Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 65 (2006) 2248–2259
Nonlinear Analysis
www.elsevier.com/locate/na

Analysis and design of switched normal systems

Guisheng Zhai[a,*], Xuping Xu[b], Hai Lin[c], Anthony N. Michel[c]

[a] *Department of Mechanical Engineering, Osaka Prefecture University, Gakuen-cho 1-1, Sakai, Osaka 599-8531, Japan*
[b] *Department of Electrical and Computer Engineering, Penn State Erie, Benson 182, 5091 Station Road, Erie, PA 16563-1701, USA*
[c] *Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA*

Abstract

In this paper, we study the stability property for a class of switched linear systems whose subsystems are *normal*. The subsystems can be continuous-time or discrete-time ones. We show that when all the continuous-time subsystems are Hurwitz stable and all the discrete-time subsystems are Schur stable, a common quadratic Lyapunov function exists for the subsystems and thus the switched system is exponentially stable under arbitrary switching. We show that when unstable subsystems are involved, for a desired decay rate of the system, if the activation time ratio between stable subsystems and unstable ones is less than a certain value (calculated using the decay rate), then the switched system is exponentially stable with the desired decay rate.

Keywords: Switched normal system; Stability; Common Lyapunov function; Arbitrary switching; Activation time ratio between stable and unstable subsystems

1. Introduction

In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems; see the survey papers [9,2,18], the recent books [10,19] and the references cited therein. The motivation for studying switched systems arises from many aspects. It is known that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior, which may depend on various environmental factors. Since these systems are essentially switched systems, powerful analysis or

* Corresponding author. Tel.:+81 72 254 9218; fax: +81 72 254 9904.
E-mail address: zhai@me.osakafu-u.ac.jp (G. Zhai).

doi:10.1016/j.na.2006.01.034

design results for switched systems are helpful for dealing with real systems. Another important observation is that switching among a set of controllers for a specified system can be regarded as a switched system, and that switching has been used in adaptive control to assure stability in situations where stability can not be proved otherwise [3,13], or to improve the transient response of adaptive control systems [14]. Also, the methods of intelligent control design are based on the idea of switching among different controllers [12,7]. Therefore, study of switched systems contributes greatly in switching controller and intelligent controller design.

When focusing on stability analysis of switched systems, there are three basic problems in stability and design of switched systems: (i) find conditions for stability/stabilizability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching laws; and (iii) construct a stabilizing switching law. There are many existing works on these problems for the case where the switched systems are composed of continuous-time subsystems. For Problem (i), Ref. [15] showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. Ref. [8] extended this result from the commutation condition to a Lie-algebraic condition. Ref. [24] showed that a class of symmetric switched systems are asymptotically stable under arbitrary switching since a common Lyapunov function, in the form of $V(x) = x^{\mathrm{T}}x$, exists for all the subsystems. Refs. [5,1,22,26] considered Problem (ii) using piecewise Lyapunov functions, and Refs. [6,30] considered Problem (ii) for switched systems with pairwise commutation or Lie-algebraic properties. Ref. [17] considered Problem (iii) by dividing the state space associated with appropriate switching depending on the state, and Ref. [23] considered quadratic stabilization, which belongs to Problem (iii), for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. Related to both Problems (ii) and (iii), Ref. [21] presented the convergence rate evaluation for simultaneously triangularizable switched systems, and Ref. [20] investigated the controllability and reachability of switched linear control systems. As regards the robustness stability/stabilization issue, Ref. [28] considered quadratic stabilizability of switched linear systems with polytopic uncertainties, and Ref. [16] dealt with robust quadratic stabilization for switched LTI systems by using piecewise quadratic Lyapunov functions so that the synthesis problem can be formulated as a matrix inequality feasibility problem. Refs. [4,24,25,27] extended the consideration to stability analysis problems for switched systems composed of discrete-time subsystems.

Motivated by the observation that all these papers deal with switched systems composed of only continuous-time subsystems or only discrete-time ones, the authors considered in a recent paper [29] a new type of switched systems which are composed of both continuous-time and discrete-time dynamical subsystems. It was pointed out there that it is very easy to find many applications involving such switched systems. For example, for a continuous-time plant, if we design a set of continuous-time controllers and a set of discrete-time controllers, and we choose an appropriate controller at every time instant, then the entire feedback system is in fact a switched system composed of both continuous-time and discrete-time subsystems. A cascaded system composed of a continuous-time plant, a set of discrete-time controllers and switchings among the controllers is also a good example. Another example of a system of this kind is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one (and a rule of switching between them). Ref. [29] gave some analysis and design results for several kinds of such switched systems, for example, the case where commutation condition holds, and the case of switched symmetric systems.

This paper aims to extend the results for switched symmetric systems in [29] to switched *normal* systems. It can be seen later that normal systems include symmetric systems, skew-symmetric systems, orthogonal systems and some other cases, and thus the extension in this paper is not trivial. For such switched systems, we show that when all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, a common quadratic Lyapunov function exists for the subsystems and thus the switched system is exponentially stable under arbitrary switching. We also discuss the applicability of the result to the switching control problem. We show that when unstable subsystems are involved, if the total activation time ratio between unstable subsystems and stable ones is less than a specified value (which is computed by using a desired decay rate), then the switched system is exponentially stable with the desired decay rate. Three numerical examples are given to show the effectiveness of the results.

2. Preliminaries and switched system description

In this section, we give some definitions and lemmas concerning normal systems, and then describe the switched system we consider in this paper.

Definition 1. A continuous-time system

$$\dot{x}(t) = Ax(t) \tag{1}$$

or a discrete-time system

$$x(k+1) = Ax(k) \tag{2}$$

is said to be *normal* if

$$A^{\mathrm{T}}A = AA^{\mathrm{T}}. \tag{3}$$

Definition 2. A real square matrix Q is said to be *orthogonal* if $Q^{\mathrm{T}}Q = I$.

The following lemma characterizes a normal system matrix by orthogonally equalizing it to a block-diagonal matrix consisting of its eigenvalues (Theorem 4.10.69 in [11]).

Lemma 1. *Suppose that $A \in \Re^{n\times n}$ is normal, its real eigenvalues are $\lambda_1, \ldots, \lambda_r$, and its complex eigenvalues are $a_1 \pm b_1\mathrm{i}, \ldots, a_s \pm b_s\mathrm{i}$, where the a_i's and b_i's are real, $b_i \neq 0$, $r + 2s = n$. Then, there exists an orthogonal matrix Q such that*

$$Q^{\mathrm{T}}AQ = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1, \ldots, \Lambda_s\}, \tag{4}$$

where

$$\Lambda_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}, \quad i = 1, \ldots, s. \tag{5}$$

The following two lemmas play a key role in the next section.

Lemma 2. *If the continuous-time system (1) is normal and Hurwitz stable, then*

$$A^{\mathrm{T}} + A < 0. \tag{6}$$

Proof. We obtain from (4) that

$$Q^{\mathrm{T}}(A^{\mathrm{T}} + A)Q = \mathrm{diag}\{2\lambda_1, \ldots, 2\lambda_r, 2a_1, 2a_1, \ldots, 2a_s, 2a_s\}. \tag{7}$$

Since A is Hurwitz stable, we obtain that $\lambda_i < 0$ $(1 \le i \le r)$ and $a_j < 0$ $(1 \le j \le s)$ and thus $Q^{\mathrm{T}}(A^{\mathrm{T}} + A)Q < 0$. This completes the proof. ∎

Lemma 3. *If the discrete-time system (2) is normal and Schur stable, then*

$$A^{\mathrm{T}}A - I < 0. \tag{8}$$

Proof. We obtain from (4) that

$$\begin{aligned} Q^{\mathrm{T}}(A^{\mathrm{T}}A)Q &= (Q^{\mathrm{T}}A^{\mathrm{T}}Q)(Q^{\mathrm{T}}AQ) \\ &= \mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1^{\mathrm{T}}, \ldots, \Lambda_s^{\mathrm{T}}\}\mathrm{diag}\{\lambda_1, \ldots, \lambda_r, \Lambda_1, \ldots, \Lambda_s\} \\ &= \mathrm{diag}\{\lambda_1^2, \ldots, \lambda_r^2, a_1^2 + b_1^2, a_1^2 + b_1^2, \ldots, a_s^2 + b_s^2, a_s^2 + b_s^2\}. \end{aligned} \tag{9}$$

Since A is Schur stable, we obtain $|\lambda_i| < 1$ $(1 \le i \le r)$ and $\sqrt{a_j^2 + b_j^2} < 1$ $(1 \le j \le s)$ and thus $Q^{\mathrm{T}}(A^{\mathrm{T}}A)Q < I$, which is equivalent to (8). ■

In this paper, we consider the switched system which is composed of a set of continuous-time subsystems

$$\dot{x}(t) = A_{ci}x(t), \quad i = 1, \ldots, N_c \tag{10}$$

and a set of discrete-time subsystems

$$x(k+1) = A_{dj}x(k), \quad j = 1, \ldots, N_d \tag{11}$$

where $x(t), x(k) \in \Re^n$ are the subsystem states, the A_{ci}'s and A_{dj}'s are constant matrices of appropriate dimension, and $N_c(N_d)$ denotes the number of continuous-time (discrete-time) subsystems. To discuss the stability of the overall switched system, we assume for simplicity that the sampling period of all the discrete-time subsystems is τ (the discussion can be easily extended to the case where the discrete-time subsystems have different sampling periods). Since the states of the discrete-time subsystems can be viewed as piecewise constant vectors between sampling points, we can consider the value of the system states in the continuous-time domain. For example, if subsystem A_{c1} is activated on $[t_0, t_1]$ and then subsystem A_{d1} is activated for m steps and subsystem A_{c2} is activated from then to t_2, the time domain is divided into

$$[t_0, t_2] = [t_0, t_1] \cup [t_1, t_1 + m\tau] \cup [t_1 + m\tau, t_2] \tag{12}$$

and the system state takes the form

$$x(t) = \begin{cases} \mathrm{e}^{A_{c1}(t-t_0)}x(t_0), & t \in [t_0, t_1] \\ A_{d1}^{k-1}x(t_1), & t \in [t_1 + (k-1)\tau, t_1 + k\tau)\ (1 \le k \le m) \\ A_{d1}^{m}x(t_1) & t = t_1 + m\tau \\ \mathrm{e}^{A_{c2}(t-t_1-m\tau)}x(t_1 + m\tau), & t \in [t_1 + m\tau, t_2]. \end{cases} \tag{13}$$

Although $x(t)$ is not continuous with respect to time t due to the existence of discrete-time subsystems, the solution $x(t)$ is uniquely defined at all time instants, and thus various stability properties in the time domain can be discussed.

Throughout this paper, we make the following assumption.

Assumption 1. All the subsystems in (10) and (11) are *normal*, i.e.,

$$A_{ci}^{\mathrm{T}}A_{ci} = A_{ci}A_{ci}^{\mathrm{T}}, \qquad A_{dj}^{\mathrm{T}}A_{dj} = A_{dj}A_{dj}^{\mathrm{T}} \tag{14}$$

hold for all $i = 1, \ldots, N_c$, and $j = 1, \ldots, N_d$.

Remark 1. For switched symmetric systems, it is assumed in [25,29] that $A_{ci}^{\mathrm{T}} = A_{ci}$ and/or $A_{dj}^{\mathrm{T}} = A_{dj}$. Obviously, Assumption 1 includes such symmetric systems. Furthermore, it also covers the cases of $A_*^{\mathrm{T}} A_* = I$ (orthogonal), $A_*^{\mathrm{T}} = -A_*$ (skew symmetric) and some other cases.

3. Arbitrary switching

In this section, we discuss the case where arbitrary switching is possible for the switched system composed of (10) and (11). Since arbitrary switching includes the case of dwelling on a certain subsystem for all time, we make the following necessary assumption.

Assumption 2. All A_{ci}'s are Hurwitz stable and all A_{dj}'s are Schur stable.

It is known that Assumption 2 is not enough to guarantee stability under arbitrary switching. That is, a switched system composed of stable subsystems could be unstable if the switching was not done appropriately [9,1]. However, when all subsystems are normal, we will show in the following that the switched system is exponentially stable under arbitrary switching.

Theorem 1. *Under Assumptions 1 and 2, the switched system composed of (10) and (11) is exponentially stable under arbitrary switching.*

Proof. Since all subsystems are normal and stable, according to Lemma 1, we obtain

$$\begin{aligned} &A_{ci}^{\mathrm{T}} + A_{ci} < 0, \quad i = 1, \ldots, N_c; \\ &A_{dj}^{\mathrm{T}} A_{dj} - I < 0, \quad j = 1, \ldots, N_d. \end{aligned} \tag{15}$$

This implies that $P = I$ is a common solution to the Lyapunov matrix inequalities

$$\begin{aligned} &A_{ci}^{\mathrm{T}} P + P A_{ci} < 0, \quad i = 1, \ldots, N_c; \\ &A_{dj}^{\mathrm{T}} P A_{dj} - P < 0, \quad j = 1, \ldots, N_d, \end{aligned} \tag{16}$$

and thus $V(x) = x^{\mathrm{T}} x$ is a common Lyapunov function for all the subsystems.

To show the exponential stability of the system, we first find two positive scalars α_c and $\alpha_d < 1$ such that

$$A_{ci}^{\mathrm{T}} + A_{ci} < -2\alpha_c I, \qquad A_{dj}^{\mathrm{T}} A_{dj} - \alpha_d^2 I < 0 \tag{17}$$

hold for all i and j. Then, in the period where a continuous-time subsystem is activated, we obtain $\dot{V}(x(t)) < -2\alpha_c V(x(t))$, and in the period where a discrete-time subsystem is activated, $V(x(k+1)) < \alpha_d^2 V(x(k))$.

For any time $t > 0$, we can always divide the time interval $[0, t]$ as $t = t_c + m\tau (m \geq 0)$, where t_c is the total duration time on continuous-time subsystems and $m\tau$ is the total duration time on discrete-time subsystems. It is not difficult to obtain that no matter what the activation order is,

$$V(x(t)) \leq \mathrm{e}^{-2\alpha_c t_c} \alpha_d^{2m} V(x(0)) \tag{18}$$

and thus

$$|x(t)| \leq \mathrm{e}^{-\alpha t} |x(0)| \tag{19}$$

where $\alpha = \min\{\alpha_c, \frac{\ln(\alpha_d^{-1})}{\tau}\} > 0$. This completes the proof. ■

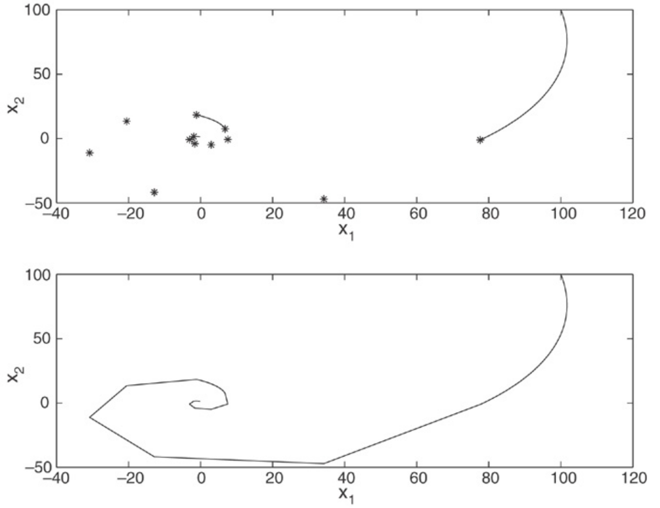

Fig. 1. The system trajectory in Example 1.

Remark 2. It has been shown in the proof of Theorem 1 that when all subsystems are normal and (Hurwitz or Schur) stable, $V(x) = x^{\mathrm{T}}x$ is a common quadratic Lyapunov function for them.

Example 1. Consider the switched system composed of one continuous-time subsystem given by

$$A_{c1} = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix} \tag{20}$$

and one discrete-time subsystem given by

$$A_{d1} = \begin{bmatrix} 0.45 & 0.6 \\ -0.6 & 0.45 \end{bmatrix}. \tag{21}$$

It is easy to confirm that both A_{c1} and A_{d1} are normal, A_{c1} is Hurwitz stable and A_{d1} is Schur stable. Suppose that the sampling period of subsystem A_{d1} is 0.1. Fig. 1 shows the convergence of the system trajectory where A_{c1} and A_{d1} are activated alternately with respectively time periods 1 and 5 steps (i.e., time period 0.5). The initial state is $[100 \quad 100]^{\mathrm{T}}$, and the mark "*" in the upper part of Fig. 1 indicates the state change when the discrete-time subsystem A_{d1} is activated. The lower part of Fig. 1 connects all the sampling points of subsystem A_{d1} into a continuous trajectory. ■

At the end of this section, we note that Theorem 1 is very useful in many switching control problems. Suppose that we have on hand an open-loop feedback system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{22}$$

where $x(t)$ is the state, $u(t)$ is the control input, A, B are constant matrices of appropriate dimension. We also suppose that we can design a set of state feedback controllers $u(t) = K_i x(t)$ $(i = 1, \ldots, N_m)$ such that each $A + BK_i$ is normal and Hurwitz stable. This is possible in many cases. For example, when

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{23}$$

it is easy to see that any $K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$, where $k_1 < -1$, $k_1 + k_4 = 3$ and $k_3 - k_2 = 5$, will make $A + BK$ normal and Hurwitz stable.

If we can (or have to) choose one from the set of controllers at every time instant, the whole system is a switched system that is composed of Hurwitz stable subsystems. Since this is a special case of Theorem 1 (no discrete-time subsystems exist), we see that the system is exponentially stable no matter how we switch among these controllers. This observation is very important in real applications when we want more flexibility to take other control specifications into consideration.

Obviously, the above discussion is also applicable to discrete-time feedback control systems, and to the case of output feedback switching control problems. Furthermore, a more interesting problem may be feedback control systems which are composed of a continuous-time plant and both continuous-time and discrete-time controllers.

Example 2. For the system (22) with (23), we set

$$K_1 = \begin{bmatrix} -1.5 & 1 \\ 6 & 4.5 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} -2 & -1 \\ 4 & 5 \end{bmatrix} \tag{24}$$

to obtain two closed-loop system matrices

$$A_{c1} = \begin{bmatrix} -0.5 & 4 \\ -4 & -0.5 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \tag{25}$$

which are normal and Hurwitz stable.

Now, we set the initial state as $x_0 = [1 \quad 1]^{\mathrm{T}}$ and randomly generate 8 positive time periods among $(0.01, 1)$ as $T_1 = 0.95$, $T_2 = 0.23$, $T_3 = 0.61$, $T_4 = 0.49$, $T_5 = 0.89$, $T_6 = 0.76$, $T_7 = 0.46$, $T_8 = 0.02$. Then, we activate subsystems A_{c1} and A_{c2} alternately with time periods $T_1, T_2, \ldots, T_7, T_8$. Fig. 2 shows the convergence of the system trajectory under such random activation periods. ■

4. Time-controlled switching

In this section, assuming that some subsystems are not stable, we consider Problem (ii) for the switched system. We propose a class of time-controlled switching laws which specify the time ratio between stable subsystems and unstable ones.

For simplicity, we suppose that A_{c1} and A_{d1} are unstable, and all the other subsystems are stable. It will be seen later that other cases can be dealt with using completely the same approach.

According to Lemma 1, there exists an orthogonal matrix Q_{c1} such that

$$Q_{c1}^{\mathrm{T}}(A_{c1}^{\mathrm{T}} + A_{c1})Q_{c1} = \mathrm{diag}\{2\lambda_1^{c1}, \ldots, 2\lambda_{r1}^{c1}, 2\eta_1^{c1}, 2\eta_1^{c1}, \ldots, 2\eta_{s1}^{c1}, 2\eta_{s1}^{c1}\}, \tag{26}$$

where $\lambda_1^{c1}, \ldots, \lambda_{r1}^{c1}$ are A_{c1}'s real eigenvalues, and $\eta_1^{c1}, \ldots, \eta_{s1}^{c1}$ are the real parts of A_{c1}'s complex eigenvalues. Since A_{c1} is not Hurwitz stable, there is at least one nonnegative number among them. For design purposes, we define the scalar

$$\beta_c = \max\{\lambda_1^{c1}, \ldots, \lambda_{r1}^{c1}, \eta_1^{c1}, \ldots, \eta_{s1}^{c1}\}. \tag{27}$$

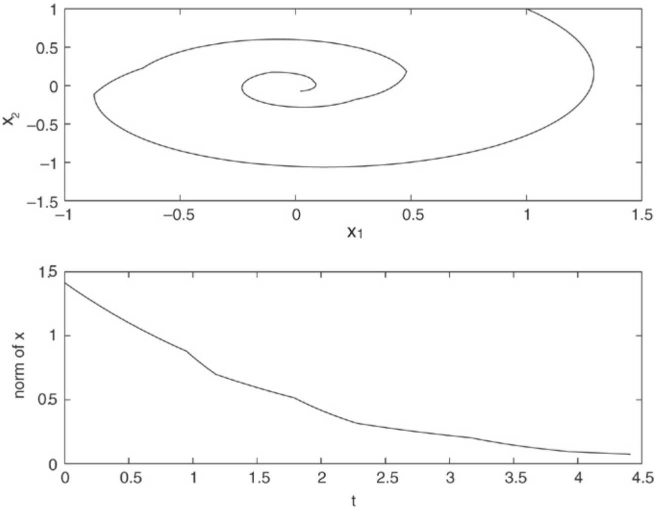

Fig. 2. The system trajectory in Example 2

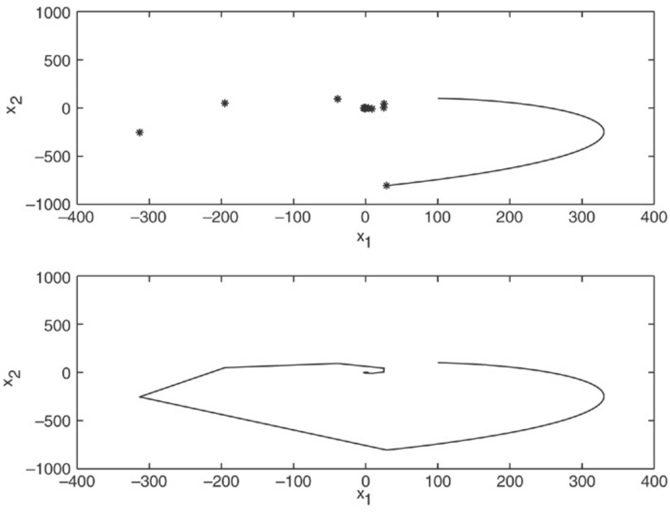

Fig. 3. The system trajectory in Example 3.

It is then easy to see that $\beta_c \geq 0$ and

$$A_{c1}^{\mathrm{T}} + A_{c1} \leq 2\beta_c I. \tag{28}$$

Similarly, for Schur unstable A_{d1}, there exists an orthogonal matrix Q_{d1} such that

$$Q_{d1}^{\mathrm{T}}(A_{d1}^{\mathrm{T}} A_{d1}) Q_{d1} = \mathrm{diag}\{(\lambda_1^{d1})^2, \ldots, (\lambda_{u1}^{d1})^2, (\eta_1^{d1})^2, (\eta_1^{d1})^2, \cdots, (\eta_{v1}^{d1})^2, (\eta_{v1}^{d1})^2\}, \tag{29}$$

where $\lambda_1^{d1}, \ldots, \lambda_{u1}^{d1}$ are A_{d1}'s real eigenvalues, and $\eta_1^{d1}, \ldots, \eta_{v1}^{d1}$ are the absolute values of A_{d1}'s complex eigenvalues. Since A_{d1} is not Schur stable, there is at least one number among them whose absolute value is not less than 1. If we define

$$\beta_d = \max\{|\lambda_1^{d1}|, \ldots, |\lambda_{u1}^{d1}|, \eta_1^{d1}, \ldots, \eta_{v1}^{d1}\}, \tag{30}$$

then $\beta_d \geq 1$ and

$$A_{d1}^{\mathrm{T}} A_{d1} \leq \beta_d^2 I. \tag{31}$$

According to (28), when subsystem A_{c1} is activated, we obtain $\dot{V}(x(t)) \leq 2\beta_c V(x(t))$. According to (31), when subsystem A_{d1} is activated, we obtain $V(x(k+1)) \leq \beta_d^2 V(x(k))$. Since the subsystems other than these two subsystems are assumed to be stable, without loss of generality, we assume that (17) is satisfied for $i \neq 1$ and $j \neq 1$ with the same scalars α_c and α_d.

Now, for any time $t > 0$, we assume that the time interval $[0, t]$ is divided as $t = t_{c1} + t_{cs} + m_{d1}\tau + m_{ds}\tau$, where t_{c1} is the total activation time of A_{c1}, t_{cs} is the total activation time of other continuous-time subsystems, $m_{d1}\tau$ is the total activation time of A_{d1}, and $m_{ds}\tau$ is the total activation time of other discrete-time subsystems. Then, it is easy to obtain

$$V(x(t)) \leq \mathrm{e}^{2\beta_c t_{c1}} \mathrm{e}^{-2\alpha_c t_{cs}} \beta_d^{2m_{d1}} \alpha_d^{2m_{ds}} V(x(0)). \tag{32}$$

Using $\beta = \max\{\beta_c, \frac{\ln(\beta_d)}{\tau}\}$ in the above leads to

$$V(x(t)) \leq \mathrm{e}^{2\beta(t_{c1}+m_{d1}\tau)} \mathrm{e}^{-2\alpha(t_{cs}+m_{ds}\tau)} V(x(0)), \tag{33}$$

where α is the same as that defined in the proof of Theorem 1. Noting that $T_u = t_{c1} + m_{d1}\tau$ is the total activation time of unstable subsystems, and $T_s = t_{cs} + m_{ds}\tau$ is the total activation time of stable subsystems, we consider the following time-controlled switching law:

Time-controlled switching law. Let T_u and T_s be the total activation time of all unstable subsystems and stable ones, respectively, and let $\alpha^* < \alpha$ be the desired decay rate of the overall system. Keep the ratio between T_u and T_s satisfying

$$\frac{T_u}{T_s} \leq \frac{\alpha - \alpha^*}{\beta + \alpha^*}. \tag{34}$$

In fact, we obtain from (34) that

$$(\beta + \alpha^*) T_u \leq (\alpha - \alpha^*) T_s. \tag{35}$$

Then, combining this with (33), we obtain

$$|x(t)| \leq \mathrm{e}^{-\alpha^* t} |x(0)|. \tag{36}$$

We summarize the above discussion in the following theorem.

Theorem 2. *Under Assumption 1 and the time-controlled switching law (34), the switched system is exponentially stable with decay rate α^*.* ■

Remark 3. Compared with the existing result in [26], where switched systems including both stable and unstable subsystems were also dealt with, there is no requirement on *average dwell time* as regards any single subsystem. Thus, we do not have to worry about the switching frequency, which is a desirable property in real applications. It is easy to see that the time-controlled switching law (34) is not involved in the number of switchings, which usually appears in the average dwell time scheme. The reason is that for the switched normal systems under consideration, we have shown that $V(x) = x^{\mathrm{T}} x$ together with the time-controlled switching law is in fact a kind of *common quadratic Lyapunov-like function* in the present case. We used "Lyapunov-like" here since it is not a Lyapunov function in the usual sense, especially for unstable subsystems, yet it is an auxiliary scalar-valued function involved with a direct method and exponential stability of the entire system is guaranteed.

Example 3. Consider the switched system composed of one continuous-time subsystem given by

$$A_{c1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \tag{37}$$

and one discrete-time subsystem given by

$$A_{d1} = \begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 0.3 \end{bmatrix}. \tag{38}$$

It is easy to confirm that both A_{c1} and A_{d1} are normal, A_{c1} is Hurwitz unstable and A_{d1} is Schur stable. Then, it is easy to compute

$$\beta_c = 0.6, \qquad \alpha_d = 0.5. \tag{39}$$

If we choose $\tau = 1$ as the sampling period of A_{d1}, then

$$\alpha = \frac{\ln(\alpha_d^{-1})}{\tau} = 0.69, \qquad \beta = \beta_c = 0.6. \tag{40}$$

According to Theorem 2, if the desired decay rate $\alpha^* = 0.4$, then the time ratio of A_{c1} and A_{d1} should satisfy

$$\frac{T_u}{T_s} \leq \frac{\alpha - \alpha^*}{\beta + \alpha^*} = \frac{0.69 - 0.4}{0.6 + 0.4} = 0.29. \tag{41}$$

In order to meet this requirement, we activate A_{c1} and A_{d1} alternately with respectively time periods 2.9 and 10 steps. Fig. 3 shows that the system trajectory still converges to zero very quickly in this case (with the same initial state $[100 \;\; 100]^{\mathrm{T}}$). ■

Remark 4. It is finally noted that the discussion and result in this section can be applied to controller failure time analysis of feedback control systems when the original system matrix is normal and the feedback is designed so that the closed-loop system is also normal. In this case, the overall system can be viewed as a switched system composed of the original unstable system and the stable closed-loop system.

5. Concluding remarks

In this paper, we have studied the stability property for a class of switched systems which are composed of both continuous-time and discrete-time LTI normal subsystems. We have shown that when all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, a common quadratic Lyapunov function $V(x) = x^{\mathrm{T}}x$ exists for the subsystems and that the switched system is exponentially stable under arbitrary switching. We have shown that when unstable subsystems are involved, if the activation time ratio between unstable subsystems and stable ones is less than a specified value, then the switched system is guaranteed to be exponentially stable with the desired decay rate.

Comparing with the three basic problems mentioned in the introduction, we note that the results in this paper point to Problem (i) and part of Problem (ii). At present, we are trying to take advantage of the structure of switched normal systems in dealing with the stabilization problem (Problem (iii)). Furthermore, although we still have not proved it, we are conjecturing that if the expanded matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is normal, then the stable system with input and output

$$\dot{x} = Ax + Bw, \qquad z = Cx + Dw \tag{42}$$

or its discrete-time counterpart will have a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}x$ in the L_2 sense. If this is true, then we can prove that the switched system composed of normal stable subsystems with the same L_2 gain γ will be stable and keep the L_2 gain γ under arbitrary switching. This problem is part of our future research.

Acknowledgements

The authors would like to thank Prof. Joe Imae and Dr. Tomoaki Kobayashi with Osaka Prefecture University, Prof. Kazunori Yasuda with Wakayama University, for their valuable discussions. This research has been supported in part by the Japan Ministry of Education, Science and Culture under Grants-in-Aid for Scientific Research (B) 15760320 & 17760356.

References

[1] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, IEEE Transactions on Automatic Control 43 (1998) 475–482.
[2] R. DeCarlo, M.S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, Proceedings of the IEEE 88 (2000) 1069–1082.
[3] M. Fu, B.R. Barmish, Adaptive stabilization of linear systems via switching control, IEEE Transactions on Automatic Control 31 (1986) 1097–1103.
[4] S.S. Ge, Z. Sun, T.H. Lee, Reachability and controllability of switched linear discrete-time systems, IEEE Transactions on Automatic Control 46 (2001) 1437–1441.
[5] J.P. Hespanha, A.S. Morse, Stability of switched systems with average dwell-time, in: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, USA, 1999, pp. 2655–2660.
[6] B. Hu, X. Xu, A.N. Michel, P.J. Antsaklis, Stability analysis for a class of nonlinear switched systems, in: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, USA, 1999, pp. 4374–4379.
[7] B. Hu, G. Zhai, A.N. Michel, Hybrid static output feedback stabilization of second-order linear time-invariant systems, Linear Algebra and its Applications 351–352 (2002) 475–485.
[8] D. Liberzon, J.P. Hespanha, A.S. Morse, Stability of switched systems: A Lie-algebraic condition, Systems & Control Letters 37 (1999) 117–122.
[9] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Systems Magazine 19 (1999) 59–70.
[10] D. Liberzon, Switching in Systems and Control, Birkhäuser, Boston, 2003.
[11] A.N. Michel, C.J. Herget, Applied Algebra and Functional Analysis, Dover, New York, 1993.
[12] A.S. Morse, Supervisory control of families of linear set-point controllers—Part 1: Exact matching, IEEE Transactions on Automatic Control 41 (1996) 1413–1431.
[13] A.S. Morse, D.Q. Mayne, G.C. Goodwin, Application of hysteresis switching in adaptive control, IEEE Transactions on Automatic Control 37 (1992) 1343–1354.
[14] K.S. Narendra, J. Balakrishnan, Improving transient response of adaptive control systems using multiple models and switching, IEEE Transactions on Automatic Control 39 (1994) 1861–1866.
[15] K.S. Narendra, J. Balakrishnan, A common Lyapunov function for stable LTI systems with commuting A-matrices, IEEE Transactions on Automatic Control 39 (1994) 2469–2471.
[16] S. Pettersson, Synthesis of switched linear systems, in: Proceedings of the 42nd IEEE Conference on Decision and Control, Hawaii, USA, 2003, pp. 5283–5288.

[17] S. Pettersson, B. Lennartson, LMI for stability and robustness of hybrid systems, in: Proceedings of the American Control Conference, Albuquerque, USA, 1997, pp. 1714–1718.
[18] Z. Sun, S.S. Ge, Analysis and synthesis of switched linear control systems, Automatica 41 (2005) 181–195.
[19] Z. Sun, S.S. Ge, Switched Linear Systems: Control and Design, Springer, London, 2005.
[20] Z. Sun, S.S. Ge, T.H. Lee, Controllability and reachability criteria for switched linear systems, Automatica 38 (2002) 775–786.
[21] Z. Sun, R. Shorten, On convergence rates of simultaneously triangularizable switched linear systems, IEEE Transactions on Automatic Control 50 (2005) 1224–1228.
[22] M.A. Wicks, P. Peleties, R.A. DeCarlo, Construction of piecewise Lyapunov functions for stabilizing switched systems, in: Proceedings of the 33rd IEEE Conference on Decision and Control, Orlando, USA, 1994, pp. 3492–3497.
[23] M.A. Wicks, P. Peleties, R.A. DeCarlo, Switched controller design for the quadratic stabilization of a pair of unstable linear systems, European Journal of Control 4 (1998) 140–147.
[24] G. Zhai, Stability and L_2 gain analysis of switched symmetric systems, in: D. Liu, P.J. Antsaklis (Eds.), Stability and Control of Dynamical Systems with Applications, Birkhäuser, Boston, 2003.
[25] G. Zhai, X. Chen, M. Ikeda, K. Yasuda, Stability and L_2 gain analysis for a class of switched symmetric systems, in: Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, USA, 2002, pp. 4395–4400.
[26] G. Zhai, B. Hu, K. Yasuda, A.N. Michel, Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach, International Journal of Systems Science 32 (2001) 1055–1061.
[27] G. Zhai, B. Hu, K. Yasuda, A.N. Michel, Stability and L_2 gain analysis of discrete-time switched systems, Transactions of the Institute of Systems, Control and Information Engineers 15 (2002) 117–125.
[28] G. Zhai, H. Lin, P.J. Antsaklis, Quadratic stabilizability of switched linear systems with polytopic uncertainties, International Journal of Control 76 (2003) 747–753.
[29] G. Zhai, H. Lin, A.N. Michel, K. Yasuda, Stability analysis for switched systems with continuous-time and discrete-time subsystems, in: Proceedings of the American Control Conference, Boston, USA, 2004, pp. 4555–4560.
[30] G. Zhai, K. Yasuda, Stability analysis for a class of switched systems, Transactions of the Society of Instrument and Control Engineers 36 (2000) 409–415.