On Feedback Passivity of Discrete-Time Nonlinear Networked Control Systems with Packet Drops

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Abstract

We analyze the feedback passivity of a networked control system in which the control packets may be dropped by the communication channel. Specifically, we consider a discrete-time switched nonlinear system with relative degree zero that switches between two modes. At the instants when the communication link transmits the packet successfully, the system evolves in closed-loop and the storage function is bounded below the energy supplied by means of the control input. However, at the instants when a packet drop occurs, the system evolves in open loop according to the free dynamics of the closed-loop mode. At these time steps, the increase in storage function is not necessarily bounded by the supplied energy. The literature on passivity of switched systems only seems to consider the case when all the modes are passive, which is not the case here. We prove that if the ratio of the time steps for which the system evolves in closed-loop versus in open loop is lower bounded by a critical number, the system is locally feedback passive in a suitably defined sense. Moreover, this generalized definition of feedback passivity is useful since it preserves two important properties of classical passivity - that feedback passivity implies asymptotic stabilizability for zero state detectable systems and that feedback passivity is preserved in parallel and feedback interconnections.

Index Terms

Networked Control Systems; Switched Systems; Passivity; Feedback Passivity; Zero Dynamics; Relative Degree Zero; Discrete-Time Systems; Nonlinear Systems

I. Introduction

Networked control systems is now an established area of research [1]. In this paper, we consider a discrete-time nonlinear process being controlled across a communication channel that drops control packets in a non-deterministic fashion [2], [3]. In particular, we are interested in analyzing the feedback

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passivity of a networked control system whose increase in storage function may be greater than the supplied energy at some time steps due to packet drops. We assume that the process being controlled is not passive, but is feedback passive, i.e., it can be made passive through a suitable designed state feedback control law. Due to the packet drops induced by the communication channel, the networked control system evolves in two modes. At the instants when no packet is dropped, a state feedback control input is applied through the communication link and the system evolves in closed-loop. Because the process is feedback passive, the resulting increase in storage function is always bounded by the energy supplied by the control input. However, at the instants when the communication channel erases the control packets, the system evolves in open loop according to the free dynamics of the original process. At these time steps, because the process is non-passive, the storage function may increase even though no energy is being supplied by the control input. The problem we are interested in is to identify conditions on the packet drop frequency so that the resulting switched system remains feedback passive.

Passivity is one of the most useful forms of dissipativity and is widely used for analyzing the stability of interconnected dynamical systems [4]–[7]. Two properties that make passivity particularly useful are that (i) passivity implies asymptotic stability for zero state detectable (ZSD) systems using feedback [7], and (ii) both negative feedback and parallel interconnections of passive systems are passive. The classical notion of passivity has been extended to consider systems with delays [8], [9], event-triggered systems [10], switched systems [11], and hybrid systems [12]–[14]. A relaxation of passivity is the concept of feedback passivity [15], [16]. A feedback passive system is not necessarily passive for every possible input. However, it is possible to construct a control law that is a function of both the state and an external input, such that the system is now passive with respect to this external input [16]–[18].

In the framework that we are interested in, because of the packet drops, the process evolution can be modeled as a switched system. While results are available for passivity of switched systems [11], the available literature seems to only consider switched systems in which all the modes are individually passive. In our problem, this framework does not hold. The main contributions of this paper are 1) to extend the concept of feedback passivity to such a discrete-time nonlinear switched system, 2) to show that if the frequency of the time steps at which the system evolves in open loop is bounded, the networked control system is locally feedback passive, and 3) to prove that the stabilizability and compositional properties of passivity are preserved under this generalized definition. The closest work to our presentation is [11] from which we borrow the concept of allowing the storage function of switched systems to increase when a particular mode is inactive. However, unlike [11], we do not assume every mode of the system to be individually passive. Also related are [19], [20] that consider the generalized asymptotic stability of nonlinear dynamical systems where the Lyapunov function is non-increasing only on certain unbounded discrete time sets. Unlike the stability analysis in these works, passivity analysis is complicated by the fact that passivity is an input-output property and both the inputs and the outputs are time varying. Due to this difficulty, we analyze the passivity properties of the switched system based on zero dynamics ([6], [15], [16], and in particular, [18]) which is the internal dynamics of the system that is consistent with constraining the system output to zero. Note that a discrete-time nonlinear system can be rendered passive only if it has relative degree zero [16], hence we assume that the process has relative degree zero.

The remainder of the paper is organized as follows. In Section II, we define the problem framework. Section III provides the main results of this paper. Section III-A analyzes the passivity of the zero dynamics of the process. Section III-B investigates the feedback passivity of the switched system based on the results from zero dynamics. Section III-C discusses the stabilizability and interconnections of feedback passive systems. We give two examples in Section IV and conclude the paper in Section V.

Notation: An m-dimensional real vector is denoted by $\mathbb{R}^m$. The space of nonnegative real numbers is denoted by $\mathbb{R}^+$. The space of positive integers is denoted by $\mathbb{Z}^+$. By a smooth vector field, we mean a field that is in C^∞. Bold-face symbols are used for vectors. In particular, if a scalar m has value zero, we denote $m = 0$; while if a vector $\mathbf{m}$ has value zero, we denote $\mathbf{m} = \mathbf{0}$. The Kronecker delta function is denoted by δ_{rs}, which is 0 if $r \neq s$ and 1 otherwise.

II. Problem Formulation

Consider a discrete-time nonlinear system described by the equation

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \mathbf{u}(k)) \end{cases}, \tag{1}$$

where $k \in \mathbb{Z}^+$ is the time index, $\mathbf{x}(k) \in \mathbb{R}^n$ is the state, $\mathbf{y}(k) \in \mathbb{R}^m$ is the output, and $\mathbf{u}(k) \in \mathbb{R}^m$ is the control input. Both $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are in C^∞. All considerations are restricted to an open set $\mathbf{X} \times \mathbf{U} : \mathbf{X} \subset \mathbb{R}^n, \mathbf{U} \subset \mathbb{R}^m$ which is a neighbourhood of the origin $\mathbf{x}^* = \mathbf{0}, \mathbf{u}^* = \mathbf{0}$. Let the origin be an isolated equilibrium point of (1) such that $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}, h(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. System (1) is assumed to be locally zero state detectable (ZSD) [21]. We also assume that the system has local relative degree zero for all the outputs at $(\mathbf{x}^*, \mathbf{u}^*) = (\mathbf{0}, \mathbf{0})$, i.e., $\left.\frac{\partial h(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{(\mathbf{x}^*,\mathbf{u}^*)}$ is non-singular [18]. This is a reasonable assumption because as shown in [16], a discrete-time nonlinear system can be rendered passive if and only if it has relative degree zero and has passive zero dynamics[1].

Definition 2.1: ([16], [17]) A system of the form (1) is *locally passive* if there exists a positive definite function $\bar{V} : \mathbf{x} \to \mathbb{R}^+$, called the *storage function*, such that

$$\bar{V}(f(\mathbf{x}(k), \mathbf{u}(k))) - \bar{V}(\mathbf{x}(k)) \leq \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k),\ \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{u}(k) \in \mathbf{U}, k \in \mathbb{Z}^+. \tag{2}$$

We assume that process (1) is not passive and hence open loop unstable; however, if the control $\mathbf{u}(k)$ is generated by a suitable state feedback control, it can be turned passive. In other words, we assume that (1) is locally feedback passive.

Definition 2.2: ([15]–[18], [23]) A system of the form (1) is *locally feedback passive* if there exist a positive definite storage function $\tilde{V} : \mathbf{x} \to \mathbb{R}^+$ and a function $\eta(\mathbf{x}, \mathbf{v}) : \mathbf{X} \times \mathbf{U} \to \mathbf{U}$ which is in $\mathcal{C}^\infty$ and locally regular[2], such that for any sequence $\{\mathbf{v}(0), \mathbf{v}(1), \cdots\}$ (with all $\mathbf{v}(j) \in \mathbf{U}$), the system evolving with the control input $\mathbf{u}(k) = \eta(\mathbf{x}(k), \mathbf{v}(k)),\ \forall k$, satisfies the inequality

$$\tilde{V}(f(\mathbf{x}(k), \eta(\mathbf{x}(k), \mathbf{v}(k)))) - \tilde{V}(\mathbf{x}(k)) \leq \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k),\ \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{v}(k) \in \mathbf{U}, k \in \mathbb{Z}^+. \tag{3}$$

Now assume that process (1) is controlled across a communication network that erases some of the control packets transmitted across it. At the instants when the packets are successfully received, the system evolves as described in (1). We denote the system as evolving in Mode 1 at these time steps. At the instants when the channel erases the packets, we assume for concreteness that the actuator applies zero control input, so that the system evolves as

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{0}) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \mathbf{0}) \end{cases}. \tag{4}$$

We denote the system as evolving in Mode 2 at these time steps. Note that $\mathbf{x}^* = \mathbf{0}$ is an isolated equilibrium for Mode 2. Also note that (4) is merely the free dynamics of Mode 1 with $\mathbf{u}(k) = \mathbf{0},\ \forall k$. If Mode 2 is active at time k, the storage function $\tilde{V}(\mathbf{x}(k+1))$ may be larger than $\tilde{V}(\mathbf{x}(k))$ even though no energy is being supplied through the control input. We denote the switched system evolving as in Mode 1 and Mode 2 by $\mathcal{S}$. The mode switching sequence for $\mathcal{S}$ is defined by the specification of the value $d(k)$ for

[1]Recent work [22] relaxes this assumption by using the coupled differential/difference representation (DDR) of the system. However, this requires the existence of a control $\mathbf{u}$ such that $f(\mathbf{x}, \mathbf{u})$ is invertible. Extensions of our results to such a scenario is left as future work.

[2]A nonlinear state feedback control law $\eta(\mathbf{x}, \mathbf{v}) : \mathbf{X} \times \mathbf{U} \to \mathbf{U}$ is locally regular if $\frac{\partial \eta}{\partial \mathbf{v}}$ is invertible for all $(\mathbf{x}, \mathbf{v}) \in \mathbf{X} \times \mathbf{U}$.

every $k \in \mathbb{Z}^+$, where $d(k) \in \{1, 2\}$ is the mode active at time k. Consider the evolution of system $\mathcal{S}$ over T time steps. Let $\tau(T)$ denote the total number of uncontrolled (open loop) time steps when $\mathcal{S}$ is in Mode 2 during this time period, and $T - \tau(T)$ the total number of controlled (closed-loop) time steps when $\mathcal{S}$ is in Mode 1. Let the ratio between the controlled time steps and the uncontrolled time steps be $r(T) = \frac{T-\tau(T)}{\tau(T)}$. When the context is clear, we will abuse the notation and suppress the dependence of $\tau(\cdot)$ and $r(\cdot)$ on T. Without loss of generality, the system is assumed to start in Mode 1 from time step $k = 1$. If this is not the case, we can shift the time axis by defining a new time variable $k' = k_0 + k$ with an appropriately defined initial condition k_0.

The introduction of Mode 2 requires a new definition of feedback passivity. To see why this is true, let us consider the extreme case when $d(k) = 2$ identically. In this case, the set of allowed control inputs is only $\mathbf{u}(k) = \mathbf{0}$ and no energy is supplied to the networked control system. Thus, for the system to be feedback passive according to Definition 2.2 would require the existence of a positive definite storage function $\tilde{V} : \mathbf{x} \to \mathbb{R}^+$ and the control input $\mathbf{u}(k) = \mathbf{v}(k) = \mathbf{0}$ such that

$$\tilde{V}(f(\mathbf{x}, \mathbf{0})) - \tilde{V}(\mathbf{x}) \leq \mathbf{0},\ \forall \mathbf{x}(k) \in \mathbf{X}, k \in \mathbb{Z}^+.$$

However, such a storage function would be a Lyapunov function for the process given by Equation (1) in open loop. Since Mode 2 is unstable, such a storage function does not exist. Thus, the switched system $\mathcal{S}$ is not feedback passive. However, it is intuitive to consider the system to be feedback passive as long as Mode 2 occurs sufficiently infrequently. To capture this intuition, we propose new generalized definitions of local passivity and local feedback passivity. Before we do that, we need to consider one more aspect of the problem, which is that the set $\mathbf{U}$ of allowable controls may differ at different time steps. In particular, in our problem, $\mathbf{u}(k)$ (and hence $\mathbf{v}(k)$) can take any value in the set $\mathbf{U}$ if $d(k) = 1$, while $\mathbf{u}(k) = \mathbf{v}(k) = \mathbf{0}$ is the only value possible if $d(k) = 2$. We introduce this notion formally.

Definition 2.3: Consider a switched system $\mathcal{S}$ evolving as in Mode 1 given by Equation (1) and Mode 2 given by Equation (4) in which the control input $\mathbf{u}(k) \in \mathbf{U}(k)$ at any time k. The system is *locally passive* if there exists a positive definite storage function $\bar{V} : \mathbf{x} \to \mathbb{R}^+$ such that the following passivity inequality holds:

$$\bar{V}(\mathbf{x}(T)) - \bar{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k),\ \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{u}(k) \in \mathbf{U}(k), T \in \mathbb{Z}^+. \quad (5)$$

Definition 2.4: Consider a switched system $\mathcal{S}$ evolving as in Mode 1 given by Equation (1) and Mode

2 given by Equation (4) in which the control input $\mathbf{u}(k) \in \mathbf{U}(k)$ at any time k. The system is *locally feedback passive* if there exists a positive definite storage function $\tilde{V} : \mathbf{x} \to \mathbb{R}^+$ and a regular state feedback control law

$$\mathbf{u}(k) = \begin{cases} \eta(\mathbf{x}(k), \mathbf{v}(k)), \eta : \mathbf{X} \times \mathbf{U} \to \mathbf{U} & \text{if } d(k) = 1 \\ \mathbf{v}(k) = \mathbf{0} & \text{if } d(k) = 2 \end{cases} \tag{6}$$

such that the following passivity inequality holds:

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k), \ \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{v}(k) \in \mathbf{U}(k), T \in \mathbb{Z}^+, \tag{7}$$

where $\mathbf{U}(k) = \mathbf{U}$ when $d(k) = 1$ and $\mathbf{U}(k) = \mathbf{0}$ when $d(k) = 2$.

Note that a system that is locally passive (respectively locally feedback passive) according to Definition 2.1 (resp. Definition 2.2) remains passive according to Definition 2.3 (resp. Definition 2.4). However, the converse is not necessarily true. It is this freedom that will allow us to define the switched system $\mathcal{S}$ as feedback passive.

With these definitions, we answer two questions in this paper. First, we prove the intuitive result that if the system is in Mode 2 only infrequently, the switched system $\mathcal{S}$ should be expected to remain locally feedback passive. More precisely, we prove that there is a critical ratio $r^\star$, such that if for every T, $r(T) > r^\star$, then the system is locally feedback passive. Secondly, we show that this definition preserves the following two properties of classical passivity:

- A feedback passive system is asymptotic stabilizable if it is ZSD.
- Parallel or negative feedback interconnections of feedback passive systems are feedback passive.

III. Main Results

A. Passivity Analysis for Zero Dynamics

Notice that there is considerable freedom in choosing the function $\eta(\mathbf{x}(k), \mathbf{v}(k))$ in Definition 2.2 for Mode 1 as defined by Equation (1). We restrict the class of functions that are allowed to further satisfy the relation $\mathbf{v}(k) = h(\mathbf{x}(k), \eta(\mathbf{x}(k), \mathbf{v}(k)))$. By the implicit function theorem [18], [24], such an η always exists since the system in (1) is assumed to have relative degree zero and η is regular. Denote the control inputs so obtained by $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$. For any given bounded vector sequence $\mathbf{v}(k) \in \mathbf{U}$, the corresponding control inputs $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k)) \in \mathbf{U}$ are bounded. Under these inputs, the system $\mathcal{S}$ in Mode 1 evolves as

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))) \triangleq \bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))) = \mathbf{v}(k) \end{cases} . \tag{8}$$

This is referred as the feedback transformed system with the control $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$. Because $h(\mathbf{x}, \mathbf{u})\Big|_{(\mathbf{x}^*, \mathbf{u}^*) = (\mathbf{0}, \mathbf{0})}$ $= \mathbf{0}$, $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$ remains an isolated equilibrium point of (8), i.e., $\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))\Big|_{(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})} = \mathbf{0}$. Note that the evolution in Mode 2 is still governed by (4). Denote the switched system defined by Equations (8) and (4) by $\mathcal{S}_1$.

In the particular case when $\mathbf{y}(k)$, and hence $\mathbf{v}(k)$, is identically zero, let the control inputs $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$ be denoted by $\tilde{\mathbf{u}}(k)$. Under $\tilde{\mathbf{u}}(k)$, Mode 1 evolves as the zero dynamics of the closed-loop system (1)

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \tilde{\mathbf{u}}(k)) \triangleq \tilde{f}(\mathbf{x}(k)) \\ \mathbf{y}(k) = \mathbf{0} \end{cases} \tag{9}$$

Denote the switched system defined by (9) and (4) as $\mathcal{S}_2$. Since the system $\mathcal{S}$ in Mode 1 as given by Equation (1) is locally feedback passive, the zero dynamics (9) of the closed-loop mode are also locally passive and hence stable (see [16, Theorem 7.3] and [15, Remark 2.5]). Further, since for system $\mathcal{S}_2$, either the input (in Mode 2 which evolves as (4)) or the output (in Mode 1 which evolves as (9)) is identically zero at every time step, Definition 2.3 implies that system $\mathcal{S}_2$ is locally passive if there exists a positive definite storage function $V(\mathbf{x}(.))$ such that the following inequality holds:

$$V(\mathbf{x}(T)) - V(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k) = 0, \ \forall \mathbf{x} \in \mathbf{X}, T \in \mathbb{Z}^{+}. \tag{10}$$

Note that the above inequality holds for every $\mathbf{x}(1) \in \mathbf{X}$ with $d(1) = 1$. From now on, we will additionally assume that the determinant of the Hessian matrix of the storage function $V(\mathbf{x})$ in (10) at $\mathbf{x} = \mathbf{0}$ is non-zero.

Our first result shows that there is a lower bound on the frequency of the steps at which system $\mathcal{S}_2$ evolves in closed-loop as defined by Equation (9) that guarantees $\mathcal{S}_2$ to be locally passive.

Lemma 3.1: Consider the switched system $\mathcal{S}_2$ defined by Equations (9) and (4). Assume there exist a positive definite storage function $V(\mathbf{x}(\cdot))$ and constants $\zeta > 1$ and $0 < \sigma \leq 1$ such that

$$\begin{aligned} V(f(\mathbf{x}(k), \mathbf{0})) &\leq \zeta V(\mathbf{x}(k)) \\ V(\tilde{f}(\mathbf{x}(k))) &\leq \sigma V(\mathbf{x}(k)). \end{aligned} \tag{11}$$

If for any time $T \in \mathbb{Z}^+$, the ratio $r(T)$ satisfies

$$r(T) \geq \frac{(T-1)\ln\zeta}{\ln\zeta - T\ln\sigma}, \tag{12}$$

and $\mathbf{x}(T) \in \mathbf{X}$ irrespective of $d(0), \cdots, d(T-1)$, then system $\mathcal{S}_2$ is locally passive according to Definition 2.3.

Proof: For any time $T \in \mathbb{Z}^+$, (11) implies that $V(\mathbf{x}(T)) \leq \sigma^{T-\tau}\zeta^{\tau-1}V(\mathbf{x}(1))$. Since (12) implies $\sigma^{T-\tau}\zeta^{\tau-1} \leq 1$, we obtain that $V(\mathbf{x}(T)) \leq V(\mathbf{x}(1))$ for any T, if the conditions (11) in the theorem are met. From Definition 2.3, system $\mathcal{S}_2$ is locally passive. ∎

Remark 3.1: The choice of ζ and σ determines how conservative the condition (12) is. The minimum ζ and σ that satisfy the inequality (11) will result in the least conservative bound.

Remark 3.2: Note that the right hand side of (12) is an increasing function of T. Thus, the condition on the frequency of Mode 2 becomes progressively less stringent. Note also that the condition does not require a constant ratio $r(T)$.

We now prove an intuitive result on the effect of increasing $r(T)$.

Corollary 3.1: Consider system $\mathcal{S}_2$ defined by Equations (9) and (4) with the conditions (11) being satisfied. If the system is locally passive with a ratio $r(T)$, it is locally passive with a ratio $r'(T) > r(T)$. Thus, decreasing the frequency of uncontrolled time steps preserves passivity.

Proof: At time $T \in \mathbb{Z}^+$, denote the number of time steps for which the system evolves open loop with the ratio $r(T)$ by $\tau(r, T)$ and with the ratio $r'(T)$ by $\tau(r', T)$. Conditions (11) yield $V(\mathbf{x}(T)) \leq \sigma^{T-\tau(r,T)}\zeta^{\tau(r,T)-1}V(\mathbf{x}(1))$ and $V(\mathbf{x}(T)) \leq \sigma^{T-\tau(r',T)}\zeta^{\tau(r',T)-1}V(\mathbf{x}(1))$. Since the system is locally passive with ratio $r(T)$, $\sigma^{T-\tau(r,T)}\zeta^{\tau(r,T)-1} \leq 1$. The proof follows by noting that $\tau(r', T) < \tau(r, T)$ and thus, $\sigma^{T-\tau(r',T)}\zeta^{\tau(r',T)-1} < \sigma^{T-\tau(r,T)}\zeta^{\tau(r,T)-1} \leq 1$. ∎

Remark 3.3: Now define the sequence of time steps $\{k_i\}$ such that $k_0 = 1$ and $k_i =$ the least time $>$ k_{i-1} such that $d(k_i - 1) = 2$ and $d(k_i) = 1$. Assume system $\mathcal{S}_2$ is locally passive in the time period $[k_0, k_i]$ with τ uncontrolled time steps and $t_c = T - \tau$ controlled time steps. According to Remark 3.2, $r(T)$ increases with T in (12). Therefore, in the time period $[k_i, k_{i+1}]$, we must have $\tau' < \tau$ uncontrolled time steps and $t'_c > t_c$ controlled time steps. Following similar derivation of Corollary 3.1, since $\mathcal{S}_2$ is locally passive in the time period $[k_0, k_i]$, it is also locally passive in the time period $[k_i, k_{i+1}]$.

B. Feedback Passivity Analysis for the Original System

We now prove the main result of the paper.

Theorem 3.1: Let the switched system $\mathcal{S}_2$ defined by Equations (9) and (4) satisfy the inequalities (11) and (12) such that $\mathcal{S}_2$ is locally passive. Furthermore, let the switched system $\mathcal{S}$ defined by Equations (1) and (4) evolve from the same initial condition and with the same mode switching signal as $\mathcal{S}_2$. Then system $\mathcal{S}$ is locally feedback passive.

Proof: We begin by recognizing that for system $\mathcal{S}$, if $d(k) = 1$, the control $\mathbf{u}(k) = \eta(\mathbf{x}(k), \mathbf{v}(k))$ can take any value in the set $\mathbf{U}$; while if $d(k) = 2$, then $\mathbf{u}(k) = \mathbf{v}(k) = \mathbf{0}$ identically. This implies that for system $\mathcal{S}$ to be locally feedback passive according to Definition 2.4, we need to prove that there exists a positive definite storage function $\tilde{V}(\mathbf{x}(k))$ and a feedback control law $\mathbf{u}(k)$ as defined by Equation (6) such that the inequality (7) is true in a neighborhood of $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$.

To prove (7), we proceed as follows. For the case $\mathbf{x}(1) = \mathbf{0}$ and $\{\mathbf{v}(k)\}_{k=1}^{T-1} = \{\mathbf{0}\}$, the inequality (7) holds trivially. For other cases, when $d(k) = 1$, we choose η that guarantees $\mathbf{v}(k) = h(\mathbf{x}(k), \eta(\mathbf{x}(k), \mathbf{v}(k)))$ so that $\mathbf{u}(k) = \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$. When $d(k) = 2$, we have $\mathbf{u}(k) = \mathbf{v}(k) = 0$.

Since $\mathcal{S}_2$ is locally passive, there exists a positive definite storage function $V(\mathbf{x}(.))$, such that for any $T \in \mathbb{Z}^+,\ \mathbf{x}(\cdot) \in \mathbf{X}$, $V(\mathbf{x}(T)) - V(\mathbf{x}(1)) \leq 0$, when the state evolves according to the switched system $\mathcal{S}_2$ with the initial condition $\mathbf{x}(1)$. Consider the storage function $\tilde{V}(\mathbf{x}(.)) = aV(\mathbf{x}(.))$ for a constant $a > 0$ for the switched system $\mathcal{S}$ with the same initial condition $\mathbf{x}(1)$ and the mode sequence as $\mathcal{S}_2$. We prove that with a suitable choice of the constant a, this storage function guarantees (7). Since the controls $\mathbf{u}(k) = \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$ are being used, $\mathbf{y}(k) = \mathbf{v}(k)$ at every time when $d(k) = 1$. Thus, proving (7) is equivalent to proving that the following inequality holds

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{\substack{k:d(k)=1 \\ k \leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k),\ \forall \mathbf{x} \in \mathbf{X}, \mathbf{v} \in \mathbf{U}, T \in \mathbb{Z}^+. \tag{13}$$

Define the function

$$\phi(\mathbf{x}(k), \mathbf{v}(k)) = \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) + \tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) = \sum_{i=1}^{m} v_i^2(k) + \tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)). \tag{14}$$

We use the following property of $\phi(\mathbf{x}(k), \mathbf{v}(k))$ that is proved in Lemma A.1 in the Appendix: if $k : d(k) = 1$, $\phi(\mathbf{x}(k), \mathbf{v}(k))$ has a local minimum at $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$ with value 0. Therefore, for the case when $\mathbf{x}(k) = \mathbf{v}(k) = \mathbf{0}$ for $k \geq K$ such that $d(k) = 1 \; \forall k < K$, the inequality (13) holds trivially.

Except the above trivial cases, let us define

$$\tilde{a}(\mathbf{x}(1), \{\mathbf{v}(k)\}_{k=1}^{T-1}) = \min_{T} \frac{\sum_{\substack{k:d(k)=1 \\ k\leq T-1}} \phi(\mathbf{x}(k), \mathbf{v}(k))}{(\zeta - 1) \sum_{\substack{k:d(k)=2 \\ k\leq T-1}} V(\mathbf{x}(k))}, \forall T \in \mathbb{Z}^{+}. \tag{15}$$

Note that $\tilde{a}$ is a function of the initial condition $\mathbf{x}(1)$ and given control sequence $\{\mathbf{v}(k)\}_{k=1}^{T-1}$ minimized over the horizon T. Because the storage function V is positive definite and $\mathbf{x}(k) \neq \mathbf{0}$ for at least one $k : d(k) = 2$ (except the trivial cases), $\zeta > 1$, and the term ϕ has a local minimum zero at $(\mathbf{0}, \mathbf{0})$, both the numerator and the denomination are greater than zero $\forall T \in \mathbb{Z}^{+}$. Therefore, $\tilde{a}$ is guaranteed to be positive.

We now choose a in the interval $(0, \tilde{a})$ so that the following inequality is satisfied,

$$a(\zeta - 1) \sum_{\substack{k:d(k)=2 \\ k\leq T-1}} V(\mathbf{x}(k)) + \sum_{\substack{k:d(k)=1 \\ k\leq T-1}} \left[\tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) - \tilde{V}(\mathbf{x}(k))\right] \leq \sum_{\substack{k:d(k)=1 \\ k\leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k). \tag{16}$$

Now, if k is such that $d(k) = 2$, systems $\mathcal{S}$ and $\mathcal{S}_2$ evolve in an identical manner as given by Equation (4). From the assumption (11), we obtain at these time steps

$$\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k)) = a\left(V(f(\mathbf{x}(k), \mathbf{0})) - V(\mathbf{x}(k))\right) \leq a(\zeta - 1)V(\mathbf{x}(k)) \tag{17}$$

so that

$$\sum_{\substack{k:d(k)=2 \\ k\leq T-1}} \left[\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k))\right] \leq a(\zeta - 1) \sum_{\substack{k:d(k)=2 \\ k\leq T-1}} V(\mathbf{x}(k)). \tag{18}$$

Now note that

$$\sum_{\substack{k:d(k)=2 \\ k\leq T-1}} \left[\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k))\right] + \sum_{\substack{k:d(k)=1 \\ k\leq T-1}} \left[\tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) - \tilde{V}(\mathbf{x}(k))\right] = \tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)),$$

so that according to the inequalities (18) and (16), we have

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{\substack{k:d(k)=1 \\ k\leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) \tag{19}$$

if a is chosen in the interval $(0, \min(\hat{a}, \tilde{a}))$ where $\hat{a}$ is defined in Lemma A.1 in the Appendix. Thus, system $\mathcal{S}$ is locally feedback passive. ■

C. Stability and Interconnections of Feedback Passive Systems

We now prove that the definition of feedback passivity we have introduced in Definition 2.4 preserves some of the important properties of classical feedback passivity.

Theorem 3.2: If the switched system $\mathcal{S}$ defined by Equations (1) and (4) is locally feedback passive according to Definition 2.4 and locally zero state detectable, then the system is locally asymptotically stabilizable with a suitable state feedback control law.

Proof: Since System $\mathcal{S}$ is locally passive, we can follow the proof of Theorem 3.1 and construct a control law $\mathbf{u}(k)$ as defined by Equation (6) that guarantees that for any $\mathbf{v}(k) \in \mathbf{U}$, $\mathbf{y}(k) = \mathbf{v}(k)$ if $d(k) = 1$ and the inequality (13) holds.

Now, we choose $\mathbf{v}(k) = \mathbf{0},\ \forall k$. Thus, the control law is given by

$$\mathbf{u}(k) = \begin{cases} \eta(\mathbf{x}(k), \mathbf{0}) & \text{if } d(k) = 1 \\ \mathbf{0} & \text{if } d(k) = 2 \end{cases},$$

so that $\mathbf{y}(k) = \mathbf{0}$ if $d(k) = 1$. In this case, the inequality (13) reduces to

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq 0,\ \forall \mathbf{x}(\cdot) \in \mathbf{X}, \forall T.$$

In other words, there exists a function η and a positive definite storage function $\tilde{V}(\mathbf{x}(\cdot)) = aV(\mathbf{x}(\cdot))$ such that the inequality (13) holds.

Recall the sequence of time steps $\{k_i\}$ such that $k_0 = 1$ and $k_i =$ the least time $> k_{i-1}$ such that $d(k_i - 1) = 2$ and $d(k_i) = 1$. Choosing $T = k_1$ yields in particular $\tilde{V}(\mathbf{x}(k_1)) - \tilde{V}(\mathbf{x}(1)) \leq \mathbf{0}, \forall \mathbf{x}(\cdot) \in \mathbf{X}$ with $\mathbf{x}(1) \in \mathbf{X}$ and $d(1) = 1$. Following Remark 3.3, we can repeat the same argument starting from time k_i with $\mathbf{x}(k_i)$ as the initial condition. Thus we obtain the series of inequalities

$$\tilde{V}(\mathbf{x}(k_{i+1})) - \tilde{V}(\mathbf{x}(k_i)) \leq 0,\ \forall i = 0, 1, \cdots, \forall \mathbf{x}(\cdot) \in \mathbf{X}.$$

Since Mode 1 is active infinitely often, $\{k_i\}$ is an infinite sequence. Then $\tilde{V}(\mathbf{x}(\cdot))$ is a Lyapunov function for system $\mathcal{S}$ which implies that the system is Lyapunov stable with the given control law.

The asymptotic stability then follows from ZSD. Observe that all the trajectories of the closed-loop system eventually approach the invariant set $I = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}(k+1)) = V(\mathbf{x}(k))\}$. Since $\mathbf{y}(k) = 0$

and by ZSD $\lim_{k\to\infty}\mathbf{x}(k)=\mathbf{0}$. The system is thus locally asymptotically stable with the given control law. ■

Theorem 3.3: If two switched nonlinear systems $\mathcal{S}^1$ and $\mathcal{S}^2$ are both locally feedback passive according to Definition 2.4, then their parallel and negative feedback interconnections (as defined in Figure 1) are both locally feedback passive.

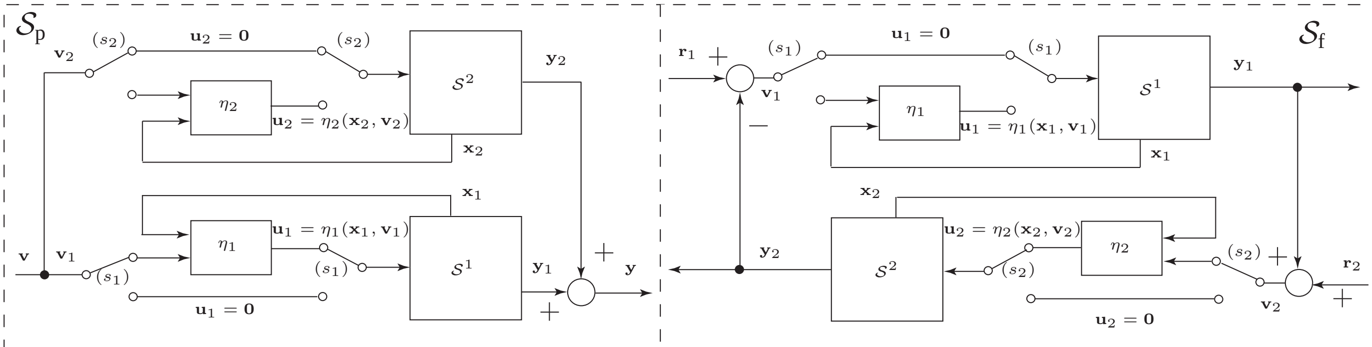


Fig. 1. (a) Parallel, and (b) negative feedback interconnections for two locally feedback passive switched nonlinear systems $\mathcal{S}^1$ and $\mathcal{S}^2$. Note that the switches marked with a same notation $(s_i),\ i=1$ or 2 switch simultaneously.

Proof: If System $\mathcal{S}^1$ (respectively $\mathcal{S}^2$) is locally feedback passive, then there exist a control law $\mathbf{u}_1(k)=\eta_1(\mathbf{x}_1(k),\mathbf{v}_1(k))$ when $d_1(k)=1$ and $\mathbf{u}_1(k)=\mathbf{0}$ when $d_1(k)=2$ (resp. $\mathbf{u}_2(k)=\eta_2(\mathbf{x}_2(k),\mathbf{v}_2(k))$ when $d_2(k)=1$ and $\mathbf{u}_2(k)=\mathbf{0}$ when $d_2(k)=2$) and a positive definite storage function $\tilde{V}_1(\mathbf{x}_1(\cdot))$ (resp. $\tilde{V}_2(\mathbf{x}_2(\cdot))$) such that the inequality (7) is satisfied for any sequence $\mathbf{u}(k)\in\mathbf{U}(k)$. For the parallel interconnection, the extrinsic control sequence $\mathbf{v}(k)$ is the same for both systems and the output $\mathbf{y}(k)=\mathbf{y}_1(k)+\mathbf{y}_2(k)$. Consider the control law $\mathbf{u}(k)=[\mathbf{u}_1^\mathrm{T}(k)\ \ \mathbf{u}_2^\mathrm{T}(k)]^\mathrm{T}$ and the storage function $\tilde{V}(\mathbf{x}_1(k),\mathbf{x}_2(k))=\tilde{V}_1(\mathbf{x}(k))+\tilde{V}_2(\mathbf{x}(k))$. For any time $T\in\mathbb{Z}^+$, we have $\tilde{V}(\mathbf{x}(T))-\tilde{V}(\mathbf{x}(1))=(\tilde{V}_1(\mathbf{x}(T))-\tilde{V}_1(\mathbf{x}(1)))+(\tilde{V}_2(\mathbf{x}(T))-\tilde{V}_2(\mathbf{x}(1)))\leq\sum_{k=1}^{T-1}\mathbf{v}^\mathrm{T}(k)\mathbf{y}_1(k)+\sum_{k=1}^{T-1}\mathbf{v}^\mathrm{T}(k)\mathbf{y}_2(k)\leq\sum_{k=1}^{T-1}\mathbf{v}^\mathrm{T}(k)\mathbf{y}(k)$.

Similarly, for the negative feedback interconnection, the control inputs and outputs are as $\mathbf{r}_1(k)=\mathbf{v}_1(k)+\mathbf{y}_2(k)$ and $\mathbf{r}_2(k)=\mathbf{v}_2(k)-\mathbf{y}_1(k)$. Consider the control law $\mathbf{u}(k)=[\mathbf{u}_1^\mathrm{T}(k)\ \ \mathbf{u}_2^\mathrm{T}(k)]^\mathrm{T}$ and the storage function $\tilde{V}(k)=\tilde{V}_1(k)+\tilde{V}_2(k)$. For any time $T\in\mathbb{Z}^+$, we have $\tilde{V}(\mathbf{x}(T))-\tilde{V}(\mathbf{x}(1))\leq\sum_{k=1}^{T-1}(\mathbf{r}_1^\mathrm{T}(k)\mathbf{y}_1(k)+\mathbf{r}_2^\mathrm{T}(k)\mathbf{y}_2(k))$. ■

IV. EXAMPLES

A. Example 1

In this example, we passify a nonlinear switched system by applying a regular state feedback control law across a network with packet drops. Consider a system of the form

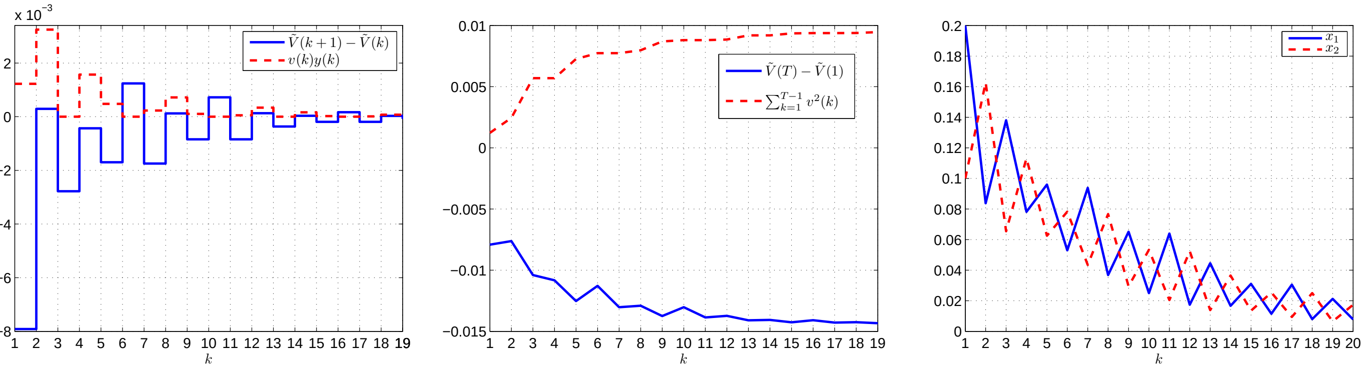
$$\begin{aligned} x_1(k+1) &= -0.3x_1^2(k)x_2(k) + 1.2x_2(k) + u(k) \\ x_2(k+1) &= 0.82x_1(k) - u^2(k) \\ y(k) &= 0.7x_2(k) + u(k), \end{aligned} \tag{20}$$

with initial states $x_1(1) = 0.2, x_2(1) = 0.1$. Note that system (20) is locally ZSD and has relative degree zero. As discussed earlier, we construct $\eta(\mathbf{x}(k), v(k))$ by imposing $v(k) = y(k)$. This leads to $u(k) = \eta(\mathbf{x}(k), v(k)) = v(k) - 0.7x_2(k)$. The resulting feedback transformed system has a passive zero dynamics with $v(k) = 0$, and hence the system is feedback passive for any possible $v(k)$. For the purpose of numerical illustration, we choose the external input as $v(k) = 0.35x_2(k)$, which leads to the controller $u(k) = -0.35x_2(k)$. The evolution of the system in Mode 2 is given by Equation (4) with $u(k) = 0$. In Mode 1, the transformed dynamics and the zero dynamics of system (20) can be obtained as in Equations (8) and (9). Given the zero dynamics, we choose a quadratic storage function $V(\mathbf{x}(k)) = \mathbf{x}(k)^{\mathrm{T}}P\mathbf{x}(k) = x_1^2(k) + 0.5x_2^2(k)$. We can verify that the determinant of the Hessian matrix of $V(\mathbf{x}(k))$ at $\mathbf{x}(k) = [0\ 0]^{\mathrm{T}}$ is not zero. The parameters in the condition (11) are $\zeta = 2.88$ and $\sigma = 0.5516$. According to (12) then, choosing the ratio $r(T)$ to satisfy

$$r(T) \geq \frac{1.0578(T-1)}{1.0578 + 0.5949T} \tag{21}$$

would guarantee system passivity. This condition is satisfied, e.g., by a periodic system in which at every third time step (i.e., at $k = 3, 6, 9, \cdots$) the system is in Mode 2. However, the system need not be periodic to satisfy (21). If the system starts in Mode 1, then any communication protocol that guarantees that out of every 3 consecutive control packets, at most one packet is not delivered would guarantee passivity. Thus, another way to interpret the result is to say that the maximum allowable transmission interval (MATI) [25], [26] is 2. The storage function $\tilde{V}(\mathbf{x}(k))$ for the transformed system is chosen as $0.32V(\mathbf{x}(k))$ with $\hat{a} = 0.49$ and $\tilde{a} = 1.9996$.

More insight can be obtained if we consider the system to operate over a finite horizon. Consider



(a) (b) (c)

Fig. 2. (a) Passivity check for the switched system in the time interval $[1, 20]$ according to classical passivity Definition 2.2, (b) Passivity check for the switched system according to the generalized feedback passivity Definition 2.4, and (c) State dynamics of the switched system.

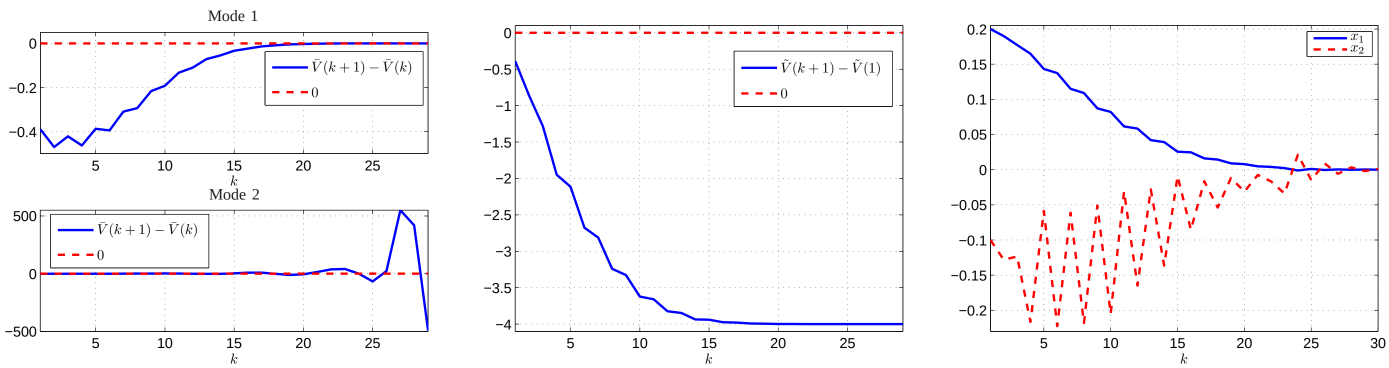
the system operation from $k = 1$ to 20. We consider the system to be in Mode 2 at time steps $k = 3, 6, 10, 13, 16, 19$ as shown in Figure 2(a). Thus, the classical feedback passivity inequality (3) does not necessarily hold at these time steps. Figure 2(b) shows the corresponding generalized feedback passivity inequality (7) for the system. We can see that unlike the classical case, the storage function is now allowed to be greater than the supplied energy instantaneously; however, the general passivity inequality is satisfied at every time till T. Figure 2(c) shows the evolution of the state dynamics of the switched system. If we choose the control to be $\mathbf{u}(k) = -0.7x_2(k)$, since the system is locally ZSD, it can achieve locally asymptotic stability.

B. Example 2

Consider the following nonlinear mass-damper-spring system which is controlled through a network with packet drops. A negative damper is used so that the system is non-passive and open loop unstable. We use the proposed method to passify and stabilize the system.

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + Tx_2(k) \\
x_2(k+1) &= -\frac{K}{m}Tx_1(k) + \left(1 - \frac{c}{m}T\sin(x_1(k))\right)x_2(k) + \frac{T}{m}u(k) \\
y(k) &= 18x_2(k) + u(k),
\end{aligned}
$$

where x_1 and x_2 are the displacement and velocity and $u(k)$ is the force. We set the sampling period $T = 0.1$s, mass $m = 0.5$kg, stiffness $K = 1$N/m, viscous damping coefficient $c = 3$N $\cdot$ s/m and initial conditions $x_1 = 0.2$m, $x_2 = -0.1$m/s. We choose the controller by imposing $f_2(\mathbf{x}, u) = \bar{f}^{\mathbf{0}}(\mathbf{x})$. The resulting controller is $u(k) = -5x_1(k) - 10x_2(k)$ with $v(k) = 0$ which renders the system locally passive and stable. The evolution of the system in Mode 2 is given in Figure 3 when $u(k) = \mathbf{0}$. We choose a storage function $V(\mathbf{x}(k)) = 100x_1^2 + 0.01x_2^2$. We can also verify that the determinant of the Hessian matrix of $V(\mathbf{x}(k))$ at $\mathbf{x}(k) = [0\ 0]^{\mathrm{T}}$ is not zero. The parameters in condition (11) are $\zeta = 1.23$ and $\sigma = 0.92$. We consider the system to operate from $k = 1$ to 30 and with $d(k) = 2$ at time steps $k = 2, 11, 20, 29$. Figure 3(a) shows the corresponding passivity inequality for Mode 1 and 2, respectively. Figure 3(b) shows the generalized passivity inequality according to 7. Figure 3(c) shows the evolution of the state dynamics of the switched system. Both states are locally asymptotic stable at the origin.



(a) (b) (c)

Fig. 3. (a) Passivity check for Mode 1 and 2 according to classical feedback passivity definition 2.2, (b) Passivity check for the switched system according to the generalized feedback passivity definition **??**, and (c) State dynamics of the switched system.

V. Conclusions

We analyzed feedback passivity for a class of discrete-time switched nonlinear systems that switch between two modes - an uncontrolled mode in which the system evolves open loop, and a controlled mode in which a control is applied to the system. This situation is of interest in, e.g., networked control systems where the communication network can erase control packets transmitted to the plant. We give a new generalized definition of feedback passivity for such a system and show that if the ratio of the time steps for which the system evolves closed-loop versus the time steps for which the system evolves open loop is bounded above a critical ratio, then the system is locally feedback passive in this sense. Moreover, we show that this generalized definition is useful since it preserves two important properties of classical passivity - that feedback passivity implies asymptotic stabilizability for zero state detectable systems and that feedback passivity is preserved in parallel and feedback interconnections.

VI. Acknowledgement

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Appendix

Lemma A.1: Consider the set up of Theorem 3.1. When $k : d(k) = 1$, the function (14) has a local minimum at $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$ with value 0. Besides, there exists a constant $\hat{a} > 0$ such that the storage function $\tilde{V}(\mathbf{x}(k))$ equals to $aV(\mathbf{x}(k))$ with $a \in (0, \hat{a})$.

Proof: For notational convenience, we suppress the dependence on k of the terms in (14) and denote the pair $(\mathbf{x}^*, \mathbf{v}^*)$ by $(\mathbf{0}, \mathbf{0})$. Thus, consider the first order derivatives of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$. We have for $i = 1, \cdots, n$, $r = 1, \cdots, m$,

$$\begin{aligned}\left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} &= \left[\frac{\partial \tilde{V}}{\partial x_i} - \sum_{h=1}^{n} \frac{\partial \tilde{V}}{\partial \bar{f}_h^{\mathbf{v}}} \frac{\partial \bar{f}_h^{\mathbf{v}}(\mathbf{x})}{\partial x_i}\right]_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} \\ \left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial v_r}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} &= \left[2v_r - \sum_{h=1}^{n} \frac{\partial \tilde{V}}{\partial \bar{f}_h^{\mathbf{v}}} \frac{\partial \bar{f}_h^{\mathbf{v}}(\mathbf{x})}{\partial v_r}\right]_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}}.\end{aligned}$$

The storage function $V(\mathbf{x}(k))$, and hence the function $\tilde{V}(\mathbf{x}(k)) = aV(\mathbf{x}(k))$, has a local minimum at $\mathbf{x}^* = \mathbf{0}$ because V is positive definite with $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Moreover, the origin is an isoalated local equilibrium of the system; thus, at $\mathbf{x}^* = \mathbf{v}^* = \mathbf{0}$, $\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) = \mathbf{0}$. Combining these facts, we see that

$$\left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} = 0,\ i = 1, \cdots, n, \qquad \left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial v_r}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} = 0,\ r = 1, \cdots, m.$$

Next, we check the elements of the Hessian matrix of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$. We have for $i, j = 1, \cdots, n$ and $r, s = 1, \cdots, m$,

$$\begin{aligned}\left.\frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} &= a\left[\frac{\partial^2 V}{\partial x_j \partial x_i} - \sum_{h,l=1}^{n} \frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}} \partial \bar{f}_l^{\mathbf{v}}} \frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial x_i} \frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial x_j}\right]_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}=^*\mathbf{0}} \\ \left.\frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial v_r \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} &= -a\left[\sum_{h,l=1}^{n} \frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}} \partial \bar{f}_l^{\mathbf{v}}} \frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial x_i} \frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial v_r}\right]_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} \\ \left.\frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial v_s \partial v_r}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} &= 2\delta_{rs} - a\left[\sum_{h,l=1}^{n} \frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}} \partial \bar{f}_l^{\mathbf{v}}} \frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial v_r} \frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial v_s}\right]_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}}.\end{aligned}$$

Denote $\tilde{\phi}(\mathbf{x}(k)) = \phi(\mathbf{x}(k), \mathbf{0}) = a\left(V(\mathbf{x}(k)) - V(\bar{f}^{\mathbf{0}}(\mathbf{x}(k)))\right)$, so that

$$\left.\frac{\partial^2 \phi(\mathbf{x}, \mathbf{v})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} = \left.\frac{\partial^2 \tilde{\phi}(\mathbf{x})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}}. \tag{22}$$

The zero dynamics (9) are locally passive and hence satisfy the passivity inequality (10). Because $\bar{f}^{\mathbf{0}}(\mathbf{x}(k)) = \tilde{f}(\mathbf{x}(k))$, the term $\tilde{\phi}(\mathbf{x}(k))$ has a local minimum at $\mathbf{x}^* = \mathbf{0}$. By assumption, the determinant of Hessian matrix of the storage function $V(\mathbf{x})$ at $\mathbf{x}^* = \mathbf{0}$ is non-zero, we obtain that the eigenvalues of the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}^* = \mathbf{0}$ are all positive. Denote these eigenvalues by λ_i, $\forall i = 1, 2, \cdots, n$. Furthermore, the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}^* = \mathbf{0}$ is symmetric and can be diagonalized. Thus, with an appropriate choice of coordinates, the Hessian matrix of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$ can be evaluated to be of the form

$$\begin{bmatrix} a\lambda_1 & \cdots & 0 & ab_{11} & \cdots & ab_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a\lambda_n & ab_{n1} & \cdots & ab_{nm} \\ ab_{11} & \cdots & ab_{n1} & 2 + ac_{11} & \cdots & ac_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ ab_{1m} & \cdots & ab_{nm} & ac_{m1} & \cdots & 2 + ac_{mm} \end{bmatrix}. \tag{23}$$

Now, we apply [18, Lemma 12] which states that for $\lambda_i > 0$ and $\forall a = (0, \hat{a}),\ \hat{a} = \min_j a_j^u$ where

$$a_j^u = \min\left\{1, \frac{2^j \lambda_1 \cdots \lambda_n - \epsilon}{|\alpha_1| + \cdots + |\alpha_j|}\right\},\ j = 1, \cdots, m \tag{24}$$

with $0 < \epsilon \ll 1$ and $\alpha_l,\ l = 1, \cdots, j$ being some constants related to λ_i, b_{il} and $c_{rl},\ i = 1, \cdots, n$, $r = 1, \cdots, j$, $l = 1, \cdots, j$, the determinant of matrix (23) is greater than zero. Sylester's criterion now readily yields that the Hessian matrix of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$ as evaluated in (23) is positive definite. Therefore, $\phi(\mathbf{x}, \mathbf{v})$ has a local minimum at $(\mathbf{0}, \mathbf{0})$. Because the storage function V is positive definite and $\left.\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))\right|_{\mathbf{x}^*=\mathbf{0}, \mathbf{v}^*=\mathbf{0}} = 0$, we obtain $\phi(\mathbf{x}, \mathbf{v}) = 0$ at $(\mathbf{0}, \mathbf{0})$. ■

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