The Control Handbook

Hybrid Dynamical Systems: Stability and Stabilization

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Chapter 1

Hybrid Dynamical Systems: Stability and Stabilization

1.1 Introduction

Hybrid systems are heterogeneous dynamical systems, the behaviors of which are determined by interacting continuous variables and discrete switching logics [1, 2]. By heterogeneity, we mean hybrid systems containing two different kinds of dynamics. One is time-driven continuous variable dynamics, usually described by differential or difference equations; the other is event-driven discrete logic dynamics, whose evolutions depend on "if-then-else" type of rules and may be described by automata or Petri nets. In addition, these two kinds of dynamics interact with each other and generate complex dynamical behaviors, such as switching once certain continuous variable passes through a threshold, or state jumping when certain discrete event occurs. As a simple example, the temperature regulation in an air-conditioned room can be considered as a hybrid system; the room temperature evolution forms the continuous variable dynamics following thermophysical laws, whereas the on-off evolution of the air conditioner can be modeled as a discrete event process.

Hybrid systems have been identified in a wide variety of applications; in the control of mechanical systems, in process control, in automotive industry, power systems, aircraft and traffic control, among many other fields. Specifically, hybrid systems have a central role in embedded control systems, where program codes interact with the physical world. In particular, the logic rules programmed in the embedded devices, which can be modeled as discrete event systems, are affecting and being influenced by the continuous variable physical processes, such as spatial location, temperature, and pressure evolutions. Studies in hybrid systems could provide a unified modeling framework for embedded systems, and systematic methods for performance analysis, verification, and embedded micro-controller design. Therefore, hybrid systems have attracted the attention of researchers not only from control engineering, but also from computer science and mathematics. Topics, such as modeling, verification, stability, controllability, optimal control and supervisory control, have been extensively studied in the hybrid system literature, and the interested readers may refer to [1, 2, 5, 4, 13] and the references therein. In this chapter, we will focus on the stability issues of hybrid systems.

It is known that the stability of hybrid systems includes several interesting phenomena due to the interaction of continuous variable dynamics and discrete switching logics [6, 3, 8]. For example, even when all the continuous variable subsystems are exponentially stable, the hybrid system may have divergent trajectories under certain discrete switching logic. On the other hand, one may carefully switch between unstable continuous variable subsystems to make the overall hybrid system exponentially stable. As these examples suggest, the stability of hybrid systems depends not only on the continuous variable dynamics of each subsystem but also on the properties of discrete switching logics. Therefore, the stability study of hybrid systems can be roughly divided into two kinds of problems. One is the stability analysis of hybrid systems under given discrete switching logics; the other is the synthesis of stabilizing switching logics for a given collection of continuous variable dynamical systems.

We mainly focus on a subclass of hybrid systems that consist of a finite number of continuous-variable subsystems and a discrete logical rule, which orchestrates switching between these subsystems. The systems are usually called switched systems in the literature [6, 8]. In this chapter, we will use the terms hybrid systems and switched systems interchangeably. One convenient way to classify hybrid/switched systems is based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear and so on. In this chapter, we will focus our attention to hybrid/switched systems where all subsystems are linear time-invariant systems. The generalization of these results to nonlinear switched systems or more general cases are well-documented in the literature, see e.g., survey papers [6, 3, 8] for further references.

The rest of this chapter is organized as follows. First, we focus on the stability analysis of hybrid systems under given discrete switching logics. In particular, some results on the stability analysis for hybrid systems under arbitrary switching are introduced in Section 1.2, while the stability under slow switching (like dwell time and average dwell time) is studied in Section 1.3. The general case of hybrid system stability under restricted switching is investigated in Section 1.4 through multiple Lyapunov functions. Then, we turn to the synthesis of stabilizing switching logic for a given collection of continuous variable dynamical systems in Section 1.5, where several stabilization conditions and design methods are described. Finally, the chapter concludes with remarks and a list of references.

1.2 Arbitrary Switching

In this section, we first consider the stability analysis problem where there are no restrictions on the discrete event dynamics of the hybrid system. This may be due to our lack of knowledge of the discrete event logic, or of the partitions of the state space, or of the constraints in the hybrid system of concern. Under these circumstances, one usually tends to be more conservative and assumes that all possible discrete switchings are possible; this is called *arbitrary switching* in the literature. On the other hand, when the stability under arbitrary switching is guaranteed, this could provide us with flexibility in the discrete logic design, where one may focus on improving the performances, since the closed-loop system stability is not a problem any longer.

1.2.1 Common Lyapunov Function

We know that a hybrid system may become unstable even when all subsystems are exponentially stable. Therefore, to identify conditions under which a hybrid system is stable under arbitrary switchings is nontrivial and interesting. For this, it is necessary to require that all the subsystems are asymptotically stable, since if one subsystem were unstable, one switching strategy would have been to always select that subsystem all time, which is a valid 'switching logic' as well, and that would make the system unstable. In general, the above subsystems' stability assumption is not sufficient to ensure stability for the hybrid systems under arbitrary switchings. However, if there exists a *common Lyapunov function* for all the subsystems, i.e., a continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, for which the derivative $\dot{V}(x,t)$ is negative definite along all subsystems' trajectories, then the stability of the hybrid systems is guaranteed under arbitrary switchings. This provides us with a possible way to solve this problem, and a lot of research efforts have been focused on finding common quadratic Lyapunov functions.

Common Quadratic Lyapunov Functions

First, we consider a collection of continuous-time linear time-invariant (LTI) systems

$$\dot{x}(t) = A_i x(t), \quad t \in \mathbb{R}^+, \quad i \in \mathfrak{I}, \tag{1.1}$$

where $\mathfrak{I}$ stands for a finite index set. For all $i \in \mathfrak{I}$, the state matrices $A_i \in \mathbb{R}^{n\times n}$. Note that the origin $x_e = 0$ is a common equilibrium for the systems described in (1.1). The hybrid system of interest is built by allowing arbitrary switching among these LTI systems (1.1).

A *Common Quadratic Lyapunov Functions* (CQLF) for (1.1) is a special class of Lyapunov functions of the form

$$V(x) = x^T P x, \tag{1.2}$$

where $P = P^T$ (symmetric) and $P > 0$ (positive definite). In addition, its time derivative along any trajectory of systems (1.1) is negative definite, or alternatively

$$A_i^T P + P A_i = -Q_i, \quad i \in \mathfrak{I}, \tag{1.3}$$

where Q_i are symmetric and positive definite for all $i \in \mathfrak{I}$. The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures the quadratic stability of the hybrid system. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice.

A CQLF for (1.1) can be obtained by solving a set of linear matrix inequalities (LMIs). Namely, there exists a positive definite symmetric matrix P, $P \in \mathbb{R}^{n\times n}$, such that

$$PA_i + A_i^T P < 0, \quad \forall i \in \mathcal{I}, \tag{1.4}$$

hold simultaneously. However, the standard interior point methods for LMIs may become ineffective as the number of subsystems increases. This motivates researchers to identify easily verifiable conditions that guarantee the existence of a CQLF for (1.1). Here, we take a look at a well-studied special case, interested readers may refer to [6, 8] for further references.

Commutative Systems

Let us first look at a special case, where the subsystems' state matrices are pairwise commutative, i.e., $A_iA_j = A_jA_i$ for all $i, j \in \mathfrak{I}$. Because of the commutativity, it is easy to show that

$$A_i^{k_1} A_j^{k_2} = A_j^{k_2} A_i^{k_1},$$

for any nonnegative integer k_1 and k_2, and

$$e^{A_i t_1} e^{A_j t_2} = e^{A_j t_2} e^{A_i t_1},$$

for any nonnegative real number t_1 and t_2. By direct computation, it is straightforward to verify that in this case the arbitrary switching system is stable if and only if all its subsystems are stable.

Theorem 1 For a collection of LTI systems (1.1) with the index set $\mathfrak{I} = \{1, \cdots, N\}$, if all subsystem matrices are stable (i.e., all eigenvalues of A_i have negative real part) and commute pairwise ($A_i A_j = A_j A_i, \forall i, j \in \mathfrak{I}$), then the hybrid system with subsystems given by (1.1) is asymptotically stable under arbitrary switching.

A common quadratic Lyapunov function exists in this case, and can be determined by solving a collection of chained Lyapunov equations as shown in the following:

Theorem 2 Assume that the index set $\mathfrak{I} = \{1, \cdots, N\}$. Let $P_1, \cdots, P_N$ be the unique symmetric positive definite matrices that satisfy the Lyapunov equations

$$A_1^T P_1 + P_1 A_1 = -I, \tag{1.5}$$

$$A_i^T P_i + P_i A_i = -P_{i-1}, \quad i = 2, \cdots, N \tag{1.6}$$

then the function $V(x) = x^T P_N x$ is a common Quadratic Lyapunov function for systems $\dot{x}(t) = A_i x(t)$, $i = 1, \cdots, N$.

In addition, the matrix P_N can be expressed in integral form as

$$P_N = \int_0^\infty e^{A_N^T t_N} \cdots \left(\int_0^\infty e^{A_1^T t_1} e^{A_1 t_1} dt_1\right) \cdots e^{A_N t_N} dt_N.$$

It is not difficult to extend this result to the discrete-time case.

Theorem 3 Let $P_1, \cdots, P_N$ be the unique symmetric positive definite matrices that satisfy the Lapunov equations

$$A_1^T P_1 A_1 + P_1 = -I, \tag{1.7}$$

$$A_i^T P_i A_i + P_i = -P_{i-1}, \quad i = 2, \cdots, N \tag{1.8}$$

then the function $V(x) = x^T P_N x$ is a common Lyapunov function for the systems $x[k+1] =$

$A_i x[k]$, $i = 1, \cdots, N$.

In the literature, there exist several interesting necessary and also sufficient algebraic conditions for the existence of a CQLF for more general cases but usually for low dimensional systems, and interested readers may consult [6, 8] for further references. Note that Lie algebraic conditions were proposed in the literature, see e.g. [6], for arbitrary switching systems, which are based on the solvability of the Lie algebra generated by the subsystems' state matrices. It was shown that if the Lie algebra generated by the set of state matrices A_i is solvable, then there exists a CQLF, and the hybrid system is stable under arbitrary switching.

1.2.2 Switched Quadratic Lyapunov Functions

It should be pointed out that the existence of a common quadratic Lyapunov function is only sufficient for the stability of arbitrary switching systems. Therefore, in general, the existence of a common quadratic Lyapunov function is only sufficient for the asymptotic or exponential stability of hybrid systems under arbitrary switchings, and could be rather conservative. Hence, some attention has been paid to a less conservative class of Lyapunov functions, called *switched quadratic Lyapunov functions.*

In this subsection, we investigate the stability of the following discrete-time arbitrary switching LTI systems

$$x[k+1] = A_i x[k], \quad t \in \mathbb{Z}^+, \tag{1.9}$$

where $x \in \mathbb{R}^n$, and $i \in \mathfrak{I}$. Basically, since every subsystem is stable, there exists a positive definite symmetric matrix P_i that solves the Lyapunov equation for each i-th subsystem

$$A_i^T P_i A_i - P_i < 0,$$

for all $i \in \mathcal{I}$. Next, these matrices P_i are patched together based on the switching signals to construct a global Lyapunov function as

$$V(k, x[k]) = x^T[k]P_{\sigma(k)}x[k], \tag{1.10}$$

where $\sigma(k) : k \rightarrow \mathcal{I}$ stands for the switching signal at step k. Since all P_i are positive definite, it is clear that the function $V(k, x[k]) = x^T[k]P_{\sigma(k)}x[k]$ is also positive definite. If it further holds that $\Delta V(k, x[k]) = V(k+1, x[k+1]) - V(k, x[k])$ is negative definite along the solution of (1.9), then the origin of the system (1.9) is globally asymptotically stable. In particular, a sufficient condition for the stability of the arbitrary switching system (1.9) is given as follows.

Theorem 4 If there exist positive definite symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(P_i = P_i^T)$ for $i \in \mathcal{I}$, satisfying

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0 \tag{1.11}$$

for all $i, j \in \mathcal{I}$, then the linear system (1.9) with arbitrary switching is asymptotically stable.
□

The stability checking for arbitrary switching linear systems can be performed by solving linear matrix inequalities (LMIs).

It is clear that when $P_i = P_j$ for all $i, j \in \mathcal{I}$, the switched quadratic Lyapunov function becomes the CQLF. Therefore, the stability criteria based on the switched quadratic Lyapunov function generalizes the CQLF approach and usually gives us less conservative results. However, it is worth pointing out that the switched quadratic Lyapunov function method is still a sufficient only condition.

1.2.3 Necessary and Sufficient Conditions

In the sequel, we will provide some necessary and sufficient conditions for the asymptotic stability of arbitrary switching linear systems. It is shown that the asymptotic stability problem for hybrid linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems, for which several strong stability conditions exist.

Theorem 5 *[8, 10]* The following statements are equivalent:

1. The arbitrary switching system

$$\dot{x}(t) = A_{\sigma(t)}x(t),$$

   where $A_{\sigma(t)} \in \{A_1, A_2, \cdots, A_N\}$, is asymptotically stable;

2. the linear time-variant system

$$\dot{x}(t) = A(t)x(t),$$

   where $A(t) \in \mathcal{A} \hat{=} \mathrm{Conv}\{\ A_1, A_2, \cdots, A_N\}$, where $\mathrm{Conv}\{\cdot\}$ stands for the convex combination, is asymptotically stable;

3. there exist a full column rank matrix $L \in \mathbb{R}^{m \times n}$, $m \geq n$, and a family of matrices $\{\bar{A}_i \in \mathbb{R}^{m \times n} :\ i \in \mathcal{I}\}$ with strictly negative row dominating diagonal, i.e., for each $\bar{A}_i$, $i \in \mathcal{I}$ its elements satisfying

$$\bar{a}_{kk} + \sum_{k \neq l} |\bar{a}_{kl}| < 0, \quad k = 1, \cdots, m,$$

   such that the matrix relations

$$LA_i = \bar{A}_i L$$

   are satisfied. □

It is interesting to notice that the nice property of $\bar{A}_i$ $(i \in \mathcal{I})$ implies the existence of a common quadratic Lyapunov function for the higher dimensional arbitrary switching system. Unfortunately, applying the above Theorem is still difficult because, in general, the numerical search for the matrix L is not simple. However, this equivalence bridges two research fields, namely the fields of hybrid system and robust stability. Therefore, existing results in the robust stability area, which has been extensively studied for over two decades, can be directly introduced to study the arbitrarily switching systems and vice versa. For example, it is known in the robust stability literature that the global attractiveness, (global) asymptotic stability, and (global) exponential stability are all equivalent for the polytopic uncertain linear time-variant systems [10]. Hence, these stability concepts are also equivalent for arbitrary switching systems. Similar results can be developed for the discrete-time case as it is shown below.

Theorem 6 *[8, 10]* The following statements are equivalent:

1. The arbitrary switching system $x[k+1] = A_{\sigma(k)}x[k]$ where $A_{\sigma(k)} \in \{A_1, A_2, \cdots, A_N\}$, is asymptotically stable;
2. the linear time-variant system $x[k+1] = A(k)x[k]$, where $A(k) \in \mathcal{A} \hat{=} \text{Conv}\{\, A_1, A_2, \cdots, A_N\}$, is robustly asymptotically stable;
3. there exists an integer $m \geq n$ and $L \in \mathbb{R}^{n\times m}$, $rank(L) = n$ such that for all A_i, $i \in \mathcal{I}$, there exists $\bar{A}_i \in \mathbb{R}^{m\times m}$ with the following properties:
   (a) $A_i^T L = L\bar{A}_i^T$,
   (b) each column of $\bar{A}_i$ has no more than n nonzero elements and

$$\|\bar{A}_i\|_\infty = \max_{1\leq k\leq m} \sum_{l=1}^{m} |\bar{a}_{kl}| < 1.$$

□

Based on the equivalence between the asymptotic stability of arbitrary switching linear systems and the robust stability of polytopic uncertain linear time-variant systems, some well established converse Lyapunov theorems can be introduced for arbitrary switching linear systems. For example, the following results were taken from [10].

Theorem 7 If the arbitrary switching system is exponentially stable, then it has a strictly convex, homogenous (of second order) common Lyapunov function of a quasi-quadratic form $V(x) = x^T L(x)x$, where $L(x) = L^T(x) = L(\tau x)$ for all nonzero $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. $\square$

Furthermore, we may restrict our search to include only polyhedral Lyapunov functions (also known as piecewise linear Lyapunov functions) as the following result pointed out.

Theorem 8 If an arbitrary switching linear system is asymptotically stable, then there exists a polyhedral Lyapunov function, which is monotonically decreasing along the switched system's trajectories. $\square$

This converse Lyapunov theorem holds for both discrete-time and continuous-time cases, which suggests that the existence of a common Lyapunov function (not necessarily quadratic) is not only sufficient but also necessary for the stability of a hybrid system under arbitrary switching.

Before we move on to another topic, let's take a look at the following example, which is taken from the robust stability literature.

Example 1.1 Consider an arbitrary switching system, $\dot{x} = A_i x$, $i \in \{1, 2\}$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.06 & -1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1.94 & -1 \end{bmatrix}.$$

It is known that no CQLF exits. However, the arbitrary switching system is asymptotically stable, which is assured by the existence of a piecewise quadratic Lyapunov function; a particular piecewise linear Lyapunov function is also suggested in the robust literature.

1.3 Slow Switching

Hybrid systems may fail to preserve stability under arbitrary switching. On the other hand, one may have some knowledge about the occurrence of possible discrete event dynamics in the hybrid systems and this knowledge can be translated into restrictions on the switching signals. For example, there must exists certain bound on the time interval between two successive switchings, which may be due to the fact that the state trajectories have to spend some finite period of time in traveling from the initial set to certain boundary sets before switching, if these two sets are separated. With such kind of *prior* knowledge about the switching signals, we may derive stronger results on the stability for a given hybrid system instead of just using the worst case arguments of the previous section.

By studying the cases where divergent trajectories are generated through switching between two stable systems, one may notice that the unboundedness is caused by the failure to absorb the energy increase caused by frequent switchings. In addition, when there is an unstable subsystem present (e.g., controller failure or sensor fault), if one either stays too long on it or switches too frequently to it, this may cause instability . Therefore, a natural question is what if we restrict the switching signals to some constrained subclasses. Intuitively, if one stays at stable subsystems long enough and switches less frequently, i.e., slow switching, one may trade off the energy increase caused by switching or unstable modes, and it should perhaps become possible to attain stability. These ideas are proved to be reasonable and are captured by concepts like *dwell time* and *average dwell time* [4] between switchings that are introduced below.

The simplest way to characterize the concept of slow switching is perhaps to request a lower bound on two consecutive switching times.

Definition 1 A positive scalar τ_d is called the dwell time if the time interval between any two consecutive switchings is no smaller than τ_d.

Assume that all subsystems of the hybrid system are exponentially stable. Then, it can be shown that there exists a scalar $\tau_d > 0$ such that the hybrid system remains exponentially stable if the dwell time is larger than τ_d. In addition, one may give an estimate on the bound of the dwell time and decay rate.

It fact, it really does not matter if one occasionally have a smaller dwell time between switching, provided this does not occur too frequently. This concept is captured by the concept of "average dwell-time."

Definition 2 A positive constant τ_a is called the average dwell time if $N_\sigma(t) \leq N_0 + \frac{t}{\tau_a}$ holds for all $t > 0$ and some scalar $N_0 \geq 0$, where $N_\sigma(t)$ denotes the number of discontinuities of a given switching signal σ over $[0, t)$.

Here the constant τ_a is called the *average dwell time* and N_0 the *chatter bound*. The reason for calling a class of switching signals that satisfy

$$N_\sigma(t) \leq N_0 + \frac{t}{\tau_a}$$

having an average dwell no less than τ_a is because

$$N_\sigma(t) \leq N_0 + \frac{t}{\tau_a} \Leftrightarrow \frac{t}{N_\sigma(t) - N_0} \geq \tau_a.$$

This means that on average the 'dwell time' between any two consecutive switchings is no smaller than τ_a. The idea is that there may exist consecutive switching separated by less than τ_a, but the average time interval between consecutive switchings is not less than τ_a.

Theorem 9 Assume that all subsystems, $\dot{x} = A_i x$ for $i \in \mathfrak{I}$, in the hybrid system are exponentially stable. Then, there exists a scalar $\tau_a > 0$ such that the hybrid system is exponentially stable if the average dwell time is larger than τ_a.

Moreover, we can also obtain a bound on the decay rate.

Theorem 10 Given a positive scalar λ_0 such that $A_i + \lambda_0 I$ is stable for all $i \in \mathfrak{I}$. Then, for any given $\lambda \in (0, \lambda_0)$, there exists a finite constant τ_a such that the hybrid system is exponentially stable with decay rate λ provided that the average dwell time is no less than τ_a.

The stability results for slow switching can be extended to discrete-time case, where the dwell time τ_d or average dwell time τ_a are counted as the number of sampling periods. In particular,

Definition 3 A positive constant τ_a is called the average dwell time if $N_\sigma(k) \leq N_0 + \frac{k}{\tau_a}$ holds for all $k > 0$ and some scalar $N_0 \geq 0$, where $N_\sigma(k)$ denotes the number of switchings of a given switching signal σ over $[0, k)$.

Theorem 11 Given a positive scalar λ_0 such that A_i/λ_0 is stable for all $i \in \mathfrak{I}$. Then, for any given $\lambda \in (\lambda_0, 1)$, there exists a finite constant τ_a such that the hybrid system, consisting of $x[k+1] = A_i x[k]$ as its subsystems, is exponentially stable with decay rate λ provided the average dwell time is no less than τ_a.

Interested readers may refer to the survey papers [6, 4, 8] for further references on the stability of hybrid systems under slow switchings.

1.4 Multiple Lyapunov Functions

We will continue our study of the stability of hybrid systems under restricted switchings in this section. It should be pointed out that not all restrictions on switching signals can be captured by the dwell time or average dwell time. For example, it is difficult to transform the invariant set constraints, guard set constraints and so on, which determine the switching signals, into only dwell-time or average dwell time restrictions on switching signals. The main difficulty comes from the fact that most constraints in hybrid systems are state dependent and in the form of partitions of the state space, and so it is hard to transform them into pure time dependent constraints like dwell time etc. This calls for a more general tool to study hybrid system stability, and we will introduce a powerful tool,*multiple Lyapunov functions*.

1.4.1 Multiple Lyapunov Function Theorem

The stability analysis under constrained switching has been usually pursued in the framework of multiple Lyapunov functions (MLF). The basic idea is to use multiple Lyapunov or Lyapunov-like functions, each of which may correspond to a single subsystem or certain region in the state space, concatenated together to produce a non-traditional Lyapunov function. The non-traditionality is in the sense that the MLF may not be monotonically decreasing along the state trajectories, may have discontinuities and be piecewise differentiable. The reason for considering non-traditional Lyapunov functions is that traditional Lyapunov function may not exist for hybrid systems with restricted switching signals. For such cases, one still may construct a collection of Lyapunov-like functions, which only requires non-positive Lie-derivative for certain subsystem in a certain region of the state space instead of globally negativity conditions.

Lyapunov-like functions are defined as a family of real-valued functions $\{V_i, i = 1, \cdots, N\}$ with certain properties, each associated with the vector field $\dot{x} = f_i(x)$ that represents the continuous dynamics for the hybrid system under the i-th discrete mode.

Definition 4 (Lyapunov-like function) By saying that a subsystem has an associated Lyapunov-like function V_i in region $\Omega_i \subseteq \mathbb{R}^n$, we mean that

1. There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that

$$\alpha_i \|x\|^2 \leq V_i(x) \leq \beta_i \|x\|^2$$

   holds for any $x \in \Omega_i$;

2. For all $x \in \Omega_i$ and $x \neq 0$, $\dot{V}_i(x) < 0$.

Here $\dot{V}_i(x) = \frac{\partial V_i(x)}{\partial x} f_i(x)$. The first condition implies positiveness and radius unboundedness for $V_i(x)$ when $x \in \Omega_i$, while the second condition guarantees the decreasing of the abstracted energy, value of function $V_i(x)$, along trajectories of subsystem i inside Ω_i.

Suppose that these regions Ω_i cover the whole state space, and so a cluster of Lyapunov-like functions is obtained. By concatenating these Lyapunov-like functions together, we obtain a non-traditional Lyapunov function, called multiple Lyapunov function (MLF), which can be used to study the global stability of hybrid systems. MLF are proved to be a powerful tool for studying the stability of switched systems and hybrid systems; see for example [9, 3, 6, 8]. There are several versions of MLF results in the literature. A very intuitive MLF result [3] is illustrate in Figure 1.1, where the Lyapunov-like function is decreasing when the corresponding mode is active and does not increase its value at each switching instant. Formally, this result can be stated by the following theorem [3].

Theorem 12 Suppose that each subsystem has an associated Lyapunov-like function V_i in its active region Ω_i, each with equilibrium point $x = 0$. Also, suppose that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\sigma(t)$ be a class of piecewise-constant switching sequences such that $\sigma(t)$ can take value i only if $x(t) \in \Omega_i$, and in addition

$$V_j(x(t_{i,j})) \leq V_i(x(t_{i,j}))$$

where $t_{i,j}$ denotes the time that the switched system switches from subsystem i to subsystems j, i.e., $x(t_{i,j}^-) \in \Omega_i$ while $x(t_{i,j}) \in \Omega_j$. Then, the switched linear system (1.1) is exponentially stable under the switching signals $\sigma(t)$. $\square$

The above MLF theorem requires that at each switching instant the Lyapunov-like function does not increase its value, which is quite conservative. Actually, one may obtain less conservative results. For example, the switching signals may be restricted in such a way that, at every time when we exit (switch from) a certain subsystem, its corresponding Lyapunov-like function value is smaller than its value at the previous exiting time. Then the switched system is asymptotically stable. In other words, for each subsystem the corresponding Lyapunov-like function values at every exiting instant form a monotonically decreasing sequence. Alternatively, the decreasing tendency is captured by the Lyapunov-like function's value at the entering instant instead. This case is illustrate in Figure 1.2. This result can be presented as follows.

Theorem 13 [3] Assume that there exists a family of Lyapunov functions $\{V_i : i \in \mathfrak{I}\}$

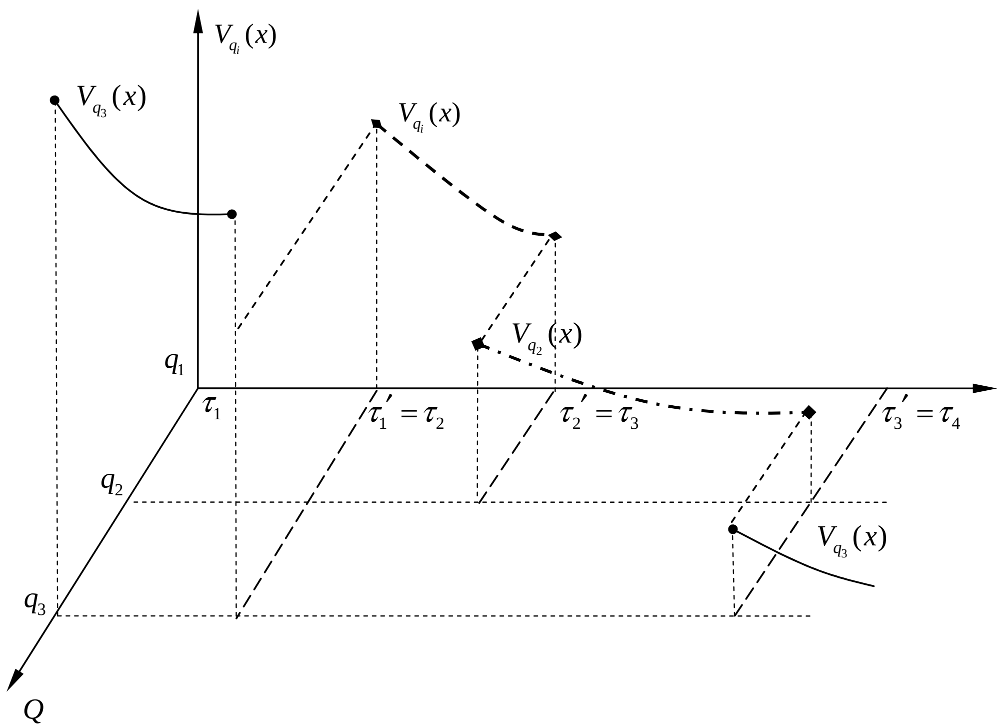

Figure 1.1: The hybrid system is asymptotically stable if the Lyapunov-like functions' values at the switching instants form a decreasing sequence.

for each stable subsystem. If for any two switching instants t_i and t_j such that $i < j$ and $\sigma(t_i) = \sigma(t_j)$ we have

$$V_{\sigma(t_j)}(x(t_{j+1})) - V_{\sigma(t_i)}(x(t_{i+1})) \leq -\rho \|x(t_{i+1})\|^2,$$

for some constant $\rho > 0$, then the switched system is asymptotically stable.

Furthermore, as shown in [9], the Lyapunov-like function may increase its value during a time interval, only if the increment is bounded by certain kind of continuous functions as illustrated in Figure 1.3. Interested readers may refer to the survey papers [3, 6, 9] and their references. Note that all the arguments for continuous-time hybrid/switched systems can be extended to the discrete-time case without essential differences.

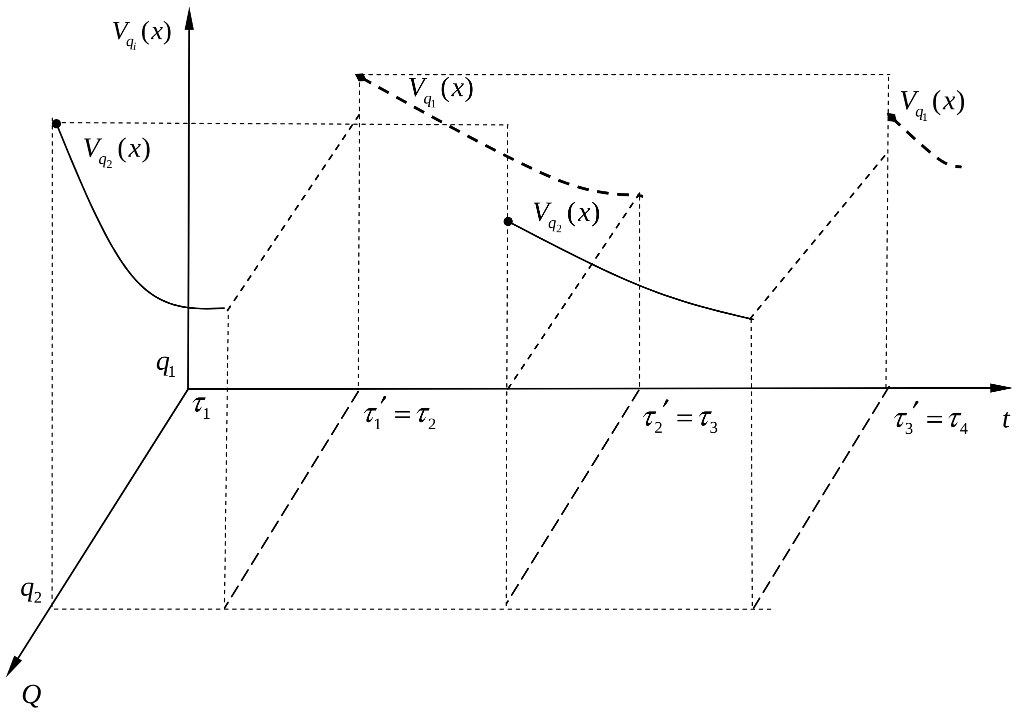

Figure 1.2: For every subsystem, its Lyapunov-like function's value V_i at the start point of each interval exceeds the value at the start point of the next interval on which the i-th subsystem is activated, then the hybrid system is asymptotically stable.

1.4.2 Piecewise Quadratic Lyapunov Functions

The critical challenge of applying the MLF theorems to practical switched/hybrid systems is how to construct a proper family of Lyapunov-like functions. Usually this is a hard problem. However, if one focuses on the linear case, piecewise quadratic Lyapunov-like functions could be attractive candidates, since the stability conditions in the MLF theorems can be formulated as LMIs [3, 5], for which efficient software solution packages exist.

Considering the hybrid system with LTI subsystem, $\dot{x}(t) = A_i x(t)$, since we do not assume that the subsystem is stable, there may not exist a quadratic Lyapunov function in a classical sense. However, it is still possible to restrict our search to certain regions of the state space, say $\Omega_i \subset \mathbb{R}^n$, and the energy of the i-th subsystem could be decreasing along the trajectories inside this region (there is no decreasing requirements outside Ω_i). Suppose that the union of all these regions Ω_i covers the whole state space; then we obtain a cluster of Lyapunov-

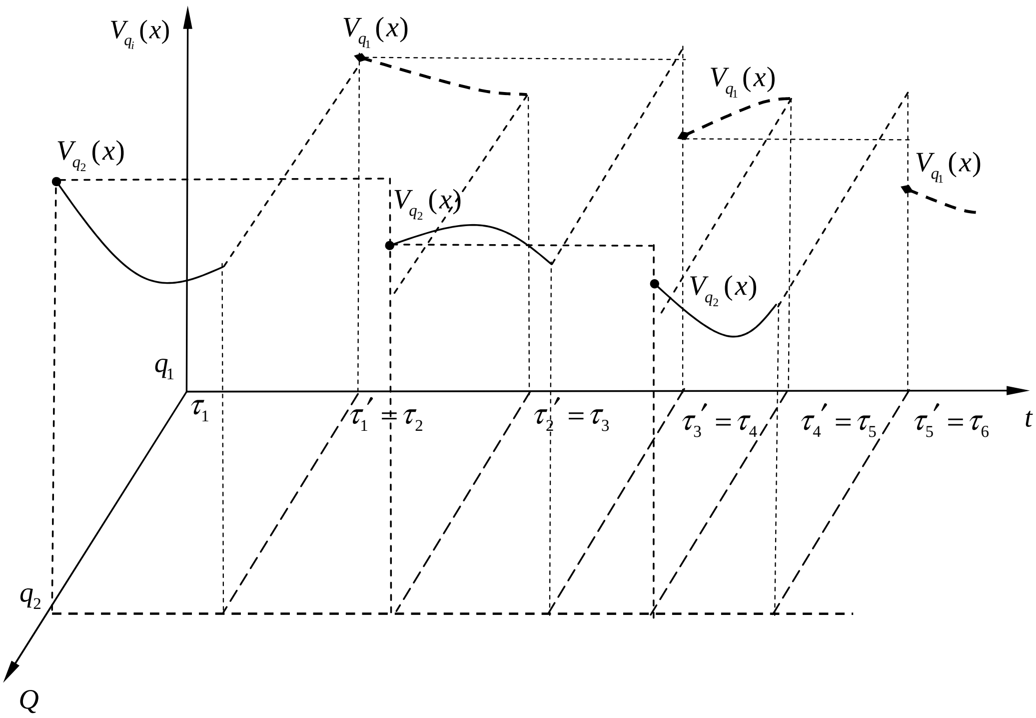

Figure 1.3: The hybrid system can remain stable even when the Lyapunov-like function increases its value during certain period.

like functions. Broadly speaking, the problem entails searching for Lyapunov-like functions whose associated Ω-region cover the state space.

Assume that the state space $\mathbb{R}^n$ has a partition given by $\{\Omega_1, \cdots, \Omega_N\}$, and these regions Ω_i are defined *a priori* as a restriction of the possible switching signals (state-dependent). In this subsection, we present LMI conditions for the existence of quadratic Lyapunov-like functions of the form of $V_i(x) = x^T P_i x$, assigned to each region Ω_i. A Lyapunov-like function $V_i(x) = x^T P_i x$ needs to satisfy the following two conditions:

Condition 1: There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that

$$\alpha_i \|x\|^2 \leq V_i(x) \leq \beta_i \|x\|^2$$

hold for all $x \in \Omega_i$.

Consider a quadratic Lyapunov-like function candidate, $V_i(x) = x^T P_i x$, and require that

$$\alpha_i x^T I x \leq x^T P_i x \leq \beta_i x^T I x,$$

holds for any $x \in \Omega_i$. That is

$$\begin{cases} x^T(\alpha_i I - P_i)x \leq 0 \\ x^T(P_i - \beta_i I)x \leq 0 \end{cases}$$

holds for all $x \in \Omega_i$.

Condition 2: For all $x \in \Omega_i$ and $x \neq 0$, $\dot{V}_i(x) < 0$.

This negativeness of the Lyapunov-like function's derivative along the trajectories of a subsystem can be represented as: $\exists P_i$, $(P_i = P_i^T)$ such that

$$x^T[A_i^T P_i + P_i A_i]x < 0 \tag{1.12}$$

for $x \in \Omega_i$.

Switching Condition: In addition, based on the MLF theorem of [3], for stability it is also required that the Lyapunov-like functions' values at switching instant are non-increasing, which can be expressed by

$$x^T P_j x \leq x^T P_i x$$

for $x \in \Omega_{i,j} \subseteq \Omega_i \bigcap \Omega_j$. The region $\Omega_{i,j}$ stands for the states where the trajectory passes from region Ω_i to Ω_j.

Note that all the above matrix inequalities are constrained in a local region, such as $x \in \Omega_i$ or $\Omega_{i,j}$. A technique called $\mathcal{S}$-procedure can be applied to replace a constrained matrix inequality condition by a condition without constraints. To employ the $\mathcal{S}$-procedure, the regions Ω_i and $\Omega_{i,j}$ need to be expressed by or be contained in regions characterized by quadratic forms. For simplicity, we assume here that each region Ω_i has a quadratic representation or approximation, that is

$$\Omega_i = \{x |\ x^T Q_i x \geq 0\},$$

and regions $\Omega_{i,j}$ can be expressed or approximated by

$$\Omega_{i,j} = \{x|\ x^T Q_{i,j} x \geq 0\}.$$

Then the above matrix inequalities can be transformed into unconstrained ones based also on the $\mathcal{S}$-procedure, namely

Theorem 14 The system (1.1) is (exponentially) stable if there exist matrices P_i ($P_i = P_i^T$) and scalars $\alpha > 0$, $\beta > 0$, $\mu_i \geq 0$, $\nu_i \geq 0$, $\vartheta_i \geq 0$ and $\eta_{i,j} \geq 0$, such that

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A_i^T P_i + P_i A_i + \vartheta_i Q_i \leq -I \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i \end{cases} \tag{1.13}$$

are satisfied. □

The above theorem is an adaptation of a result in [3]. If there is a solution to the above LMI problem, the exponential stability is verified. In addition, a bound on the convergence rate can be estimated:

$$\|x(t)\| \leq \sqrt{\frac{\beta}{\alpha}} e^{-\frac{1}{2\beta}t} \|x_0\|$$

where $x(t)$ is the continuous trajectory with initial state x_0, and the constants α, β are solutions of the LMI (1.13). Based on similar arguments, LMI based sufficient conditions for the discrete-time case can be derived, see e.g., [8].

An example is now presented to illustrate Theorem 14.

Example 1.2 Consider a hybrid system,

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t), & if\ x(t) \in \Omega_1 = \{x|x^T Q_1 x \geq 0\}, \\ \dot{x}(t) = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix} x(t), & if\ x(t) \in \Omega_2 = \{x|x^T Q_2 x \geq 0\}, \end{cases} \tag{1.14}$$

where $Q_1 = \begin{bmatrix} -0.25 & -0.25 \\ -0.25 & 2 \end{bmatrix}$ and $Q_2 = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & -2 \end{bmatrix}$. Since $Q_1 = -Q_2$, it is straightforward to verify that $\Omega_1 \bigcup \Omega_1 = \mathbb{R}^2$.

Solving the LMI problem in Theorem 14 results in a solution

$$P_1 = \begin{bmatrix} 0.1000 & -0.4500 \\ -0.4500 & 41.1167 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.3792 & 3.8292 \\ 3.8292 & 6.8833 \end{bmatrix}$$

with a value of $\beta = 41.12$. Hence the hybrid system is exponentially stable. Interested readers may refer to [3] for details and illustration of trajectories and Lyapunov level curves.

Notice that the above conditions are all based on MLF theorems, so the results developed in this subsection are sufficient only. To reduce the possible conservativeness, a new kind of polynomial Lyapunov functions has been introduced and investigated for the stability analysis of hybrid systems. The computation of such polynomial Lyapunov functions can be efficiently performed using convex optimization, based on the sum of squares (SOS) decomposition of multivariate polynomials. It is also possible to use SOS techniques together with the $\mathcal{S}$-procedure to construct piecewise polynomial Lyapunov functions, with each polynomial as a SOS while incorporating the state constraints, so to generalize piecewise quadratic Lyapunov functions. Interested readers may refer to the survey paper [8] for further references.

1.5 Switching Stabilization

Implicitly, the above MLF results provide methodologies for the design of switching logics between vector fields so to achieve a stable trajectory, since MLF results characterize the conditions on switching signals, under which the hybrid system is stable. In this section, we will explicitly consider the design of stabilizing switching logics for hybrid systems. The formulation of the problem can be stated as follows.

- Given a collection of LTI systems $\dot{x} = A_i x$, design switching logics so that the induced hybrid system is stable.

This is usually called *switching stabilization problem* in the literature. It is known that even when all subsystems are unstable, there still may exist stabilizing switching signals.

1.5.1 Quadratic Switching Stabilization

In the switching stabilization literature, most of the work has focused on quadratic stabilization for certain classes of systems. A hybrid system is called quadratically stabilizable when there exist switching signals which stabilize the system along a quadratic Lyapunov function, $V(x) = x^T P x$.

It is known that a necessary and sufficient condition for a pair of LTI systems to be quadratically stabilizable is the existence of a stable convex combination of the two subsystems' matrices. Specifically,

Theorem 15 A hybrid system that contains two LTI subsystems, $\dot{x}(t) = A_i x(t)$, $i = 1, 2$, is quadratically stabilizable *if and only if* the matrix pencil $\gamma_\alpha(A_1, A_2) = \{A_\alpha \mid A_\alpha = \alpha A_1 + (1-\alpha)A_2,\ 0 \le \alpha \le 1\}$ contains a stable matrix. □

A generalization to more than two LTI subsystems was suggested by using a "min-projection strategy", i.e.,

$$\sigma(t) = \arg\min_{i \in \mathcal{I}} x(t)^T P A_i x(t). \tag{1.15}$$

Theorem 16 If there exist constants $\alpha_i \in [0, 1]$, and $\sum_{i \in \mathcal{I}} \alpha_i = 1$ such that

$$A_\alpha = \sum_{i \in \mathcal{I}} \alpha_i A_i,$$

is stable, then the min-projection strategy (1.15) quadratically stabilizes the switched system.

However, the existence of a stable convex combination matrix A_α is only sufficient for switched LTI systems with more than two subsystems. There are example systems for which no stable convex combination state matrix exists, yet the system is quadratically stabilizable using certain switching signals. A necessary and sufficient condition for the quadratic stabilizability of switched controller systems is as follows.

Theorem 17 [12] The switched system is quadratically stabilizable *if and only if* there exists a positive definite real symmetric matrix $P = P^T > 0$ such that the set of matrices $\{A_iP + PA_i^T\}$ is strictly complete, i.e., for any $x \in \mathbb{R}^n/\{0\}$, there exists $i \in \mathcal{I}$ such that $x^T(A_iP + PA_i^T)x < 0$. In addition, a stabilizing switching signal can be selected as $\sigma(t) = \min_i\{x^T(t)(A_iP + PA_i^T)x(t)\}$. □

Analogously, for the discrete-time case, it is necessary and sufficient for quadratic stabilizability to check whether there exists a positive symmetric matrix P such that the set of matrices $\{A_i^TPA_i - P\}$ is strictly complete. Obviously, the existence of a convex combination of state matrices A_α automatically satisfies the above strict completeness conditions due to convexity, while the inverse statement is not true in general. Unfortunately, checking the strict completeness of a set of matrices is NP hard [12]. Interested readers may refer to survey papers [6, 3, 8] for further references.

Quadratic stability means that there exists a positive constant ϵ such that $\dot{V}(x) \leq -\epsilon x^T x$. All of these methods guarantee stability by using a common quadratic Lyapunov function, which is conservative in the sense that there are switched systems that can be asymptotically (or exponentially) stabilized in case when a common quadratic Lyapunov function does not exist. Therefore, we will turn our attentions to multiple Lyapunov functions, and describe constructive synthesis methods based on piecewise quadratic Lyapunov function in the next section, which are mainly based on [11].

1.5.2 Piecewise Quadratic Switching Stabilization

According to Theorem 14, if there exist real matrices P_i ($P_i = P_i^T$) and scalars $\alpha > 0$, $\beta > 0$, $\mu_i \geq 0$, $\nu_i \geq 0$, $\vartheta_i \geq 0$ and $\eta_{i,j} \geq 0$, satisfying

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A^T P_i + P_i A + \vartheta_i Q_i \leq -I \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i \end{cases}$$

then the switched linear system (1.1) is exponentially stable.

In contrast to the stability analysis problem, here the state space partitions Ω_i are not given *a priori* any more. Actually, the state partitions Ω_i, which induce the state-dependent switching signals, are to be designed. Moreover, the state space cannot be partitioned in an arbitrary way. The partition of the state space should facilitate the search of proper quadratic Lyapunov-like functions, and satisfy the non-increasing conditions when switching occurs. This will be discussed in detail in the following.

State Space Partition

Once again, the purpose of dividing the whole state space $\mathbb{R}^n$ into pieces, denoted as Ω_i, is to facilitate the search for Lyapunov-like functions for one of these subsystems. After successfully obtaining these Lyapunov-like functions associated with each region Ω_i, one may patch them together, following the conditions of the above MLF theorem, so to guarantee global stability.

For this purpose, it is necessary to require that these regions Ω_i cover the whole state space, i.e., the following *covering property* holds.

$$\Omega_1 \bigcup \Omega_2 \bigcup \cdots \bigcup \Omega_N = \mathbb{R}^n.$$

This condition merely says that there are no regions in the state space where none of the subsystems is activated.

Since we will restrict our attention to quadratic Lyapunov-like functions for purpose of computational efficiency, we will consider regions given (or approximated) by quadratic forms

$$\Omega_i = \{x \in \mathbb{R}^n | \; x^T Q_i x \geq 0\},$$

where $Q_i \in \mathbb{R}^{n\times n}$ are symmetric matrices, and $i \in \{1, \cdots, N\}$.

The following lemma gives a sufficient condition for the covering property.

Lemma 1.1 [11] If for every $x \in \mathbb{R}^n$

$$\sum_{i=1}^{N} \theta_i x^T Q_i x \geq 0 \tag{1.16}$$

where $\theta_i \geq 0$, $i \in \mathcal{I}$, then $\bigcup_{i=1}^{N} \Omega_i = \mathbb{R}^n$. □

Switching Condition

In order to guarantee exponential stability we also need to make sure that

1. Subsystem i is active only when $x(t) \in \Omega_i$,

2. When switching occurs, it is required to guarantee that the Lyapunov-like function values are not increasing.

To verify the first requirement, we consider the *largest region function strategy*, i.e.,

$$\sigma(x(t)) = \arg\left(\max_{i\in\mathcal{I}} x(t)^T Q_i x(t)\right). \tag{1.17}$$

This is due to the selection of subsystems (at state $x(t)$) corresponding to the largest value of the region function $x(t)^T Q_i x(t)$.

Suppose that the covering condition (1.16) holds, i.e.,

$$\sum_{i=1}^{N} \theta_i x^T Q_i x \geq 0$$

for some $\theta_i \geq 0$, $i \in \mathcal{I}$. Then, based on the largest region function strategy, the state x with the current active mode i satisfies $x^T Q_i x \geq 0$. This implies that $x \in \Omega_i$. So the first condition holds for the largest region function strategy (1.17).

To satisfy the second energy decreasing condition at switching instants, we need to know in which direction the state trajectory $x(t)$ is passing through the switching surfaces. However, the switching surface is to be designed, and so such information is lacking in general. Then we make a compromise and require that

$$x^T P_i x = x^T P_j x$$

for states at the switching plane, i.e., $x \in \Omega_i \cap \Omega_j$. Assume that the set $\Omega_i \cap \Omega_j$ can be represented by the following quadratic form

$$\Omega_i \cap \Omega_j = \{x | x^T (Q_i - Q_j) x = 0\}.$$

Again, applying the $\mathcal{S}$-procedure, we obtain

$$P_i - P_j + \eta_{i,j}(Q_i - Q_j) = 0,$$

for an unknown scalar $\eta_{i,j}$, as the switching condition.

Synthesis Condition

The above discussion can be summarized by the following sufficient conditions for the collection of continuous-time systems (1.1) to be exponentially stabilized.

Theorem 18 [11] If there exist real matrices P_i ($P_i = P_i^T$) and scalars $\alpha > 0$, $\beta > 0$, $\mu_i \geq 0$,

$\nu_i \geq 0$, $\theta_i \geq 0$, $\vartheta_i \geq 0$ and $\eta_{i,j}$, solving the optimization problem:

$$\min \beta$$
$$s.t. \begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A^T P_i + P_i A + \vartheta_i Q_i \leq -I \\ P_j = P_i + \eta_{i,j}(Q_i - Q_j) \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \geq 0 \end{cases}$$

for all $i, j \in \{1, \cdots, N\}$, then the switched linear system (1.1) can be exponentially stabilized (with decay rate $\frac{1}{2\beta}$) by the largest region function strategy (1.17). $\square$

The extension of the synthesis method for continuous-time switched linear systems to discrete-time counterpart is not obvious. The main difficulty is that, unlike the continuous-time case, discrete-time switched systems do not have the nice property that the switching occurs exactly on the switching surface. Instead, the switching happens in a region around the switching surface. As a result, we can not simply capture the switching instants for discrete-time switched systems as the time instants when the state trajectories cross the switching surfaces. Therefore, in order to guarantee the non-increasing requirement at the switching instants for the discrete-time case, we need to include more constraints involving state transitions for the discrete-time switched systems around the switching surfaces. This makes the switching stabilization problem for discrete-time switched systems more challenging.

Some remarks are in order. First, for both the continuous-time and discrete-time cases, the optimization problem above is a Bilinear Matrix Inequality (BMI) problem, due to the product of unknown scalars and matrices. BMI problems are NP-hard, and not computationally efficient. However, practical algorithms for optimization problems over BMIs exist and typically involve approximations, heuristics, branch-and-bound, or local search. One possible way to solve the BMI problem is to grid up the unknown scalars, and then solve a set of LMIs for fixed values of these parameters. It is argued in [11] that the gridding of the unknown scalars can be made quite sparsely.

Example 1.3 [11] To illustrate the synthesis procedure, consider the case of two unstable subsystems given by

$$A_1 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}.$$

It can be shown that there is no stable convex combination of these two matrices, which means that the system cannot be quadratically stabilized. However, solving the BMI in Theorem 18 through gridding up the unknown parameters results in a solution

$$\beta = 3.7941, \quad \alpha = 0.2101$$

and

$$Q_1 = -Q_2 = \begin{bmatrix} -0.08242 & 0.8648 \\ 0.8648 & 0.8053 \end{bmatrix}, \tag{1.18}$$

$$P_1 = \begin{bmatrix} 1.1896 & 1.1440 \\ 1.1440 & 3.2447 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.3325 & -1.1044 \\ -1.1044 & 1.1509 \end{bmatrix}. \tag{1.19}$$

Hence, the switched linear system can be exponentially stabilized by the largest region function strategy (1.17), and the estimate of the exponential convergence becomes $\|x(t)\| \leq 4.2495e^{-0.1318t}\|x_0\|$.

So far, we have only derived sufficient conditions for the existence of stabilizing switching signals for a given collection of linear systems. A more difficult problem has been the necessity part of the switching stabilizability problem, and a particularly challenging part has been the problem of finding necessary and sufficient conditions for switching stabilizability. In [7], a necessary and sufficient condition was proposed for the existence of a switching control law (in static state feedback form) for asymptotic stabilization of continuous-time switched linear systems.

1.6 Conclusion

In this chapter, we gave, by necessity, a brief introduction to the basic concepts and results of the field of stability and stabilizability of hybrid systems. For further references, we would suggest several survey papers on the stability of hybrid and switched systems, for example, [6, 9, 3, 8].

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