Hybrid Output Feedback Stabilization for LTI Systems with Single Output

Hai Lin

***Abstract*— This note presents a hybrid control scheme for a class of continuous-time LTI systems that cannot be stabilized by a single static output feedback (SOF) controller. The main idea here is to design multiple SOF gains and proper logic rules that orchestrate switching among these gains so as to achieve the global stability. One challenge, however, is that the switching logic should be in the output feedback form as well. This may seriously restrict the possible choices of switching surfaces, especially when the output is just a scalar. To overcome this difficulty, a multirate sampling control scheme is proposed. Under this framework, a hybrid output feedback stabilizing controller is designed, and sufficient controller synthesis conditions are proposed as linear matrix inequalities based on multiple Lyapunov function theorems. The note concludes with a discussion on possible extensions and future research topics.**

***Index Terms*— Hybrid Systems, Static Output Feedback, Stabilization, Multiple Lyapunov Functions.**

I. INTRODUCTION

Static output feedback (SOF) stabilization is a well-known open problem in systems and control theory [4], [25]. The problem is motivated by the fact that it is not always possible to have access to the full state vector for practical plants, and that only a partial information through the measured output is available in most cases. In addition, compared with a dynamic controller/observer, the static output feedback controller has advantages like ease of implementation and maintenance, higher reliability and better cost-efficiency. Furthermore, many problems involving the synthesis of dynamic controllers can be reformulated as a SOF control design involving augmented plants [25]. Therefore, the SOF problem has been attracting a lot of researchers' attention, and various approaches have been developed. For example, constructive approaches based on the resolution of Riccati equations [14], pole or eigenstructure assignment techniques [24], optimization methods based on matrix inequalities [8], [22], [3], and so on. For a comprehensive review, please refer to [25].

Although the SOF stabilization problem can be simply formulated, it is non-convex and NP-complete [5]. Because of the non-convexity, the existing necessary and sufficient conditions for the SOF stabilizability are not numerically tractable. Moreover, it is known that the systems that can be stabilized by a (single) SOF controller are limited [25]. This motivates us to consider a hybrid output feedback controller scheme, which switches among multiple SOF controllers to fulfill the stabilization task [15], [9]. In particular, this note will focus on the following continuous-time LTI system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t), \end{array} \right. \tag{1}$$

where the state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$ (assume $p = 1$, i.e., single output), control input $u \in \mathbb{R}^m$. It is assumed that the system (1) cannot be stabilized by a single SOF gain F to make the problem nontrivial. The basic idea here is to design multiple SOF gains, F_i, which "partially stabilize" the system (1), and to design a switching logic, which orchestrates switching among these F_i's so as to stabilize the system (1) globally.

Notice that the switching logic should also depend solely on the measured output, i.e., in an output feedback form. Since only partial information on the state vector is available, we can only detect the state trajectories passing through certain specific surfaces, which seriously restricts the choice of possible switching surfaces. Contrary to the traditional switching stabilization problem [15], [21], [12], [18], where the switching surfaces are totally free design variables, the design of switching logic for multiple SOF controllers presents a new challenge.

There are some related works in the switched system literature on using a hybrid controller to stabilize linear systems. In [19], it was shown that if a continuous-time LTI system is controllable and observable, then it admits a stabilizing hybrid output feedback that uses a countable number of discrete states. Then, a natural question is whether it is possible to stabilize such a system by using a hybrid output feedback with only a *finite* number of discrete modes. This question was explicitly raised by Artstein in [2] via an interesting planar system example, harmonic oscillator, which cannot be stabilized by a SOF, but can be stabilized by a hybrid controller. In [11], a conic switching law was proposed to exponentially stabilize a class of single input and single output (SISO) second order LTI systems, which is reachable and observable. Also for the second order SISO LTI system, a root locus based analysis was presented in [23] for the existence of a switched output feedback controller. For higher order LTI systems, a two mode hybrid output feedback control method, based on piecewise quadratic Lyapunov functions, was proposed in [16]. Unfortunately, all the proposed switching rules in [16], [2], [11], [23] may need the state information, which are usually not available for the output feedback design. Hence, [11] proposed a periodic time-controlled switching implementation via explicitly calculating the time period between two successive switchings. An interesting work based on periodically switched output feedback stabilization was reported in [1] for SISO LTI systems, where necessary and sufficient conditions for stabilizability were given as bilinear matrix inequalities. Note that the class of switched systems that can be stabilized by periodic switching laws is not generic [1]. In addition, periodically time-controlled switching is actually an open-loop switching logic, which is not an ideal solution, especially in the face of disturbances or uncertainties.

This note aims to design a hybrid output feedback controller, including the design of multiple SOF gains and an output feedback based switching logic, so as to stabilize the system (1). The rest of the note is organized as follows. In Section II, a detectable state space partition is obtained based on the multirate sampling method, where the detectablility is with respect to the output measurements. Then, the multiple SOF gains and switching logic co-design problem is investigated in Section III based on a well-known multiple Lyapunov function theorem. Then, the harmonic oscillator example is revisited for illustration. Finally, the note concludes with some remarks and possible future extensions.

Notation: The notations used throughout the note are quite standard. The relation $A > B$ ($A < B$) means that the matrix $A - B$ is positive (negative) definite, similar for $A \geq B$. The superscript $(\cdot)^T$ stands for matrix transposition, and the notation M^{-1} denotes the right inverse matrix of M, i.e., $MM^{-1} = I$. The matrix I stands for identity matrix of proper dimensions.

II. MULTIRATE SAMPLING

To carry out the hybrid output feedback design, a partition of the state space is necessary. In addition, the partition should be based on the output signal $y(t)$ only. However, this is a very challenging task for the single output case, since $y(t)$ is just a scalar and only provides very limited information about the position of the state $x(t)$ (such as on which side of the hyperplane $Cx = 0$ the state $x(t)$ lies, or whether $x(t)$ passes through the hyperplane $Cx = 0$). This is obviously not enough to design a meaningful partition of the state

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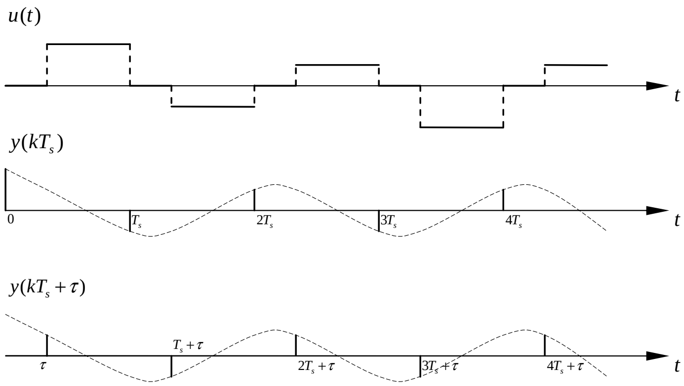

Fig. 1. Multirate sampling mechanism illustrated for the single input and single output case.

space. Hence, a method based on multirate sampling techniques is proposed in this section. The multirate sampling technique has been used in the literature of SOF to augment input/output so as to make the arbitrary pole placement possible, e.g., [10]. Here, a modified version of the multirate sampling scheme is employed, and the main purpose here is to generate a partition of the state space $\mathbb{R}^n$.

Using the sampled data control scheme and connecting a sampler and a zero-order hold with basic sampling period $T_s > 0$ to (1)'s output and input respectively, the input is given as

$$u(t) = \begin{cases} 0, & kT_s \le t < kT_s + \tau \\ u(kT_s+\tau), & kT_s + \tau \le t < (k+1)T_s, \end{cases} \tag{2}$$

where $0 < \tau < T_s$. Substituting $u(t)$ into (1), we obtain

$$x(kT_s+\tau) = e^{A\tau}x(kT_s) \tag{3}$$

and

$$x(kT_s+T_s) = e^{AT_s}x(kT_s) + \int_{\tau}^{T_s} e^{As}ds B u(kT_s+\tau). \tag{4}$$

Detecting the output at every kT_s and $kT_s+\tau$, the sampled value is given by $y(kT_s) = Cx(kT_s)$ and

$$y(kT_s+\tau) = Cx(kT_s+\tau) = Ce^{A\tau}x(kT_s)$$

respectively. Define a new output

$$\bar{y}[k] = \begin{bmatrix} y(kT_s) \\ y(kT_s+\tau) \end{bmatrix},$$

then

$$\bar{y}[k] = \begin{bmatrix} C \\ Ce^{A\tau} \end{bmatrix} x[k],$$

where $x[k] = x(kT_s)$. Furthermore, denote $u(kT_s+\tau)$ as $u[k]$, $\bar{A} = e^{AT_s}$, $\bar{B} = \int_{\tau}^{T_s} e^{As}dsB$, and

$$\bar{C} = \begin{bmatrix} C \\ Ce^{A\tau} \end{bmatrix}.$$

Then, we obtain the following sampled data system of (1) as

$$\begin{cases} x[k+1] &= \bar{A}x[k] + \bar{B}u[k] \\ \bar{y}[k] &= \bar{C}x[k]. \end{cases} \tag{5}$$

The multirate sampling mechanism is illustrated in Figure 1.

Note that the introduction of a new row to the output matrix makes it possible to partition the state space based on the output $\bar{y}[k]$ alone. Here, we propose to use the hyper-planes that are generated by the row vectors of matrix $\bar{C}$, namely hyper-planes $Cx = 0$ and $Ce^{A\tau}x = 0$, as switching surfaces. For notational simplicity, denote C and $Ce^{A\tau}$ as c_1 and c_2 respectively in the sequel. Notice that these two hyper-planes both pass through the origin. Therefore, the following two cones can be generated:

$$\Omega_1 = \{x | c_1x \ge 0 \wedge c_2x \ge 0\} \bigvee \{x | c_1x \le 0 \wedge c_2x \le 0\},$$

and

$$\Omega_2 = \{x | c_1x \ge 0 \wedge c_2x \le 0\} \bigvee \{x | c_1x \le 0 \wedge c_2x \ge 0\}.$$

It is straightforward to verify that $\Omega_1 \bigcup \Omega_2 = \mathbb{R}^n$, and they have mutually exclusive interiors. Hence, we obtain a conic partition of the state space.

It is also interesting to note that the discrete event, like "switching into" or "switching out of" a region Ω_i $(i = 1, 2)$ occurring within the sampling period $(k-1)T_s < t \le kT_s$, can be easily detected by observing the sign changes of a proper multiplication of elements in $\bar{y}[k]$. For example, following the previous expression

$$\begin{aligned} x(kT_s) \in \Omega_1 \quad &\Leftrightarrow \quad x[k]^T(c_1^Tc_2 + c_2^Tc_1)x[k] \ge 0 \\ &\Leftrightarrow \quad \bar{y}_1^T[k]\bar{y}_2[k] \ge 0, \end{aligned}$$

where $\bar{y}_1[k] = y(kT_s)$ and $\bar{y}_2[k] = y(kT_s+\tau)$. On the other hand, $x(kT_s - T_s) \in \Omega_2$ if and only if $\bar{y}_1^T[k-1]\bar{y}_2[k-1] \le 0$. Therefore, by simply observing the sign changes of the product $\bar{y}_1^T[k]\bar{y}_2[k]$, we can determine whether a switching from region Ω_2 to region Ω_1 occurred within the period $(k-1)T_s < t \le kT_s$. Hence, this kind of conic partition can be easily implemented solely based on the measured output $\bar{y}[k]$.

In addition, these cones can be represented in a quadratic form. For example, the quadratic form characterizing the region Ω_1 defined above is obtained by multiplying the two half-planes together, i.e., $\Omega_1 = \{x : x^TQ_1x \ge 0\}$, where $Q_1 = c_1^Tc_2 + c_2^Tc_1$. Similarly, $Q_2 = -Q_1$ for Ω_2.

III. Stabilization

This section aims to design the multiple SOF gains $F_i \in \mathbb{R}^{m\times 2}$ and a proper switching law, $\sigma(\bar{y}[k]) : \mathbb{R}^2 \to \{1, 2\}$, such that the following discrete-time switched system

$$\begin{cases} x[k+1] &= (\bar{A} + \bar{B}F_{\sigma(\bar{y}[k])}\bar{C})x[k] \\ \bar{y}[k] &= \bar{C}x[k] \end{cases} \tag{6}$$

is exponentially stable. Note that the exponential stability of (6) implies the asymptotic stability of the continuous-time system (1) since for any $t > 0$ there exists k such that $kT_s \le t < (k+1)T_s$, and

$$\begin{aligned} \|x(t)\| &= \|e^{\bar{A}_{\sigma(\bar{y}[k])}(t-kT_s)}x(kT_s)\| \le \|e^{\bar{A}_i(t-kT_s)}\|\|x(kT_s)\| \\ &= \|e^{\bar{A}_i(t-kT_s)}\|\|x[k]\| \le \kappa\xi^k\|x[0]\| = \kappa\xi^k\|x(0)\|, \end{aligned}$$

where $\kappa > 0$ and $0 < \xi < 1$ are positive scalars, and

$$e^{\bar{A}_i(t-kT_s)} = \begin{cases} e^{A(t-kT_s)}, & t - kT_s < \tau \\ e^{A(t-kT_s)} + \int_{\tau}^{t-kT_s} e^{As}dsBF_i\bar{C}, & t - kT_s \ge \tau. \end{cases}$$

A. Multiple Lyapunov Function Theorem

Since the subsystems of (6) may be unstable, there may not exist Lyapunov functions for the subsystems. However, it is still possible to restrict our concern in certain regions of the state space, say $\Omega_i \subset \mathbb{R}^n$, and the abstracted energy of the i-th subsystem may be decreasing along the trajectories inside this region. This idea is captured by the concept of Lyapunov-like function.

Definition 1: By saying that the i-th subsystem has an associated Lyapunov-like function V_i in a region Ω_i, we mean that

1) There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that
$$\alpha_i \|x[k]\|^2 \leq V_i(x[k]) \leq \beta_i \|x[k]\|^2$$
hold for any $x[k] \in \Omega_i$;
2) For all $x[k] \in \Omega_i$ and $x[k] \neq 0$,
$$\Delta V_i(x[k]) = V_i(x[k+1]) - V_i(x[k]) \leq -\nu \|x[k]\|^2,$$
holds for some positive scalar ν.

The first condition implies the positiveness and radius unboundedness for $V_i(x)$ when $x \in \Omega_i$, while the second condition guarantees the decreasing of the abstracted energy, value of $V_i(x)$, along trajectories of the subsystem i inside Ω_i. For convenience, the region Ω_i is called the active region for the i-th subsystem.

Suppose that all these regions Ω_i cover the whole state space, then we get a cluster of Lyapunov-like functions. To study the global stability of a switched system, one needs to concatenate these Lyapunov-like functions together and form a non-traditional Lyapunov function, called multiple Lyapunov function (MLF). The non-traditionality is in the sense that the MLF may not be monotonically decreasing along the state trajectories, may have discontinuities, and may be piecewise differentiable. The MLF is proved to be a powerful tool for studying the stability of switched systems, see e.g. [7], [20], [15], [9]. Here, we recall a MLF theorem from [9], [21].

Theorem 1: Suppose that each subsystem of (6) has an associated Lyapunov-like function V_i in its active region Ω_i, each with equilibrium point $x = 0$. Also, suppose that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $\mathcal{S}$ be a class of switching sequences such that any switching signal $\sigma(k) \in \mathcal{S}$ can take value i only if $x[k] \in \Omega_i$, and in addition
$$V_j(x[k_{i,j}]) \leq V_i(x[k_{i,j}])$$
where $k_{i,j}$ denotes that a switching from the subsystem i to the subsystem j occurs during the period, $(k_{i,j}T_s - T_s, k_{i,j}T_s]$, i.e., $x[k_{i,j} - 1] \in \Omega_i$ while $x[k_{i,j}] \in \Omega_j$. Then, the switched system (6) is exponentially stable under the switching signals $\sigma(k) \in \mathcal{S}$. □

The basic idea is that the MLF's value would decrease when a subsystem is activated, i.e., no switching, while at every switching instant the MLF's value is also non-increasing. It is worth pointing out that there exist various versions of MLF theorems [20], [15], [9], [17], which could be less conservative than the version presented here. However, the current form of MLF theorem guarantees the exponential stability and simplifies the controller design below.

In the sequel, we will restrict our attention to multiple quadratic Lyapunov-like functions [13], [21], which are attractive due to their computational tractability and the existence of software packages.

B. Quadratic Lyapunov-like Functions

Using the conic partition derived in Section II, the existence of a quadratic Lyapunov-like function $V_i(x) = x^T P_i x$ for each Ω_i needs to satisfy the following two conditions:

1) Condition 1: There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that
$$\alpha_i \|x\|^2 \leq V_i(x) \leq \beta_i \|x\|^2$$
holds for all $x \in \Omega_i$.

Consider a quadratic Lyapunov-like function candidate, $V_i(x) = x^T P_i x$, and require that
$$\alpha_i x^T I x \leq x^T P_i x \leq \beta_i x^T I x \tag{7}$$
hold for all $x^T Q_i x \geq 0$. Applying the $\mathcal{S}$-procedure [6], the above constrained inequalities (7) are implied by the following LMIs
$$\begin{cases} \alpha_i I - P_i + \eta_i Q_i \leq 0 \\ P_i - \beta_i I + \rho_i Q_i \leq 0, \end{cases}$$
where $\eta_i \geq 0$ and $\rho_i \geq 0$ are unknown scalars. Define two scalars, $\alpha = \min_i\{\alpha_i\}$ and $\beta = \max_i\{\beta_i\}$. Notice that $0 < \alpha \leq \beta$.

2) Condition 2: For all $x[k] \in \Omega_i$, $x[k] \neq 0$,
$$\Delta V_i(x[k]) = V_i(x[k+1]) - V_i(x[k]) \leq -\nu \|x[k]\|^2,$$
where $x[k+1] = \bar{A}_i x[k]$ and $\bar{A}_i = \bar{A} + \bar{B} F_i \bar{C}$.

This is equivalent to
$$x[k]^T (\bar{A}_i^T P_i \bar{A}_i - P_i + \nu I) x[k] \leq 0, \tag{8}$$
for $x[k] \in \Omega_i$.

Let's recall the Finsler's Lemma [6], which has been used previously in the control literature mainly with the purpose of eliminating design variables in matrix inequalities.

Lemma 1 (Finsler's Lemma): Let $\zeta \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that $rank(H) = r < n$. The following statements are equivalent:

1) $\zeta^T P \zeta < 0$, for all $\zeta \neq 0$, $H\zeta = 0$;
2) $\exists X \in \mathbb{R}^{n\times m}$ such that $P + XH + H^T X^T < 0$. □

Applying the Finsler's Lemma to a strict inequality version of (8), with
$$P = \begin{bmatrix} -P_i & 0 \\ 0 & P_i \end{bmatrix},\ \zeta = \begin{bmatrix} x[k] \\ x[k+1] \end{bmatrix},\ X = \begin{bmatrix} E_i \\ G_i \end{bmatrix},$$
and $H = \begin{bmatrix} \bar{A}_i & -I \end{bmatrix}$, then (8) is implied by
$$\zeta^T \begin{bmatrix} \bar{A}_i^T E_i^T + E_i \bar{A}_i - P_i + \nu I & \bar{A}_i^T G_i^T - E_i \\ G_i \bar{A}_i - E_i^T & P_i - G_i - G_i^T \end{bmatrix} \zeta < 0,$$
for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \zeta \geq 0$. Here $E_i,\ G_i \in \mathbb{R}^{n\times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure [6], the above constrained stability condition is implied by the following unconstrained condition for unknown matrices $P_i = P_i^T$, E_i, $G_i \in \mathbb{R}^{n\times n}$, and scalars $\nu > 0$, $\vartheta_i \geq 0$,
$$\begin{bmatrix} \bar{A}_i^T E_i^T + E_i \bar{A}_i - P_i + \nu I + \vartheta_i Q_i & \bar{A}_i^T G_i^T - E_i \\ G_i \bar{A}_i - E_i^T & P_i - G_i - G_i^T \end{bmatrix} < 0. \tag{9}$$

The above matrix inequality (9) is implied by the following LMI in matrices P_i ($P_i = P_i^T$), M_i, N_i, and scalars $\nu > 0$, $\vartheta_i \geq 0$,
$$\begin{cases} \begin{bmatrix} -P_i + \nu I + \vartheta_i Q_i & \bar{A}^T G_i^T + \bar{C}^T N_i^T \bar{B}^T \\ G_i \bar{A} + \bar{B} N_i \bar{C} & P_i - G_i - G_i^T \end{bmatrix} < 0 \\ \bar{B} M_i = G_i \bar{B} \end{cases} \tag{10}$$
To see this, plug in $\bar{B} = G_i \bar{B} M_i^{-1}$, set $F_i = M_i^{-1} N_i$, and note that $\bar{A}_i = \bar{A} + \bar{B} F_i \bar{C}$, then it implies (9) with $E_i = 0$.

C. Switching Condition

Following Theorem 1, in order to guarantee the exponential stability, we also need to make sure that

1) the i-th subsystem is active only when $x[k] \in \Omega_i$;
2) when a switching occurs, the value of the MLF is not increasing.

The design of switching laws to satisfy the first condition is not straightforward, since checking whether $x[k] \in \Omega_i$ needs to detect both $y(kT_s)$ and $y(kT_s + \tau)$. However, $y(kT_s + \tau)$ is not available at the beginning of each sampling period $t = kT_s$. To overcome this difficulty, the sampled data control scheme is implemented as follows. The input signal $u(t)$ is reset to 0 during the interval $kT_s \leq t < kT_s + \tau$, and updated at $t = kT_s + \tau$ when $y(kT_s + \tau)$ is measured. Actually, the controller needs to check which region Ω_i the state $x[k] = x(kT_s)$ lies in, and then updates the control input accordingly. For example, if $x[k] \in \Omega_1$, then the continuous input signal is updated as

$$u(t) = F_1 \begin{bmatrix} y(kT_s) \\ y(kT_s + \tau) \end{bmatrix} = u[k] = F_1 \bar{C} x[k],$$

for $kT_s + \tau \le t < (k+1)T_s$.

For the second condition, assume that a switching, $i \to j$, occurs within the time interval, $(k-1)T_s < t \le kT_s$, i.e., $x[k] \in \Omega_j$ while $x[k-1] \in \Omega_i$ for $i \ne j$. It is required that $V_j(x[k]) \le V_i(x[k])$, which means that

$$x[k]^T (P_j - P_i) x[k] \le 0 \tag{11}$$

holds for $x[k-1] \in \Omega_i$, $x[k] = \bar{A}_i x[k-1] \in \Omega_j$.

Applying the Finsler's Lemma with $P = \begin{bmatrix} 0 & 0 \\ 0 & P_j - P_i \end{bmatrix}$, $\zeta = \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix}$, $X = \begin{bmatrix} E_{ij} \\ G_{ij} \end{bmatrix}$, and $H = \begin{bmatrix} \bar{A}_i & -I \end{bmatrix}$, (11) is implied by

$$\zeta^T \begin{bmatrix} \bar{A}_i^T E_{ij}^T + E_{ij} \bar{A}_i & \bar{A}_i^T G_{ij}^T - E_{ij} \\ G_{ij} \bar{A}_i - E_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T \end{bmatrix} \zeta < 0 \tag{12}$$

for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & Q_j \end{bmatrix} \zeta \ge 0$. Here E_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure, the above constrained stability condition (12) is implied by the existence of unknown matrices $P_i = P_i^T$, E_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$, and scalars $\vartheta_{ij} \ge 0$, such that

$$\begin{bmatrix} \bar{A}_i^T E_{ij}^T + E_{ij} \bar{A}_i + \vartheta_{ij} Q_i & \bar{A}_i^T G_{ij}^T - E_{ij} \\ G_{ij} \bar{A}_i - E_{ij}^T & P_j - P_i - G_{ij} - G_{ij}^T + \vartheta_{ij} Q_j \end{bmatrix}$$

is negative definite.

This is implied by the following LMI in matrices P_i ($P_i = P_i^T$), P_j ($P_j = P_j^T$), M_j, N_j, G_i, and scalar $\vartheta_{ij} \ge 0$,

$$\begin{cases} \begin{bmatrix} \vartheta_{ij} Q_i & \bar{A}^T G_i^T + \bar{C}^T N_i^T \bar{B}^T \\ G_i \bar{A} + \bar{B} N_i \bar{C} & P_j - P_i - G_i - G_i^T + \vartheta_{ij} Q_j \end{bmatrix} < 0 \\ \bar{B} M_i = G_i \bar{B} \end{cases} \tag{13}$$

and $F_i = M_i^{-1} N_i$.

D. Synthesis Condition

In summary, a sufficient condition for the continuous-time LTI system (1) being asymptotically stabilized by the proposed switched multiple SOF controller can be presented as the following theorem.

Theorem 2: For a given sampling period T_s and detecting time $0 < \tau < T_s$, if there exist matrices P_i ($P_i = P_i^T$), G_i, M_i, N_i, and scalars $\alpha > 0$, $\beta > 0$, $\nu > 0$, $\eta_i \ge 0$, $\rho_i \ge 0$, $\vartheta_i \ge 0$, $\vartheta_{ij} \ge 0$, that make the the following LMI

$$\begin{cases} \alpha I + \eta_i Q_i \le P_i \le \beta I - \rho_i Q_i \\ \begin{bmatrix} -P_i + \nu I + \vartheta_i Q_i & \bar{A}^T G_i^T + \bar{C}^T N_i^T \bar{B}^T \\ G_i \bar{A} + \bar{B} N_i \bar{C} & P_i - G_i - G_i^T \end{bmatrix} < 0 \\ \bar{B} M_i = G_i \bar{B} \\ \begin{bmatrix} \vartheta_{ij} Q_i & \bar{A}^T G_i^T + \bar{C}^T N_i^T \bar{B}^T \\ G_i \bar{A} + \bar{B} N_i \bar{C} & P_j - P_i - G_i - G_i^T + \vartheta_{ij} Q_j \end{bmatrix} < 0 \end{cases} \tag{14}$$

feasible for all $i, j \in \{1, 2\}$, $i \ne j$, where

$$\bar{A} = e^{AT_s}, \ \bar{B} = \int_{\tau}^{T_s} e^{As} ds B, \ \text{and} \ \bar{C} = \begin{bmatrix} C \\ Ce^{A\tau} \end{bmatrix},$$

then the proposed hybrid output feedback controller with

$$F_i = M_i^{-1} N_i$$

asymptotically stabilizes the continuous-time LTI system (1). □

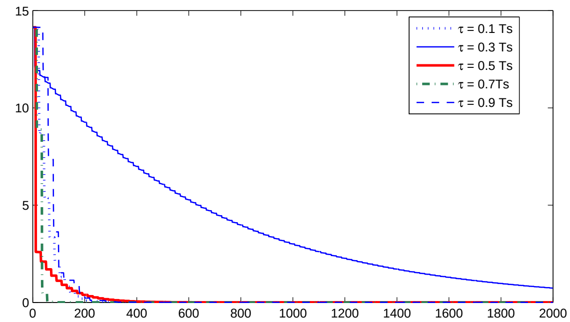

Fig. 2. The plot of $\|x(t)\|$ for system (15) under the switched multiple SOF control for various detecting time τ.

The feasibility of (14) guarantees the exponential stability of the sampled data system (6), which implies the asymptotic stability of the continuous-time linear system (1).

If (14) is not feasible, one may choose to change the sampling period T_s or the detecting time τ, which further changes the state matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and the partition of the state space, i.e., Q_i's. With these new state matrices and Q_i's, it may become possible to find a feasible solution for (14). Actually, the selection of T_s, τ and the design of static feedback gains are coupled together. So iterative design process may be necessary for some cases.

As an example to illustrate the proposed method, the harmonic oscillator with position measurements [2], [16] is revisited here, which is described as

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases} \tag{15}$$

Although this system is both controllable and observable, it cannot be stabilized by a (single) SOF [2]. Applying the multirate sampled data control scheme with $T_s = 0.1s$ and $\tau = 0.5T_s$, we obtain $\bar{A} = \begin{bmatrix} 0.9950 & 0.0998 \\ -0.0998 & 0.9950 \end{bmatrix}$ $\bar{B} = \begin{bmatrix} 0.0012 \\ 0.0500 \end{bmatrix}$, and $\bar{C} = \begin{bmatrix} 1 & 0 \\ 0.9988 & 0.0500 \end{bmatrix}$, then the proposed switched multiple SOF controller with $F_1 = \begin{bmatrix} 396.1171 & -395.1271 \end{bmatrix}$ and $F_2 = \begin{bmatrix} 240.3499 & -239.7491 \end{bmatrix}$ asymptotically stabilize (15). This is illustrated in Figure 2. In addition, the procedure is tested for some different values of τ as shown in Figure 2, which suggests that the system (15) can be stabilized by the proposed hybrid output feedback control scheme for a wide range of variations in the detecting time τ.

IV. Concluding Remarks

In this note, a new hybrid output feedback control scheme was proposed to stabilize a class of continuous-time LTI systems with single output. The arguments were based on the multirate sampling technique and the Multiple-Lyapunov-Function theorem. While this note focused only on the single output case, the proposed design procedure could be extended to the case of multi-output (i.e., $p > 1$) without essential changes. In addition, the multirate sampling scheme can be extended via detecting the output $y(t)$ more than once within a sampling period T_s, e.g., over a sequence of detecting time $0 < \tau_1 < \tau_2 < \cdots < \tau_k < T_s$. Then, with more information on $y(t)$, it becomes possible to further partition the state space and design more multiple output feedback gains correspondingly, and hence improve the chance to stabilize the system. A natural question is how generic the method could be. We ask whether it is always possible to find a pair of sampling period T_s and detecting time τ (or a sequence of detecting time $0 < \tau_1 < \tau_2 < \cdots < \tau_k < T_s$) such that the system (assumed to be reachable and observable) can be stabilized by the proposed multiple SOF controller scheme. If not, what conditions the state matrices $\{A, B, C, D\}$ should satisfy?

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