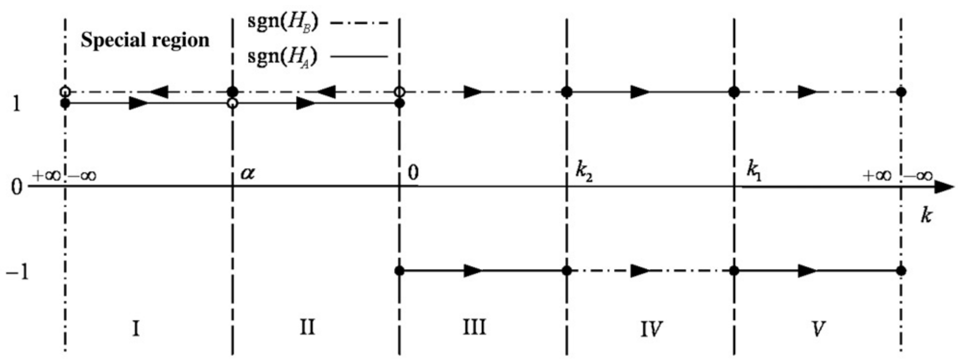

Figure A2. S_{12}: $\det(P_2)<0$, $\alpha<0<k_2<k_1$, the switched system is unstabilisable.

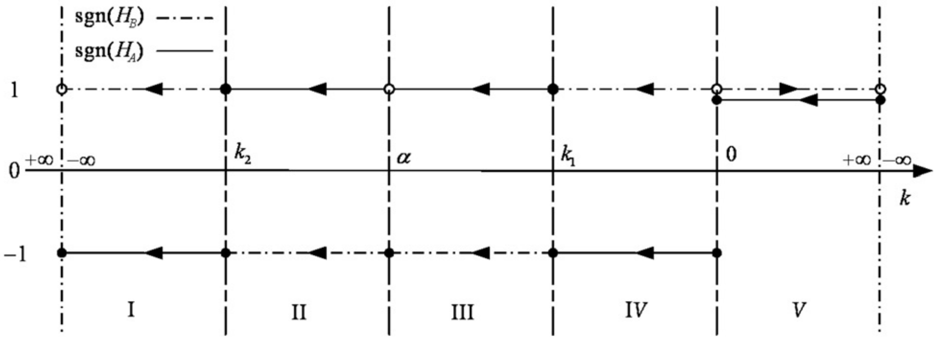

Figure A3. S_{12}: $\det(P_2)>0$, the best case trajectory rotates around the origin clockwise.

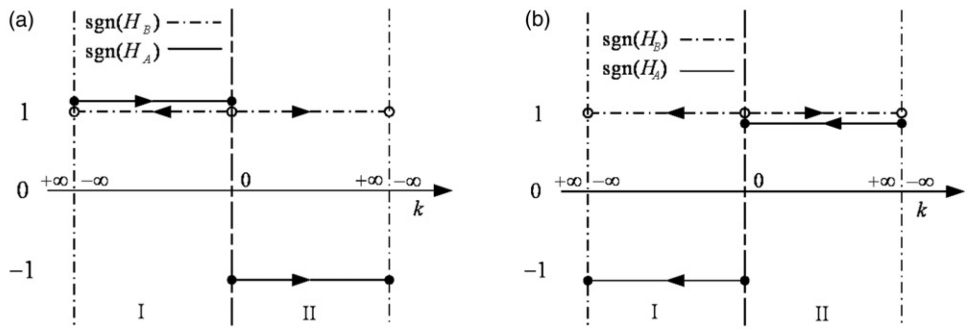

Figure A4. S_{13}: $N(k)$ does not have two distinct real roots, the switched system is unstabilisable: (a) $\det(P_3)<0$ and (b) $\det(P_3)>0$.

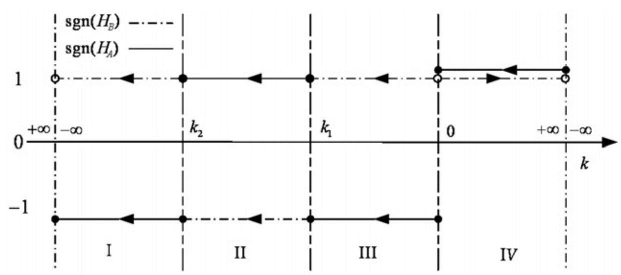

Figure A5. S_{13}: $\det(P_3)>0$, the best case trajectory rotates around the origin clockwise.

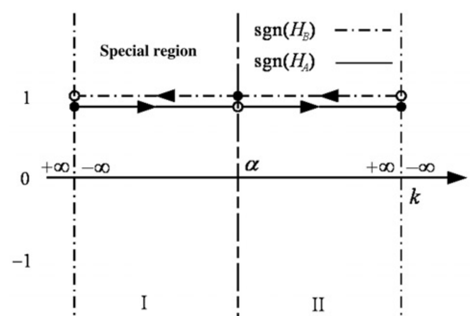

Figure A6. S_{22}: $N(k)$ does not have two distinct real roots, the switched system is unstabilisable.

In this case, we have $\beta<0$. Similarly, we obtain the BCSS as (62) with reference to Figure A5.

E3: Proof of $S_{ij}=S_{22}$

$$A_2=\begin{bmatrix}\lambda_a & 0\\ -1 & \lambda_a\end{bmatrix},\quad B_2=\frac{1}{\beta}\begin{bmatrix}\beta\lambda_b+\alpha & -1\\ \alpha^2 & \beta\lambda_b-\alpha\end{bmatrix}, \tag{E10}$$

where $\lambda_a,\lambda_b>0$. Substituting A_2 and B_2 into (32)–(35), it follows that $\mathrm{sgn}(H_A(k))=\mathrm{sgn}(\beta)\mathrm{sgn}(\bar{N}(k)), \mathrm{sgn}(H_B(k))=\mathrm{sgn}(\bar{N}(k)), \mathrm{sgn}(Q_A(k))=-1, \mathrm{sgn}(Q_B(k))=\mathrm{sgn}(\beta)$, where

$$\bar{N}(k)=k^2-\frac{2\lambda_a\alpha+1}{\lambda_a}k+\frac{\lambda_a\alpha^2+(\beta\lambda_b+\alpha)}{\lambda_a}. \tag{E11}$$

Case 1: $\bar{N}(k)$ does not have two distinct real roots.

(1.1) $\beta<0$: It follows that

$$\Delta_{22}=\left(\frac{2\lambda_a\alpha+1}{\lambda_a}\right)^2-4\frac{\lambda_a\alpha^2+(\beta\lambda_b+\alpha)}{\lambda_a}=\frac{1-4\beta\lambda_a\lambda_b}{\lambda_a^2}>0, \tag{E12}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta<0$ is impossible in this case.

(1.2) $\beta>0$: With reference to Figure A6, the switched system is unstabilisable.

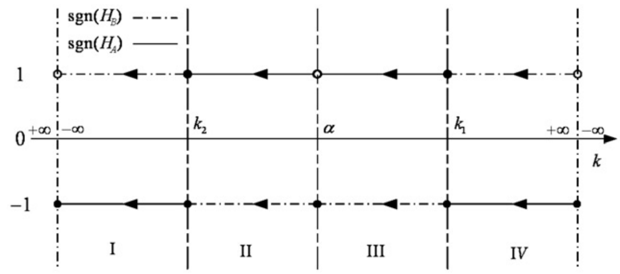

Figure A7. S_{22}: $\det(P_2)>0$, the best case trajectory rotates around the origin clockwise.

Case 2: $\bar{N}(k)$ has two distinct real roots and $\det(P_2)<0$.

In this case, we have $\beta>0$. Then both $H_A(k)$ and $H_B(k)$ are negative when $k\in(k_2,k_1)$. Based on Lemma 3, the switched system is regionally stabilisable as long as k_1 and k_2 exist. In addition, it can be shown that the existence of k_1 and k_2 implies $\alpha<k_2<k_1$ in S_{22} as follows.

$$k_2-\alpha=\frac{\frac{2\lambda_a\alpha+1}{\lambda_a}-\sqrt{\Delta_{22}}}{2}-\alpha=\frac{1-\sqrt{1-4\beta\lambda_a\lambda_b}}{2\lambda_a}>0. \quad \text{(E13)}$$

Hence, it can be concluded that $\alpha<k_2<k_1$ is necessary and sufficient for the stabilisability in Case 2.

Case 3: $\bar{N}(k)$ has two distinct real roots and $\det(P_2)>0$.

$\beta<0$, Similarly, we have the BCSS as (62) with reference to Figure A7.

E4: Proof of $S_{ij}=S_{23}$

$$A_2=\begin{bmatrix}\lambda_a & 0\\ -1 & \lambda_a\end{bmatrix},\quad B_3=\frac{\omega}{\beta}\begin{bmatrix}\beta\xi-\alpha & 1\\ -(\alpha^2+\beta^2) & \beta\xi+\alpha\end{bmatrix}, \quad \text{(E14)}$$

where $\mu>0$, $\omega<0$, and $\xi=\frac{\mu}{\omega}<0$. So we have $\operatorname{sgn}(H_A(k))=\operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))$, $\operatorname{sgn}(H_B(k))=\operatorname{sgn}(\bar{N}(k))$, $\operatorname{sgn}(Q_A(k))=-1$, $\operatorname{sgn}(Q_B(k))=\operatorname{sgn}(\beta)$, where $\bar{N}(k)=k^2-\frac{2\lambda_a\alpha+1}{\lambda_a}k+\frac{\lambda_a(\alpha^2+\beta^2)-(\beta\xi-\alpha)}{\lambda_a}$.

Case 1: $\bar{N}(k)$ does not have two distinct real roots.

(1.1) $\beta<0$: $H_A(k)$ is negative and $H_B(k)$ is positive for all regions, then Σ_B is the BCSS for all regions. On the boundary, which is the eigenvector of σ_A, the BCSS is still Σ_B. Therefore Σ_B is the BCSS for the whole phase plane and it is trivial to show that the switched system is unstabilisable.

(1.2) $\beta>0$: Both $H_A(k)$ and $H_B(k)$ are positive, since the only boundary is the real eigenvector of A, the trajectory alone A goes to its real eigenvector and cannot go out of this region. Hence Σ_B is the BCSS for the whole phase plane and the switched system is unstabilisable.

Case 2: $\bar{N}(k)$ has two distinct real roots and $\det(P_3)<0$.

It follows from $\det(P_3)=-\beta<0$ that $\beta>0$. Both $H_A(k)$ and $H_B(k)$ are negative when $k\in(k_2,\,k_1)$, thus the switched system is regionally stabilisable as long as $k_2<k_1$ exists. It proves the first inequality of Theorem 1 because $M=+\infty$ and $L=-\infty$ for S_{23} with reference to (46).

Case 3: $\bar{N}(k)$ has two distinct real roots and $\det(P_3)>0$.

In this case, $\beta<0$. Similarly, the BCSS is the same as (62) with reference to Figure A8.

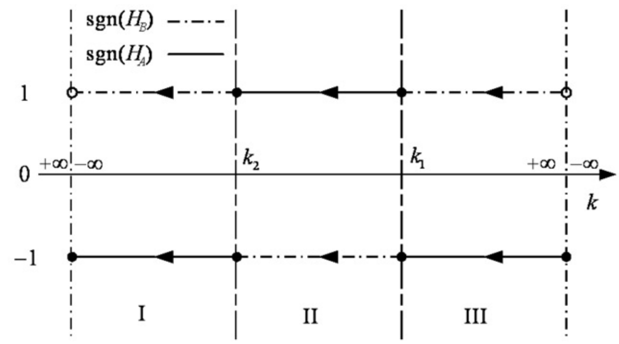

Figure A8. S_{23}: $\det(P_3)>0$, the best case trajectory rotates around the origin clockwise.

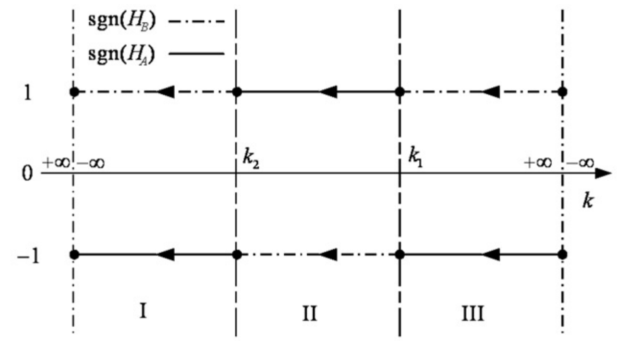

Figure A9. S_{33}: $\det(P_3)>0$, the best case trajectory rotates around the origin clockwise.

E5: Proof of $S_{ij}=S_{33}$

$$A_3=\begin{bmatrix}\mu_a & 1\\ -1 & \mu_a\end{bmatrix}\quad B_3=\frac{\omega_b}{\beta}\begin{bmatrix}\beta\xi-\alpha & 1\\ -(\alpha^2+\beta^2) & \beta\xi+\alpha\end{bmatrix},$$

where μ_a, $\mu_b>0$, $\omega_b<0$ and $\xi=\frac{\mu_b}{\omega_b}<0$. Similarly, we have $\operatorname{sgn}(H_A(k))=\operatorname{sgn}(N(k))$, $\operatorname{sgn}(H_B(k))=\operatorname{sgn}(\beta)\operatorname{sgn}(N(k))$, $\operatorname{sgn}(Q_A(k))=-1$, $\operatorname{sgn}(Q_B(k))=\operatorname{sgn}(\beta)$, where

$$\begin{aligned}N(k)=\frac{\omega_b}{\beta}\{&[(\beta\xi+\alpha)-\mu_a]k^2+[1+2\mu_a\alpha-(\alpha^2+\beta^2)]k\\ &+(\beta\xi-\alpha)-\mu_a(\alpha^2+\beta^2)\}\triangleq p_2k^2+p_1k+p_0.\end{aligned} \quad \text{(E15)}$$

Case 1: $N(k)$ does not have two distinct real roots.

(1.1) $\beta<0$: One of $H_A(k)$ and $H_B(k)$ is negative, and the other one is positive for all k. The BCSS is one of the subsystems for the whole phase plane. So the switched system is unstabilisable.

(1.2) $\beta>0$ and $p_2>0$: Both $H_A(k)$ and $H_B(k)$ are positive for the whole phase plane, then switched system is unstabilisable.

(1.3) $\beta>0$ and $p_2<0$: With reference to (15), we have $p_2=\frac{\omega_b}{\beta}[(\beta\xi+\alpha)-\mu_a]$ and $p_0=\frac{\omega_b}{\beta}[(\beta\xi-\alpha)-\mu_a(\alpha^2+\beta^2)]$. If $p_2<0$, it follows from $\beta>0$, $\mu_a<0$ and $\xi<0$ that $\alpha>0$, which leads to $p_0 > 0$, which contradicts the condition that $N(k)$ does not have two distinct real roots. So this case will not happen.

(1.4) $\beta > 0$ and $p_2 = 0$: The case $p_2 = 0$ has been excluded by Assumption 3.4.

Case 2: $N(k)$ has two distinct real roots and $\det(P_3) < 0$.

Note that the sign of $N(k)$ is positive when $k \in (k_2, k_1)$ because p_2, the leading coefficient of $N(k)$, was assumed to be negative by Assumption 3.5. It follows from $\det(P_3) = -\beta < 0$ that $\beta > 0$. Both $H_A(k)$ and $H_B(k)$ are negative when $k \in (k_2, k_1)$, thus the switched system is regionally stabilisable as long as the two roots $k_2 < k_1$ exists, which is equivalent to the first inequality of Theorem 1.

Case 3: $N(k)$ has two distinct real roots and $\det(P_3) > 0$.

In this case, $\beta < 0$. With reference to Figure A9, the BCSS can be derived that is the same as (62).