DYNAMICS OF BIMEROMORPHIC MAPS OF SURFACES.

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ABSTRACT. We classify bimeromorphic self-maps $f : X \circlearrowleft$ of compact kähler surfaces X and classify them in terms of their actions $f^* : H^{1,1}(X) \circlearrowleft$ on cohomology. We observe that the growth rate of $||f^{n*}||$ is invariant under bimeromorphic conjugacy, and that by conjugating one can always arrange that $f^{n*} = f^{*n}$. We show that the sequence $||f^{n*}||$ can be bounded, grow linearly, grow quadratically, or grow exponentially. In the first three cases, we show that after conjugating, f is an automorphism virtually isotopic to the identity, f preserves a rational fibration, or f preserves an elliptic fibration, respectively. In the last case, we show that there is an unique (up to scaling) expanding eigenvector θ_+ for f^*, that θ_+ is nef, and that f is bimeromorphically conjugate to an automorphism if and only if $\theta_+^2 = 0$. We go on in this case to construct a dynamically natural positive current representing θ_+, and we study the growth rate of periodic orbits of f. We conclude by illustrating our results with a particular family of examples.

INTRODUCTION

Holomorphic dynamics on compact complex manifolds have received a lot of attention in the past few years. The use of pluripotential theory by Hubbard-Papadopol [HP1], Bedford-Smillie [BS1], Fornæss-Sibony [FS1], [FS2], Briend-Duval [BD], and others have much clarified our view of such systems, especially for endomorphisms of $\mathbb{P}^n$ and polynomial automorphisms of $\mathbb{C}^2$.

One can more generally consider the dynamics of meromorphic maps. The indeterminacy sets for such maps introduce a singular behavior which makes the pluripotential analysis much harder. Meromorphic maps appear in a natural way when extending a polynomial map from $\mathbb{C}^n$ to $\mathbb{P}^n$ (see [Sib], [Gue]), in Newton's method in several variables (see [HP2]) and also in physical problems (see [AABHM2], [AABM], [AABHM1] and Section 9).

In the present paper, we examine dynamics of invertible meromorphic maps (bimeromorphic maps) $f : X \circlearrowleft$ on a compact Kähler surface X focusing on the behavior of curves under iteration by f. The case of automorphisms has been previously studied by Cantat (see [Can2]). There are, however, many bimeromorphic maps which cannot be bimeromorphically conjugated to automorphisms—for instance, most birational maps of $\mathbb{P}^2$ (see examples in Section 9).

Gromov established a relationship between the dynamical complexity of meromorphic maps and the growth of volumes of complex subvarieties. Our work expands on this idea, the context allowing us to restrict attention to complex curves. On a Kähler surface X, area growth of curves is controlled in turn by the actions $(f^n)^* : H^{1,1}_{\mathbb{R}}(X) \circlearrowleft$—that is, by the sequence of norms $\{||(f^n)^*||\}_{n \geq 0}$. We observe that the rate of growth of this sequence does not change under bimeromorphic conjugacy. Neither, therefore, does the *first dynamical degree* $\lambda_1(f) := \lim_{n \to \infty} ||(f^n)^*||^{1/n} \geq 1$ (introduced in [RS], [Fri]). This quantity dominates the topological entropy $h_{top}(f) \leq \log \lambda_1$ and equality is conjectured (see [Fri]).

To relate λ_1 to the spectral radius of f^* we need $(f^n)^* = (f^*)^n$ for any $n \geq 0$. When this happens we say following [FS2] and [Sib] that f is *analytically stable* (AS for short). It turns out that one can always arrange for a map to be AS by changing coordinates.

Theorem 0.1. *If $f : X \circlearrowleft$ is a bimeromorphic map of a compact complex surface, then there exists a proper modification $\pi : \hat{X} \to X$ that lifts f to an AS map.*

An immediate consequence is that λ_1 is an algebraic integer (compare with [BF]).

The case $\lambda_1 = 1$ of low dynamical complexity is handled by

Theorem 0.2. *Let $f : X \circlearrowleft$ be a bimeromorphic map of a Kähler surface with $\lambda_1 = 1$. Up to bimeromorphic conjugacy, exactly one of the following holds.*

- *The sequence $\|(f^n)^*\|$ is bounded, and f^n is an automorphism isotopic to the identity for some n.*
- *The sequence $\|(f^n)^*\|$ grows linearly, and f preserves a rational fibration. In this case, f cannot be conjugated to an automorphism.*
- *The sequence $\|(f^n)^*\|$ grows quadratically, and f is an automorphism preserving an elliptic fibration.*

In the last two cases, the invariant fibrations are unique.

This parallels the dynamical classification of elementary polynomial automorphisms of $\mathbb{C}^2$ in [FM] and the more recent classification of surface automorphisms with $\lambda_1 = 1$ in [Can1].

We proceed to the case $\lambda_1 > 1$. Our aim is to construct a positive closed $(1,1)$ current (the Green current) representing the pull-back of a generic curve. For this purpose, we first study the spectrum of $f^* : H^{1,1}(X) \circlearrowleft$. We let (α, β) be the intersection pairing of classes $\alpha, \beta \in H^{1,1}(X)$, and we say α is numerically effective (nef) when $(\alpha, \beta) \geq 0$ for any class β represented by a positive closed $(1,1)$ current $\beta = \{T\}$.

Theorem 0.3. *Let $f : X \circlearrowleft$ be an AS bimeromorphic map of a Kähler surface with $\lambda_1 > 1$.*

- *The spectrum of f^* outside the unit disk consists of the single simple eigenvalue λ_1.*
- *The eigenspace associated to λ_1 is generated by a nef class $\theta_+ \in H^{1,1}(X)$.*

When f is an automorphism, it is easy to verify that the self-intersection (θ_+, θ_+) vanishes. We show this condition is in fact sufficient to detect automorphisms.

Theorem 0.4. *Under the assumptions of Theorem 0.3, the following are equivalent:*

- *$(\theta_+, \theta_+) = 0$;*
- *f is bimeromorphically conjugate to an automorphism.*

Theorem 0.3 allows us to construct a natural and invariant positive closed $(1,1)$ current. Since f^* and f_* are adjoint, they have the same spectra. We let θ_- denote a nef class generating the λ_1 eigenspace of f_*.

Theorem 0.5. *Let $f : X \circlearrowleft$ be an AS bimeromorphic map of a compact Kähler surface with $\lambda_1 > 1$. Then there exists a positive closed $(1,1)$ current T_+ in the cohomology class θ_+ satisfying $f^* T_+ = \lambda_1 T_+$. It is characterized by the property that if α is any smooth closed $(1,1)$ form on X, then*

$$(1) \qquad \lim_{k \to \infty} \lambda_1^{-k} (f^k)^* \alpha = \frac{(\{\alpha\}, \theta_-)}{(\theta_+, \theta_-)} \cdot T_+$$

in the weak topology on currents.

We finally extend a theorem of [Fav1].

Theorem 0.6. *Let $f : X \circlearrowleft$ be an AS bimeromorphic map of a compact Kähler surface with $\lambda_1 > 1$. Assume f admits no curves of periodic points. Let Per_k be the number of periodic orbits of period (dividing) k. Then there is a constant $C > 0$ such that*

$$|Per_k - \lambda_1^k| \leq C$$

for all $k \geq 0$.

Throughout the paper, our main techniques come from analytic geometry. We especially rely on a detailed understanding of the relationship between f^* and the quadratic intersection form on $H^{1,1}$. Theorem 0.1 follows from the factorization of bimeromorphic maps of surfaces. Theorems 0.2, 0.3 and 0.4 are consequences of a push-pull formula that we state and prove in Section 3. Theorem 0.5 follows from modifications of fairly well-known potential theoretic arguments (see e.g. [Fav3], [Dil]). Theorem 0.6 is proven by modifying arguments from [Fav1].

We summarize our classification of bimeromorphic dynamical systems in the following table:

Growth of $\|(f^n)^*\|$	Up to bimeromorphic conjugacy	Type of surface	
$\|(f^n)^*\|$ bounded	f is an aut. and f^N is isotopic to Id	any surface	Class [1]
$\|(f^n)^*\| \sim Cn$	rational fibration preserved (f is never an aut.)	ruled surface	Class [2]
$\|(f^n)^*\| \sim Cn^2$	elliptic fibration preserved (f is an aut.)	elliptic surface	Class [3]
$\|(f^n)^*\|$ unbounded with exponential growth, $f^*\theta_+ = \lambda_1\theta_+$, with $\lambda_1 > 1$, θ_+ nef	$\theta_+^2 = 0$: f is an aut.	Tori, K3, Enriques, rational surface	Class [4]
	$\theta_+^2 > 0$: f is never an aut.	rational surface	Class [5]

TABLE 1. Classification of invertible dynamics on a Kähler surface.

The next step toward better understanding bimeromorphic dynamical systems on surfaces would be to construct an invariant measure $\mu := T_+ \wedge T_-$, where T_- is the Green current for f^{-1}. In the case of polynomial automorphisms of $\mathbb{C}^2$, [BS2] [BLS] described the statistical properties of this measure (see also [Can3] in the case of automorphisms). They show that μ is mixing, that it is the unique measure of maximal entropy and that it represents the distribution of saddle periodic orbits. All these results are open in our setting. One might also seek to generalize the main theorems in this paper to non-invertible meromorphic maps or to higher dimensions. We postpone these issues to a later paper. For now, we close our introduction with a brief outline of the paper to follow.

- Section 1 provides the necessary background. We summarize the relevant properties of mero-morphic maps and rational surfaces, define the pullback operator f^* and introduce the notions of dynamical degree and algebraic stability.
- Section 2 is occupied with the proof of Theorem 0.1. We deduce that the dynamic degree is an algebraic integer.
- In Section 3, we state and prove a push-pull formula for meromorphic maps.
- Section 4 classifies bimeromorphic maps with $\lambda_1 = 1$. We break the statement of Theorem 0.2 into several parts and prove each of them.
- Section 5 discusses the expanding part of the spectrum of f^*. We prove Theorems 0.3 and 0.4.
- Section 6 concerns the Green current. We prove Theorem 0.5 and describe some further properties of the Green current.
- In Section 7, we compare our classification of bimeromorphic dynamical systems with the Enriques-Kodaira classification of compact complex surfaces.
- Section 8 contains the proof of Theorem 0.6.
- In Section 9, we consider a class of examples studied in [AABHM2].
- In the Appendix we treat the case of automorphism with $\lambda_1 = 1$ and unbounded degrees.

1. GENERALITIES ABOUT MEROMORPHIC MAPS

1.1. **Compact complex surfaces.** In this paper, a surface X will always be a connected complex analytic two-dimensional manifold. We usually assume that X is compact. A *proper modification* $\pi : X \to Y$ is a proper surjective holomorphic map whose generic fiber $\pi^{-1}(p)$ is a point.

Given $p \in Y$ the *blowup* of Y at p is the proper modification $\pi : X \to Y$ which replaces p with the set $\pi^{-1}(p) \simeq \mathbb{P}^1$ of holomorphic tangent directions at p and is a biholomorphism elsewhere. The rational curve $\pi^{-1}(p)$ is called the exceptional set. An irreducible curve $C \subset X$ is called exceptional if it is the exceptional set for some blowup. In dimension 2, the structure of an arbitrary proper modification is quite simple thanks to the following theorem.

Theorem 1.1. *Any proper modification $\pi : X \to Y$ between compact complex surfaces is a composition of finitely many point blow-ups.*

We have the following criterion.

Theorem 1.2 (Castelnuovo's criterion). *An irreducible curve $C \subset X$ is exceptional if and only if it is a smooth rational curve of self-intersection -1.*

We say X is a minimal surface if it admits no exceptional curve. We say that a surface X is rational when one can find a surface Y and proper modifications $\pi_1 : Y \to \mathbb{P}^2$ and $\pi_2 : Y \to X$. A rational surface is always projective (see [BPV]).

To any surface X, one associates its Dolbeault cohomology groups $H^{p,q}(X)$ and the cohomology groups $H^k(X, \mathbb{Z})$, $H^k(X, \mathbb{R})$, $H^k(X, \mathbb{C})$ associated to the constant sheaves $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively. We let $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$ and $H^{1,1}_{\mathbb{R}}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{R})$.

We denote by $\mathcal{C}_1^+(X)$ the set of positive closed currents of bidegree $(1,1)$. A current $T \in \mathcal{C}_1^+(X)$ can be written locally as $T = dd^c u$ for some plurisubharmonic function u. Any (possibly singular)

complex curve defines a current of integration $[C] \in \mathcal{C}_1^+(X)$. Conversely, any positive closed current T can be weakly approximated locally by currents of integration on analytic cycles.

Let T be a positive closed current of bidegree $(1,1)$ in X. Fix a point $p \in X$ and local coordinates sending p to the origin in $\mathbb{C}^2$. Choose a local plurisubharmonic potential u for T defined around 0 in these coordinates. The function $r \to \sup_{|z|=r} u(z)$ is a convex increasing function of $\log r$. We can therefore define the *Lelong number* of u at 0 by

$$\nu(0, u) := \max\{c \geq 0 : u(z) \leq c \log |z| + O(1)\},$$

which is a finite non-negative real number. We then set $\nu(p, T) := \nu(0, u)$, which does not depend on any choice we made.

Given $T \in \mathcal{C}_1^+(X)$, we denote by $\{T\} \in H^{1,1}(X)$ its cohomology class in the Dolbeault cohomology of currents. As T is positive, $\{T\} \in H^2(X, \mathbb{R})$ is in fact real.

Definition 1.3. *We let $H_{psef}^{1,1}(X) \subset H_{\mathbb{R}}^{1,1}(X)$ be the closed convex cone of classes $\{T\}$ of currents $T \in \mathcal{C}_1^+(X)$.*

If X is projective, then $H_{psef}^{1,1}(X)$ is the closure of classes defined by analytic cycles in $H_{\mathbb{R}}^{1,1}(X)$ (see [Dem] Proposition 6.1). The cone $H_{psef}^{1,1}(X)$ is strict in the sense $H_{psef}^{1,1}(X) \cap -H_{psef}^{1,1}(X) = \{0\}$. We say $\alpha \geq \beta$ for $\alpha, \beta \in H_{\mathbb{R}}^{1,1}(X)$ if $\alpha - \beta \in H_{psef}^{1,1}(X)$.

The operation $\alpha, \beta \to \int \alpha \wedge \bar{\beta}$ on smooth 2-forms induces a quadratic *intersection* form $(\cdot, \cdot)$ on $H^2(X)$. Its structure is given by the following fundamental theorem.

Theorem 1.4 (Hodge Index theorem.). *Assume X is a compact Kähler surface. The intersection form has signature $(1, h^{1,1} - 1)$ on $H^{1,1}(X)$.*

We define the closed convex sub-cone $H_{nef}^{1,1}(X) \subset H_{psef}^{1,1}(X)$ to be the set of classes $\alpha \in H_{\mathbb{R}}^{1,1}(X)$ so that $(\alpha, \beta) \geq 0$ for all $\beta \in H_{psef}^{1,1}(X)$. When X is Kähler, one can show that $H_{nef}^{1,1}(X)$ is the closure of the set of Kähler classes (see [Dem] Proposition 6.1).

Finally we denote by $K_X := \Lambda^2 \mathcal{O}_X^*$ the canonical bundle of X and we denote by $\mathrm{kod}(X)$ the Kodaira dimension of X (see [BPV] p.23).

1.2. **Meromorphic maps.** Let X, Y be compact complex surfaces. A *meromorphic map* $f : X \dashrightarrow Y$ is defined by its graph $\Gamma(f) \subset X \times Y$, an irreducible subvariety for which the projection $\pi_1 : \Gamma(f) \to X$ onto the first factor is a proper modification (see [Fis]). We let $I(f) \subset X$ denote the *indeterminacy set*—i.e. the finite set of points where π_1 does not admit a local inverse. The map f is *dominating* when the second projection $\pi_2 : \Gamma(f) \to Y$ is surjective. We define the critical set $\mathcal{C}(f) \subset X$ to be the projection of the critical set of π_2 i.e. $\mathcal{C}(f) := \pi_1(\mathcal{C}(\pi_2))$. We single out those components of $\mathcal{C}(f)$ that are mapped onto points, setting $\mathcal{E}(f) := \pi_1(\mathcal{E}(\pi_2)) \subset \mathcal{C}(f)$, where $\mathcal{E}(\pi_2)$ is the set of points where π_2 is not a finite map.

If $g : Y \to Z$ is another dominating meromorphic map, the graph $\Gamma(g \circ f)$ of the composite map is the closure of the set $\{(x, g(f(x))) \in X \times Z : x \notin I(f) \text{ and } f(x) \notin I(g)\}$. This is equal to the larger set

$$\Gamma(g) \circ \Gamma(f) = \{(x, z) \in X \times Z : (x, y) \in \Gamma(f), (y, z) \in \Gamma(g) \text{ for some } y \in Y\},$$

if and only if the latter is irreducible.

Proposition 1.5. *$\Gamma(g) \circ \Gamma(f)$ is irreducible if and only if there are no components $V \subset \mathcal{E}(f)$ such that $f(V) \subset I(g)$.*

The proof is fairly straightforward and we leave it to the reader. Following [Sib], we call a dominating meromorphic self-map $f : X \circlearrowleft$ *analytically stable* (*AS* for short) if $\Gamma(f) \circ \Gamma(f^n) = \Gamma(f \circ f^n)$ for all n.

When $f : X \dashrightarrow Y$ admits a meromorphic inverse, we say that f is *bimeromorphic*. If in this case Γ is a desingularization of $\Gamma(f)$, the two induced projections $\pi_1 : \Gamma \to X$, $\pi_2 : \Gamma \to Y$ are proper modifications and Theorem 1.1 implies:

Theorem 1.6. *Any bimeromorphic map $f : X \dashrightarrow Y$ between smooth compact complex surfaces can be written as a composition $f = f_1 \circ \cdots \circ f_k$ where f_i is either a point blow-up or the inverse of a point blow-up.*

The following result plays a key role in this paper. A *fibration* of a compact complex surface X is a surjective holomorphic map $\rho : X \to C$ onto a compact curve C such that $\rho^{-1}(p)$ is irreducible for generic p. We note that the genus of of a generic fiber is independent of p. If this genus is zero, the fibration is called *rational*.

Proposition 1.7. *Suppose that $f : X \circlearrowleft$ is a bimeromorphic map of a compact complex surface with $\mathcal{E}(f^{-1}) \neq \emptyset$ and that $(V, V) \leq 0$ for every irreducible component $V \subset \mathcal{E}(f^{-1})$. Then either X admits a rational fibration with some generic fiber lying in $\mathcal{E}(f^{-1})$, or there exists a proper modification $\pi : X \to \check{X}$ with $\emptyset \neq \mathcal{E}(\pi) \subset \mathcal{E}(f^{-1})$. In the latter case, the induced map $\check{f} : \check{X} \circlearrowleft$ is AS if f is.*

Proof. Let Γ be the minimal desingularization of the graph of f and π_1, π_2 be the projections onto the first and second factors. Decompose each projection π_j into a sequence $\pi_{j1} \circ \cdots \circ \pi_{jk}$ of blowups. Set $V_k = \mathcal{E}(\pi_{1k}) \subset \Gamma$ and $V_j = \pi_{2j} \circ \cdots \circ \pi_{2k}(V_k)$. Note that V_k is a smooth rational curve with self-intersection -1, that $(V_{j-1}, V_{j-1}) = (V_j, V_j) + (V_j, \mathcal{E}(\pi_{2j}))$, and that V_0 is either a point or an irreducible component of $\mathcal{E}(f^{-1})$.

To rule out the first possibility, assume $V_j = \mathcal{E}(\pi_{2j})$ for some j. Then $V_l \cap \mathcal{E}(\pi_{2l}) = \emptyset$ for any $j + 1 \leq l \leq k$. We let $\pi : \Gamma \to \check{\Gamma}$ be the map contracting V_k. The projections π_1, π_2 induce holomorphic maps $\tilde{\pi}_i : \check{\Gamma} \to X$ and this shows that Γ was not a minimal desingularization. Hence $V_j \not\subset \mathcal{E}(\pi_{2j})$ for all j, and V_0 is one dimensional.

Case 1: $(V_0, V_0) = 0$.

In this case there is exactly one j for which $(V_j, \mathcal{E}(\pi_{2j})) = 1$, and for all other j the intersection vanishes. In particular, $V_0 = \pi_2(V_k)$ remains smooth. Hence X is a rational fibration (see [BPV], page 142) and V_0 is a generic fiber.

Case 2: $(V_0, V_0) = -1$.

In this case $V_j \cap \mathcal{E}(\pi_{2j}) = \emptyset$ for all j, and V_0 is again smooth. We let $\pi : X \to \check{X}$ be the map contracting V_0 to a point p. It remains to see that $\check{f} := \pi \circ f \circ \pi^{-1}$ is AS if f is.

We have $\mathcal{E}(\check{f}) = \pi(\mathcal{E}(f))$ and $I(\check{f}) \subset \pi(I(f)) \cup \{p\}$. Assume that f is AS. Consider a curve $W \subset \mathcal{E}(f)$ and let $\check{W} = \pi(W) \subset \mathcal{E}(\check{f})$. Then $\check{f}^n(\check{W}) \cap \pi(I(f)) = \pi(f^n(W) \cap I(f)) = \emptyset$ for all n. Also, if $\check{f}^n(\check{W}) = p$ then $f^{n-1}(W) = f^{-1}(V_0) \in I(f)$, which is impossible if f is AS. Hence $\check{f}^n(\check{W}) \cap I(\check{f}) = \emptyset$ for all n, and $\check{f}$ is AS. $\qquad \square$

Following [Fri] we define the *natural extension* of a dominating rational self-map $f : X \circlearrowleft$ to be the closure

$$\Gamma^\infty = \overline{\{(p_j)_{j=1}^\infty : (p_j, p_{j+1}) \in \Gamma(f), \ p_j \notin I(f)\}}$$

of the set of 'honest' orbits of f. The map f lifts to a continuous map $\sigma : \Gamma^\infty \circlearrowleft$, and we declare $h_{top}(f) := h_{top}(\sigma)$.

1.3. **Action on cohomology groups and currents.** Let $f : X \dashrightarrow Y$ be a dominating meromorphic map between compact complex surfaces, Γ a desingularization of its graph and π_1, π_2 the natural projections.

A smooth form $\alpha \in \mathcal{C}^{\infty}_{p,q}(Y)$ of bidegree (p, q) can be pulled back as a smooth form $\pi_2^* \alpha \in \mathcal{C}^{\infty}_{p,q}(\Gamma)$ and then pushed forward as a current. Hence we define

$$f^* \alpha := \pi_{1*} \pi_2^* \alpha \ .$$

This gives an L^1_{loc} form on X that is smooth outside $I(f)$. The action of f^* commutes with differentiation $d(f^* \alpha) = f^*(d\alpha)$ and so descends to a linear action on De-Rham and Dolbeault cohomology.

Definition 1.8. *Let $\{\alpha\} \in H^{p,q}(Y)$ be the Dolbeault class of some smooth form α. We set*

$$f^* \{\alpha\} := \{\pi_{1*} \pi_2^* \alpha\} \in H^{p,q}(X) \ .$$

This defines a linear map $f^* : H^{p,q}(Y) \to H^{p,q}(X)$.

In a similar way, we define the push-forward $f_* := \pi_{2*} \pi_1^* : H^{p,q}(X) \to H^{p,q}(Y)$.

Remark 1.9. *Note that when f is bimeromorphic, $f_* = (f^{-1})^*$.*

The action of f cannot be extended in a continuous way to all (positive closed) currents (see [Meo]). However it is possible to construct a natural extension for positive closed currents of bidegree $(1, 1)$. If π is a surjective holomorphic map and $T \in \mathcal{C}_1^+$ is a positive closed current with local potential $T := dd^c u$, we define locally $\pi^* T := dd^c(u \circ \pi)$. One can check that this definition does not depend on the choice of the potential and that π^* is continuous in the weak topology of currents (see [Sib]).

Definition 1.10. *Let $f : X \dashrightarrow Y$ be a dominating meromorphic map and $T \in \mathcal{C}_1^+(Y)$. Define*

$$
\begin{aligned}
f^* T &:= \pi_{1*} \pi_2^* T \in \mathcal{C}_1^+(X) \ , \\
f_* T &:= \pi_{2*} \pi_1^* T \in \mathcal{C}_1^+(Y) \ .
\end{aligned}
$$

These operators are continuous.

One checks the compatibility relations $f^*\{T\} = \{f^*T\}$, $f_*\{T\} = \{f_*T\}$.

In the sequel we will focus our attention on the action of f on H^2 and $H^{1,1}$.

Proposition 1.11. *Let $f : X \dashrightarrow Y$ be a dominating meromorphic map between compact complex surfaces.*

1. *The linear maps f_*, f^* preserve $H^2(\mathbb{R})$ and $H^2(\mathbb{Z})$.*
2. *The linear maps f_*, f^* preserve $H^{1,1}_{psef}$ and $H^{1,1}_{nef}$.*
3. *If $X = Y$, f_* and f^* are adjoint for the intersection form. That is*

$$(f^* \alpha, \beta) = (\alpha, f_* \beta)$$

   for any classes $\alpha \in H^{1,1}(Y)$, $\beta \in H^{1,1}(X)$.

Proof.

(1) We work with the pullback $f^* = \pi_{1*} \pi_2^*$, and note that both factors on the right can be described purely topologically. The action of π_2^* is straightforward, whereas π_{1*} acts on a class α by pushing forward the Poincaré dual of α and taking the Poincaré dual of the result. All of these operations preserve real and integral classes, so f^* does too.

(2) It is sufficient to show that the cones $H^{1,1}_{psef}$ and $H^{1,1}_{nef}$ are preserved by f^* and f_* for any proper holomorphic map. If $T \in \mathcal{C}_1^+(X)$, its pull-back $f^*\{T\} = \{f^*T\}$ and its push-forward

$f_*\{T\} = \{f_*T\}$ also belongs to $\mathcal{C}_1^+(X)$. Hence f_* and f^* preserve $H^{1,1}_{psef}$. Preservation of $H^{1,1}_{nef}$ now follows from (3).

(3) Let $[T, \gamma]$ denote the action of a current T on the test form γ. We have for any smooth forms α, β of bidegree $(1,1)$:

$$
\begin{aligned}
(f^*\{\alpha\}, \{\beta\}) &= [f^*\alpha, \beta] = [\pi_{1*}\pi_2^*\alpha, \beta] \\
&= [\pi_2^*\alpha, \pi_1^*\beta] = [\pi_{2*}\pi_1^*\beta, \alpha] = (f_*\{\beta\}, \{\alpha\}) .
\end{aligned}
$$

Hence f_* and f^* are adjoint operators.   □

In the sequel, we denote by $\rho(f^*)$ the spectral radius of the linear operator f^*. We include a proof of the following lemma for sake of convenience. It is a classical result which will be very useful in our analysis.

Lemma 1.12. *Assume $f : X \circlearrowleft$ is a dominating meromorphic map of a Kähler surface. Then there exists a class $\alpha \in H^{1,1}_{nef}(X)$ such that $f^*\alpha = \rho(f^*) \cdot \alpha$.*

Proof. Note first that $\rho(f^*|_{H^{1,1}(X)}) = \rho(f^*|_{H^{1,1}_{\mathbb{R}}(X)})$ because f is holomorphic and preserves real classes. As $H^{1,1}_{nef}(X)$ is a strict cone, we can choose a basis so that $H^{1,1}_{\mathbb{R}}(X) = \mathbb{R}^n$ ($n = h^{1,1}(X)$) and $H^{1,1}_{nef}(X) \subset \mathbb{R}^n_+$. We let u denote f^* in these coordinates.

Consider the affine space $V := \{x_1 + \cdots + x_n = 1\}$ and set $\mathcal{C} := H^{1,1}_{nef}(X) \cap V$. This a closed convex set with non empty interior inside V. The induced continuous map $\overline{u} : V \setminus u^{-1}(x_1 + \cdots + x_n = 0) \to V$ sends $\mathcal{C} \cap V$ into itself since nef classes are preserved by f^*. The set $H^{1,1}_{nef}(X)$ has non-empty interior in $H^{1,1}_{\mathbb{R}}(X)$, so for a generic vector $x \in H^{1,1}_{nef}(X)$ we have $\overline{u}^k(x) \underset{k\to\infty}{\longrightarrow} \bigoplus_{|\lambda|=\rho(f^*)} \ker(u - \overline{\lambda}\mathrm{Id}) \cap \mathcal{C}$ for some eigenvalue λ of modulus $\rho(f^*)$. The intersection of V with this last set is convex, closed, and fixed by $\overline{u}$, so by Brouwer's theorem, it contains a vector $\alpha \in \mathcal{C}$ fixed by $\overline{u}$. This implies that there is a constant $\tau \geq 0$ such that $u(a) = \tau a$. We therefore have $\tau = \lambda = |\lambda| = \rho(f^*) \in \mathbb{R}_+$. This concludes the proof.   □

1.4. Dynamical degrees and algebraic stability.
In this subsection, we restrict our attention to dominating meromorphic maps $f : X \circlearrowleft$ of a *Kähler* surface. We explicitly allow for non-minimal surfaces X. We first give a characterization of algebraic stability that will be crucial in the sequel. We then describe the *dynamical degrees* $\lambda_0(f), \lambda_1(f), \lambda_2(f) \geq 1$ introduced in [RS], concluding with a corollary that equates $\lambda_1(f)$ and $\rho(f^*)$ for AS maps. Let us start with

Proposition 1.13. *Let X, Y, Z be compact Kähler surfaces and $f : X \to Y$, $g : Y \to Z$ be dominating meromorphic maps. For any $\alpha \in H^{1,1}_{nef}(Z)$, we have*

$$(g \circ f)^*\alpha \leq f^*g^*\alpha. \tag{2}$$

with equality when $f(\mathcal{E}(f)) \cap I(g) = \emptyset$. Conversely, if α is a Kähler class, $(g \circ f)^*\alpha = f^*g^*\alpha$ implies $f(\mathcal{E}(f)) \cap I(g) = \emptyset$.

Proof. It is sufficient to work with a Kähler class $\alpha = \{\omega\}$. Clearly, $(g \circ f)^*\omega = f^*g^*\omega$ off of $I(f) \cup f^{-1}(I(g))$. Both currents are positive, and the former does not charge any curve, so $f^*g^*\omega - (g \circ f)^*\omega$ is positive. The first assertion follows.

Assume $f(\mathcal{E}(f)) \cap I(g) = \emptyset$. Then $I(f) \cup f^{-1}(I(g))$ is finite and we infer $(g \circ f)^*\omega = f^*g^*\omega$. By continuity, equality holds for any nef class.

To conclude, suppose that $V \subset \mathcal{E}(f)$ is a curve such that $f(V) = p \in I(g)$. By Corollary 3.5 below, the Lelong number $\nu(p, g^*\omega) > 0$ is positive. It then follows from [Fav3] that the Lelong number $\nu(q, f^*g^*\omega) > 0$ for every $q \in V$. That is, $f^*g^*\omega$ charges V and cannot equal $(g \circ f)^*\omega$. $\square$

The notion of analytic stability will play a central role in the construction of the Green current (see Remark 6.4). As the following immediate consequence of Propositions 1.5 and 1.13 shows, analytic stability implies that the action of f on $H^{1,1}(X)$ determines that of all higher iterates f^k.

Theorem 1.14. *Let $f : X \circlearrowleft$ be a dominating meromorphic map on a Kähler surface and ω a given Kähler form. Then f is AS if and only if any of the following hold:*

C1: *for any $\alpha \in H^{1,1}(X)$ and any $k \in \mathbf{N}$, one has $(f^*)^k \alpha = (f^k)^* \alpha$;*
C2: *there is no curve $V \subset X$ such that $f^k(V) \subset I(f)$ for some integer $k \geq 0$;*
C3: *for all $k \geq 0$ one has $(f^k)^* \omega = (f^*)^k \omega$.*
C4: *for all $T \in C_1^+(X)$ and all $k > 0$, one has $(f^k)^* T = (f^*)^k T$*

Given Kähler surfaces X and Y, we choose arbitrary hermitian norms $\|.\|$ on $H^{1,1}(X, \mathbb{C})$ and $H^{1,1}(Y, \mathbb{C})$. Note that there is a constant $D > 0$ such that $\alpha \leq \beta$ implies $\|\alpha\| \leq D\|\beta\|$. Indeed, since $H_{psef}^{1,1}$ is strict, we can choose a basis for $H^{1,1}(X)$ such that $H_{psef}^{1,1} \subset \mathbb{R}_+^{h^{1,1}}$. In the Euclidean norm with respect to this particular basis, we can take $D = 1$. Our observation is now justified by the fact that any other norm is comparable to this one. We do no harm to the arguments that follow by assuming that $D = 1$ in general. We have

Proposition 1.15. *Let $h : X \to Y$ be a bimeromorphic map. Then there exists a constant $C > 1$ such that for any meromorphic dominant map $f : X \circlearrowleft$*

$$(3) \qquad\qquad C^{-1}\|f^*\| \leq \|g^*\| \leq C\|f^*\|$$

with $g := h \circ f \circ h^{-1}$.

Proof. The assertion is independent of the choice of norms, so we take $\|.\|$ to be the supremum norm with respect to a basis $\{\alpha_i\}$ of nef classes. By Proposition 1.13,

$$g^*\alpha_i = (h \circ f \circ h^{-1})^*\alpha_i \leq h_* f^* h^* \alpha_i \ ,$$

for any i. Hence $\|g^*\| \leq \|h_*\| \cdot \|f^*\| \cdot \|h^*\|$. The roles of f and g are interchangeable in this argument, so the assertion follows. $\square$

Let $f : X \circlearrowleft$ be a dominating meromorphic map on a compact Kähler surface, and consider $f^* : H^{1,1}(X) \circlearrowleft$. We are interested in the growth rate of the sequence $\{\|f^{n*}\|\}_{n \geq 0}$, regarding two such sequences as equivalent if their terms are uniformly comparable via some multiplicative constant. It follows immediately from (3) that

Corollary 1.16. *If $f : X \circlearrowleft$ and $g : Y \circlearrowleft$ are bimeromorphically conjugate dominating meromorphic maps, then $\{\|f^{n*}\|\}$ and $\{\|g^{n*}\|\}$ are equivalent.*

In particular, one sequence is bounded if and only if the other is.

Remark 1.17. *Note that if ω is a Kähler class, the sequences $\{\|(f^n)^*\|\}$ and $\{\|(f^n)^*\omega\|\}$ are equivalent. Indeed, for any Kähler classes ω_1, ω_2, one has $C^{-1}\omega_1 \leq \omega_2 \leq C\omega_1$ for some $C > 0$. And since the set of Kähler classes is open, one can endow $H^{1,1}(X)$ with the Euclidean norm with respect to a basis of Kähler classes.*

It is useful to have a number that describes the exponential growth of $\|(f^n)^*\|$. Proposition 1.13 and the remarks following Theorem 1.14 yield

$$\|(f^{n+m})^*\| \leq \|(f^m)^*\| \cdot \|(f^n)^*\|$$

for any $n, m \geq 0$. We therefore have

Proposition 1.18. *If $f : X \circlearrowleft$ is a dominating meromorphic map of a Kähler surface, then the first dynamical degree*

$$\lambda_1 := \lim_{n \to \infty} \|f^{n*}\|^{1/n} \geq 1$$

exists and is invariant under bimeromorphic coordinate change.

Corollary 1.19. *Let $f : X \to X$ be an AS dominating meromorphic map on a compact Kähler surface. Then λ_1 coincides with the spectral radius $\rho(f^*)$ of $f^* : H^{1,1}(X) \circlearrowleft$.*

In a similar but more straightforward way, the actions of f on $H^0(X)$ and on $H^4(X) = H^{2,2}(X)$ lead us to define $\lambda_0(f) = 1$ and $\lambda_2(f) =$ the topological degree of f. These quantities are clearly also invariant under bimeromorphic conjugacy.

2. Good compactification of bimeromorphic maps

We have stressed that analytic stability is crucial for studying dynamics of meromorphic maps. In this section we prove Theorem 0.1—i.e. that one can always find a bimeromorphic change of coordinates that conjugates a given bimeromorphic self-map to one that is also AS. Before beginning the proof, we describe some consequences.

Corollary 2.1. *For any bimeromorphic map $f : X \circlearrowleft$ of a compact surface X, the quantity $\lambda_1(f)$ is an algebraic integer.*

Proof. By Proposition 1.18 and Theorem 0.1 we can assume f is AS—i.e. that $\lambda_1 = \rho(f^*)$. But $f^* : H^2(X, \mathbb{C}) \circlearrowleft$ preserves $H^2(X, \mathbb{Z})$ and thus has algebraic integers as eigenvalues. $\quad\square$

When $f : X \circlearrowleft$ is AS with X rational and minimal (hence isomorphic to $\mathbb{P}^2$ or to a Hirzebruch surface), the first dynamical degree $\lambda_1(f)$ is quadratic. This is not the case in general (see Remark 9.9 in Section 9).

Since $H^{1,1}(\mathbb{P}^2)$ is one dimensional, the action of a rational map $f : \mathbb{P}^2 \circlearrowleft$ on $H^{1,1}(\mathbb{P}^2)$ is simply multiplication by a positive integer $d =$ the *algebraic degree* of f. We answer for dimension two a question raised in [BV].

Corollary 2.2. *Given a birational map $f : \mathbb{P}^2 \circlearrowleft$, let d_n be the algebraic degree of f^n. Then there is $k > 0$ and a linear function $L : \mathbb{Z}^k \to \mathbb{Z}$ with integer coefficients such that*

$$d_{n+k} = L(d_{n+k-1}, \ldots, d_n)$$

for all $n \geq 0$.

Proof. Let $\pi : \hat{X} \to \mathbb{P}^2$ be a proper modification conjugating f to an AS map $\hat{f}$. Then $I(\pi) = \mathcal{E}(\pi^{-1}) = \emptyset$, so we have from Proposition 1.13 and Theorem 1.14 that

$$(f^n)^* = \pi_*(\hat{f}^*)^n \pi^*$$

for all n. Thus if $P(x) = x^k + c_{k-1}x^{k-1} + \ldots c_0$ is the characteristic polynomial for $\hat{f}^*$, it follows that $d_{n+k} = -(c_{k-1}d_{n+k-1} + \cdots + \ldots c_0 d_n)$ for all n. $\quad\square$

Proof of Theorem 0.1. Write f as a composition

$$X := X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n := X$$

where each map f_i behaves in one of two ways.

- Either f_i blows up a point $p_i := I(f_i) \in X_i$. In this case, we let $f_i(p_i) := V_{i+1} = \mathcal{E}(f_i^{-1}) \subset X_{i+1}$ be the exceptional divisor of f_i^{-1}.
- Or f_i blows down the exceptional divisor $E_i \subset X_i$. We then define $f_i(E_i) := q_{i+1} \in X_{i+1}$.

For any $j \in \mathbb{N}$, we let $X_j := X_{j \bmod n}$ and $f_j := f_{j \bmod n}$.

If f is not AS, then there are integers $1 \le i < N$ such that f_i blows down E_i and

(4) $$f_{N-1} \circ \cdots \circ f_i(E_i) = p_N \in I(f_N) .$$

We can assume by choosing a pair (i, N) of minimal length that for all $i < j \le N$

$$x_j := f_j \circ \cdots \circ f_i(E_i) = f_j \circ \cdots \circ f_{i+1}(q_{i+1}) \notin I(f_j) \cup \mathcal{E}(f_j).$$

If in particular $j_1 = j_2 \mod n$ but $j_1 \ne j_2$, then the points $x_{j_1}, x_{j_2} \in X_{j_1} = X_{j_2}$ are distinct.

We now blow up the surfaces $\{X_j\}$ at all preimages of p_N. That is we replace X_j by its blow-up at x_j. Note that modifying X_j means modifying X_{j+n}, X_{j-n}, etc, so that this process might ultimately result in blowing up the same factor at several distinct points. Nevertheless, the preceding remark shows that blowing up a point x_j does not interfere with the behavior of the map f_j around $x_{j-n}, x_{j-2n}, \cdots$, and we can blow up these points independently. We claim that the lifted map decreases the quantity $\sum \text{Card } f_j(\mathcal{E}(f_j))$ (see Equation (5) below).

In order to show this, let us explain our blow-up procedure more carefully. First we blow up X_N at $x_N = p_N$. The map f_N then lifts to a biholomorphism $\widehat{f}_N$ whereas $\widehat{f}_{N-1}$ either blows up two distinct points $\{x_{N-1}, p_{N-1}\}$ or blows up x_{N-1} and blows down $E_{N-1} \not\ni x_{N-1}$. In particular, the quantity $\sum \text{Card } f_j(\mathcal{E}(f_j)) = \sum \text{Card } \widehat{f}_j(\mathcal{E}(\widehat{f}_j))$ remains constant.

We then proceed in like fashion blowing up $x_{N-1}, x_{N-2}, \cdots, x_{i+2}$ in turn. At each step the integer $\sum \text{Card } f_j(\mathcal{E}(f_j))$ remains constant. We finish by blowing up $x_{i+1} = f_i(E_i)$, but the effect of the last blowup differs significantly from that of the previous blowups in that f_i becomes a biholomorphism $\widehat{f}_i$. We hence reduce the number of components of $\mathcal{E}(f_i)$ from one to zero, which gives us

(5) $$\sum \text{Card } \widehat{f}_j(\mathcal{E}(\widehat{f}_j)) = \left(\sum \text{Card } f_j(\mathcal{E}(f_j)) \right) - 1$$

Summarizing our construction, we have a commutative diagram:

$$
\begin{array}{ccccccc}
\widehat{X} = \widehat{X}_0 & \xrightarrow{\widehat{f}_1} & \widehat{X}_1 & \xrightarrow{\widehat{f}_2} & \cdots & \xrightarrow{\widehat{f}_n} & \widehat{X}_n = \widehat{X} \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_n = \pi_0} \\
X = X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & X_n = X
\end{array}
$$

where $\{\pi_i\}$ are proper modifications. Let $\widehat{f} := \widehat{f}_N \circ \cdots \circ \widehat{f}_1$. After repeating the above argument finitely many times, we will either produce a map $\widehat{f}$ that is AS, or thanks to Equation (5), we will eliminate all exceptional components of the factors of f. In the latter case, $\widehat{f}$ will be AS automatically. $\square$

3. A PUSH-PULL FORMULA.

The aim of this section is to give general formulas for computing $f_* f^* T$ (resp. $f_* f^* \alpha$) given a current $T \in \mathcal{C}_1^+(X)$ (resp. a class $\alpha \in H^2(X, \mathbb{C})$). These formulas will be the keystone of our study of the spectrum of f^*.

Proposition 3.1 (Holomorphic case)**.** *Let X, Y be two connected compact complex surfaces, and $f : X \to Y$ be a surjective holomorphic map. Then for any current $T \subset \mathcal{C}_+^1(Y)$ (resp. for any class $\alpha \in H^2(Y, \mathbb{C})$), the following*

$$
\begin{align}
f_* f^* T &= \lambda_2 \cdot T \tag{6} \\
f_* f^* \alpha &= \lambda_2 \cdot \alpha \tag{7}
\end{align}
$$

hold.

Proof. Assume T is a (not necessarily positive) smooth form. We have $f_* f^* T = \lambda_2 \cdot T$ outside the proper analytic subset $f(\mathcal{C}(f))$. Both sides define forms on X with L_{loc}^1 coefficients, hence the equality $f_* f^* T = \lambda_2 \cdot T$ holds everywhere. Equation (6) follows for general T by approximation, and Equation (7) follows immediately. $\qquad\square$

Proposition 3.2 (Blow-up case)**.** *Let X be a compact complex surface, p a point in X and $\pi : \widehat{X} \to X$ the blow-up of X at p. Let $E := \pi^{-1}(p)$ be the exceptional curve of π. For any current $T \subset \mathcal{C}_+^1(\widehat{X})$ (resp. for any class $\alpha \in H^2(\widehat{X}, \mathbb{C})$), the following equalities*

$$
\begin{align}
\pi^* \pi_* T &= T + E\ (T, E) \tag{8} \\
\pi^* \pi_* \alpha &= \alpha + \{E\}\ (\alpha, E) \tag{9}
\end{align}
$$

hold.

Proof. Let us first consider the case of a positive closed current T. Outside E, the map π is an isomorphism. Hence $\pi^* \pi_* T|_{\widehat{X} \setminus E} = T|_{\widehat{X} \setminus E}$. By Siu's theorem [Siu], we infer the existence of a constant $c \in \mathbb{R}$ s.t. $\pi^* \pi_* T = T + cE$. We conclude with a computation of cohomology classes, using the fact $\pi_* E = 0$:

$$
0 = (\pi_* T, \pi_* E) = (\pi^* \pi_* T, E) = (T, E) + c(E, E) = (T, E) - c\ .
$$

Hence $c = (T, E)$.

Now take $\alpha \in H^2(\widehat{X}, \mathbb{C})$. We have the following isomorphism (see [BPV] p.28)

$$
H^2(\widehat{X}, \mathbb{C}) = \pi^* H^2(X, \mathbb{C}) \oplus \mathbb{C}\{E\}\ .
$$

Hence we can find $\beta \in H^2(X, \mathbb{C})$ s.t. $\alpha = \pi^* \beta + c\{E\}$ with $c = (\alpha, E)$ as above. We have

$$
\pi^* \pi_* \alpha = \pi^* (\pi_* \pi^*) \beta = \pi^* \beta = \alpha - c\{E\}
$$

where the second equality follows from Equation (7). This concludes the proof. $\qquad\square$

Let us introduce some notation for stating our main result. Let $f : X \dashrightarrow Y$ be a dominating meromorphic map between two compact surfaces. Let Γ be a desingularization of the graph of f and π, g be the projections onto the factors X and Y.

$$
\begin{array}{ccc}
 & \Gamma & \\
\pi \downarrow & & \searrow g \\
X & \underset{f}{\dashrightarrow} & Y
\end{array}
$$

The proper modification π factors into a finite sequence of point blow-ups $\pi = \pi_1 \circ \cdots \circ \pi_n$. We set $\varpi_i := \pi_i \circ \cdots \circ \pi_n$.

Let $E(\pi_i)$ be the exceptional set of π_i. Let $V_i^\Gamma \subset \Gamma$ (resp. E_i^Γ) be the proper (resp. total) transform of $E(\pi_i)$ by ϖ_{i-1}, and set $V_i := g_* V_i^\Gamma \subset Y$ (resp. $E_i := g_* E_i^\Gamma$). Note that there exist non-negative integers $\{k_{ij}\}$ s.t.

$$(10) \qquad\qquad E_i = V_i + \sum_{j>i} k_{ij} V_j \ .$$

Theorem 3.3 (Push-pull formula). *Let X and Y be compact surfaces, and $f : X \dashrightarrow Y$ a dominating meromorphic map. For any $T \subset \mathcal{C}_+^1(Y)$ (resp. any $\alpha \in H^2(Y, \mathbb{C})$), we have*

$$(11) \qquad\qquad f_* f^* T \ = \ \lambda_2 \cdot T + \sum E_i \ (T, E_i)$$

$$(12) \qquad\qquad f_* f^* \alpha \ = \ \lambda_2 \cdot \alpha + \sum \{E_i\} \ (\alpha, E_i).$$

Proof. Proofs of Equation (11) and (12) are similar. Proposition 3.2 implies

$$
\begin{aligned}
f_* f^* T \ &= \ g_* \ \varpi_2^* \ (\pi_1^* \pi_{1*}) \ \varpi_{2*} \ g^* T \\
&= \ g_* \varpi_2^* \ [\varpi_{2*} g^* T + E(\pi_1) \ (E(\pi_1), \varpi_{2*} g^* T)] \\
&= \ g_* \varpi_2^* \varpi_{2*} g^* T + E_1 \ (E_1, T) \\
&= \ g_* \ \varpi_3^* \ (\pi_2^* \pi_{2*}) \ \varpi_{3*} \ g^* T + E_1 \ (E_1, T).
\end{aligned}
$$

By induction and by applying Proposition 3.1, one finally gets

$$f_* f^* T = g_* g^* T + \sum E_i \ (E_i, T) = \lambda_2 \cdot T + \sum E_i \ (E_i, T).$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 3.4. *There is a non-negative hermitian form Q on $H^2(X, \mathbb{C})$ such that*

$$(13) \qquad\qquad (f^*\alpha, f^*\beta) = \lambda_2 \ (\alpha, \beta) + Q(\alpha, \beta)$$

for all $\alpha, \beta \in H^2(X, \mathbb{C})$. Moreover, $Q(\alpha, \alpha) = 0$ if and only if $(\alpha, V_i) = 0$ for all i.

Proof. Equation (13) follows from Theorem 3.3 if we define $Q(\alpha, \beta) := \sum_i (\alpha, E_i)(E_i, \beta)$.

Assume now that α satisfies $Q(\alpha, \alpha) = 0$. We first infer $(\alpha, E_i) = 0$ for all i. As $E_n = V_n$, we get $(\alpha, V_n) = 0$. By (10), one has $E_{n-1} = V_{n-1} + k_{n-1,n} V_n$, hence $(\alpha, V_{n-1}) = (\alpha, E_{n-1}) = 0$. One easily concludes the proof by induction. $\qquad\qquad\qquad\qquad\qquad\square$

We used the following result in the proof of Proposition 1.13 (see Proposition 9 in [Fav2]).

Corollary 3.5. *Let $f : X \to Y$ be a meromorphic map between complex surfaces and ω a Kähler form on Y. Then $\nu(p, f^*\omega) > 0$ for every $p \in I(f)$.*

Proof. Let π, g, and Γ be as above, and let $p \in I(f)$ be given. Let $V \subset \Gamma$ be any irreducible component of $\pi^{-1}(p)$. Since $V \not\subset \mathcal{E}(g)$, we have $(V, g^*\omega) = (g_*V, \omega) > 0$. Hence by positivity of $g^*\omega$ and the push-pull formula (11), the current $\pi^* \pi_* g^*\omega$ concentrates mass on V—i.e. $\nu(q, \pi^* \pi_* g^*\omega) > 0$ at every $q \in V$. Theorem 2 of [Fav2] then allows us to conclude that $\nu(p, f^*\omega) = \nu(p, \pi_* g^*\omega) > 0$, as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4. Bimeromorphic maps with $\lambda_1 = 1$.

In this section, we classify bimeromorphic maps with first dynamical degree equal to one, proving Theorem 0.2 in several parts. For the remainder of this section, we let $f : X \circlearrowleft$ denote a bimeromorphic map of a Kähler surface X with $\lambda_1(f) = 1$. We let $|| \cdot ||$ be an Hermitian norm on $H^2(X, \mathbb{C}) \supset H^{1,1}(X)$.

Lemma 4.1. *The sequence $\{||f^{n*}||\}_{n \geq 0}$ is bounded if and only if f is conjugate to an automorphism g so that g^n is isotopic to the identity for some $n > 0$.*

Proof. The 'if' direction is clear, so suppose that $\{||f^{n*}||\}$ is a bounded sequence. By performing appropriate blowups, we can assume that f is AS. Proposition 1.16 guarantees that the sequence $||f^{n*}|| = ||(f^*)^n||$ remains bounded.

Note that for any class $\alpha \in H^{0,2}(X)$ (or $H^{2,0}(X)$), one has $Q(\alpha, \alpha) = 0$ in Equation (13). Hence f^* induces an isometry on these two vector spaces. This shows that the sequence of operators $(f^*)^n : H^2(X) \circlearrowleft$ is in fact bounded. As f^* preserves the lattice $H^2(X, \mathbb{Z})$, we have $(f^*)^{n+k} = (f^*)^k$ for some $n > 0$ and k large enough. Let ω be a Kähler form on X and $\alpha = \{f^{k*}\omega\}$. Then α is a non-zero nef class that satisfies $f^{n*}\alpha = \alpha$. By Corollary 3.4, we have $\alpha \cdot V = 0$ for every irreducible component V of $\mathcal{E}(f^{-n})$. We conclude that $(\omega, f_*^k V) = 0$ and that since $f_*^k V$ is effective, $f_*^k V = 0$. We infer (Corollary 3.4 again) that $V \cdot V \leq 0$ for any $V \subset \mathcal{E}(f^{-n})$. In fact, $V \cdot V < 0$. Otherwise $\{V\} = \alpha$ by the Hodge index theorem, and we get $f_*^k f^{k*}\{\omega\} = 0$, which is absurd.

Proposition 1.7 now gives a proper modification $\pi : \check{X} \to X$ that blows down a curve in $\mathcal{E}(f^{-1})$. The induced map $\check{f} : \check{X} \circlearrowleft$ is AS, and since $I(\pi) = \mathcal{E}(\pi^{-1}) = \emptyset$, we have by Proposition 1.13 and Theorem 1.14

$$(\check{f}^*)^j = \pi_*(f^*)^j \pi^*$$

for every $j \geq 0$. Hence, $\check{f}^{(n+k)*} = \check{f}^{k*}$ and we see as above that $\mathcal{E}(\check{f}^{-1})$ has no irreducible components with non-negative self-intersection. We can therefore continue blowing down curves unless $\mathcal{E}(\check{f}^{-1})$ is empty. Since $h^{1,1}$ decreases by one each time, we will eventually have that $\check{f}$ is an automorphism.

To conclude the proof, note that $\check{f}^*$ must be invertible, and therefore $\check{f}^{n*}$ is the identity. This implies that some iterate of $\check{f}^n$ is isotopic to the identity (see [Can1]). $\qquad\square$

Lemma 4.2. *If $\{||f^{n*}||\}_{n \geq 0}$ is unbounded, then f is conjugate to*

- *an automorphism; or*
- *a birational map preserving a rational fibration.*

In the latter case, $||f^{n*}|| = Cn(1 + o(1))$ for some $C > 0$.

Proof. As before, we can assume after appropriate blowups that f is AS. After finitely many applications of Proposition 1.7 to f, we will obtain a proper modification $\pi : X \to \check{X}$ such that either the induced AS map $\check{f} : \check{X} \circlearrowleft$ is an automorphism or $\mathcal{E}(\check{f}^{-1})$ contains an irreducible component V with $\{V\}$ nef.

In the latter case we use Lemma 1.12 to pick a nef class $\alpha \in H^{1,1}_{nef}(\check{X})$ s.t. $\check{f}^*\alpha = \lambda_1 \alpha = \alpha$. We infer from Corollary 3.4 that $(\alpha, V) = 0$ for any nef component $V \subset \mathcal{E}(\check{f}^{-1})$. By the Hodge index theorem, $\alpha = \{V\}$. In particular, all components of $\mathcal{E}(\check{f}^{-1})$ have non-positive self-intersection. We apply Proposition 1.7 again to conclude that X is a rational fibration and that α is the class of a generic fiber—i.e f preserves the fibration.

Assume $f : X \circlearrowleft$ is a birational map of a ruled surface preserving the ruling. By [BPV] p.139, we can assume X is a product $C \times \mathbb{P}^1$ for some compact curve C. The map f can be then written

$$f(x, y) = \left(g(x), \frac{A_1(x)y + B_1(x)}{A_2(x)y + B_2(x)} \right) \; ,$$

for some $g \in \text{Aut}(C)$ and some meromorphic maps $A_1, A_2, B_1, B_2 : C \to \mathbb{P}^1$ such that $A_1 B_2 - A_2 B_1 \not\equiv 0$. One then checks easily that $||f^{n*}||$ grows at most linearly. $\square$

The following result is proved in [Can1] (see [Giz] for the case of rational surfaces). For the sake of completeness we present a detailed proof in the appendix. The arguments, which rely heavily on the Enriques-Kodaira classification of compact complex surfaces, were explained to us by Cantat.

Theorem 4.3 ([Giz], [Can1])**.** *Assume $f \in \text{Aut}(X)$ and $||f^{n*}||$ is unbounded. Then f preserves an elliptic fibration.*

The following proposition is due to [Giz],[Bel]. Again, we provide a proof in the appendix due to Cantat.

Proposition 4.4 ([Giz], [Bel])**.** *Suppose that $f \in \text{Aut}(X)$ preserves an elliptic fibration and $||f^{n*}||$ is unbounded. Then $||f^{n*}|| = Cn^2(1 + o(1))$ for some $C > 0$.*

The next lemma completes the proof of Theorem 0.2. It implies that classes [2] and [3] in Table 1 are mutually exclusive and that, in either class, the invariant fibration is uniquely determined.

Lemma 4.5. *Let $f : X \circlearrowleft$ be a bimeromorphic map preserving two fibrations which are generically transverse. Then the sequence $\{||f^{n*}||\}_{n \geq 0}$ is bounded.*

Proof. Let $\pi_1 : X \to C_1$ and $\pi_2 : X \to C_2$ be two invariant fibrations which are generically transverse. The holomorphic map $\pi := \pi_1 \times \pi_2 : X \to C_1 \times C_2$ is surjective and semi-conjugates f to a product map $\check{f} : C_1 \times C_2 \circlearrowleft$. In particular, after fixing a norm on $H^{1,1}(C_1 \times C_2)$, we have that $\{||\check{f}^{n*}||\}$ is bounded. Pick a Kähler class α on X. It suffices to show that $\{||f^{n*}\alpha||\}$ is bounded (see Remark 1.17). Set $\beta = \pi^*\pi_*\alpha \geq \alpha$. Then we have that

$$||f^{n*}\alpha|| \leq ||f^{n*}\beta|| = ||(\pi \circ f^n)^*\pi_*\alpha|| = ||(\check{f}^n \circ \pi)^*\pi_*\alpha|| \leq C||\alpha||,$$

where C is a constant independent of n and of α. $\square$

5. Spectral properties of $f^* : H^{1,1}(X) \circlearrowleft$.

In this section, we describe the spectral properties of the linear operator f^* acting on $H^{1,1}(X)$. Theorem 0.3 is an immediate consequence of Corollary 1.19 and

Theorem 5.1. *Let $f : X \circlearrowleft$ be a bimeromorphic map with $\rho(f^*) > 1$, and fix a Kähler form ω on X.*

1. *The operator f^* has exactly one eigenvalue $\lambda \in \mathbb{R}_+$ of modulus $|\lambda| > 1$, and in fact $\lambda = \rho(f^*)$.*
2. *The eigenvalue λ is a simple root of the characteristic polynomial for f^*.*
3. *The one-dimensional eigenspace $\ker(f^* - \lambda Id)$ is represented by a nef class θ_+ (which we normalize so that $(\theta_+, \omega) = 1$).*
4. *If $\theta_- \in H^{1,1}_{nef}(X)$ is the corresponding eigenvector associated to $(f^{-1})^* = f_*$ (again normalized to satisfy $(\theta_-, \omega) = 1$), then $(\theta_-, \theta_+) > 0$.*

5. *For any class $\alpha \in H^{1,1}(X)$, we have*

$$\lim_{k \to \infty} \lambda^{-k} (f^*)^k \alpha = \frac{(\alpha, \theta_-)}{(\theta_+, \theta_-)} \; \theta_+.$$

In particular, the limit is non-zero if α has positive self-intersection.

Remark 5.2. *This theorem remains true for non-invertible maps with $\lambda_1^2 > \lambda_2$. Assertion 1 becomes the following. The operator f^* has exactly one eigenvalue $\lambda \in \mathbb{R}_+$ of modulus $|\lambda| > \sqrt{\lambda_2}$, and in fact $\lambda = \lambda_1$.*

Proof. 1. Let μ_1, μ_2 be two eigenvalues of modulus $|\mu_i| > 1$, and $\alpha_1, a_2 \in H^{1,1}(X)$ be eigenvectors associated to them. We claim that $(,)$ is a positive hermitian form on $\mathbb{C}\alpha_1 + \mathbb{C}\alpha_2$. This is equivalent to proving

(14) $$(\alpha_1, \alpha_1) \cdot (\alpha_2, \alpha_2) \geq (\mathrm{Re} \, (\alpha_1, \alpha_2))^2 \; .$$

We compute (α_i, α_j) using Equation (13) and we get

$$
\begin{aligned}
&(\alpha_1, \alpha_1) \cdot (\alpha_2, \alpha_2) - (\mathrm{Re} \, (\alpha_1, \alpha_2))^2 \\
&= \; \frac{1}{|\mu_1|^2 - 1} \cdot \frac{1}{|\mu_2|^2 - 1} \; Q(\alpha_1, \alpha_1) \cdot Q(\alpha_2, \alpha_2) - \frac{1}{|\mu_1\overline{\mu_2} - 1|^2}(\mathrm{Re}\, Q(\alpha_1, \alpha_2))^2 \\
&\geq \; |\mu_1\overline{\mu_2} - 1|^{-2} \cdot \left(Q(\alpha_1, \alpha_1) \cdot Q(\alpha_2, \alpha_2) - (\mathrm{Re}\, Q(\alpha_1, \alpha_2))^2 \right) \geq 0
\end{aligned}
$$

since Q defines a positive hermitian form. The first inequality follows from the elementary fact $(|\mu_1|^2 - 1)^{-1} \cdot (|\mu_2|^2 - 1)^{-1} \geq |\mu_1\overline{\mu_2} - 1|^{-2}$. The claim is now established.

By the Hodge index theorem we infer $\alpha_1 = \alpha_2$. In particular, $\mu_1 = \mu_2 = \lambda \in \mathbb{R}_+$. We also deduce that the eigenspace $\ker(f^* - \lambda\mathrm{Id})$ is one dimensional.

Assertion 3. now follows from Lemma 1.12. Let $\theta_+ \in H^{1,1}_{nef}(X)$ be the eigenvector associated to λ normalized by $(\theta_+, \omega) = 1$.

2. Assume λ is not a simple root of $\det(f^* - \lambda\mathrm{Id})$. One can then find a two-dimensional vector space V containing θ_+ s.t. $f^*\beta = \lambda\beta + l(\beta) \, \theta_+$ for any $\beta \in V$ and for some linear form l. We apply Equation (13) to compute $(f^*\theta_+, \overline{f^*\theta_+})$, $(f^*\beta, \overline{f^*\beta})$, $(f^*\theta_+, \overline{f^*\beta})$, and we get

$$(\beta, \beta) = Q \left(\beta + \frac{\lambda l(\beta)\theta_+}{\lambda^2 - 1}, \beta + \frac{\lambda l(\beta)\theta_+}{\lambda^2 - 1} \right) + \frac{l(\beta)^2 \, Q(\theta_+, \theta_+)}{(\lambda^2 - 1)^2} \geq 0.$$

Hence the intersection form is positive on V which contradicts the Hodge index theorem.

4. As θ_+, θ_- are nef classes, we have $(\theta_-, \theta_+) \geq 0$. If equality holds, then the intersection form is positive on $\mathbb{R}\theta_+ + \mathbb{R}\theta_-$. The Hodge index theorem then implies that $\theta_+ = \theta_-$ and that therefore $\theta_+^2 = 0$. Hence $Q(\theta_+, \theta_+) = 0$ and $(\theta_+, E_i) = 0$ by Equation (13). We now apply Equation (12) and conclude that

$$\lambda^2 \theta_+ = f_* f^* \theta_+ = \theta_+$$

which contradicts $\lambda > 1$.

5. This follows from the first four assertions and elementary linear algebra. $\square$

Theorem 0.4 is an immediate consequence of the following two results.

Theorem 5.3. *Assume that $f : X \circlearrowleft$ is bimeromorphic with $\lambda := \rho(f^*) > 1$ and $\theta_+^2 = 0$. Then there exists a proper modification $\pi : X \to \check{X}$ that conjugates f to an automorphism.*

Proof. From $(\theta_+, \theta_+) = 0$ we infer $(f^*\theta_+, f^*\theta_+) = \lambda^2(\theta_+, \theta_+) = 0$. Hence $Q(\theta_+, \theta_+) = 0$ by Corollary 3.4, and $(\theta_+, V) = 0$ for every $V \subset \mathcal{E}(f^{-1})$. From Equation (12), we get for all $k \geq 0$

$$(f_*)^k\theta_+ = \lambda^{-k}(f_*)^k(f^*)^k\theta_+ = \lambda^{-k}\theta_+.$$

Since f_* preserves $H^2(X, \mathbf{Z})$ and $\lambda^{-k} \to 0$, it follows that no multiple of θ_+ is an integral class. In particular, $\theta_+ \neq c\{V\}$ for any $V \subset \mathcal{E}(f^{-1})$.

Thus by the Hodge Index Theorem, $(V, V) < 0$ for every irreducible $V \subset \mathcal{E}(f^{-1})$. We apply Proposition 1.7 to contract a curve $V \subset \mathcal{E}(f^{-1})$. Let $\pi : X \to \check{X}$ denote the resulting modification and $\check{f} : \check{X} \circlearrowleft$ the induced map.

By Corollary 3.4 we have $(\pi_*\theta_+)^2 = 0$. Moreover, by (12) and the fact that $I(\pi) = \mathcal{E}(\pi^{-1}) = \emptyset$, we have

$$\check{f}^*(\pi_*\theta_+) = \pi_*f^*(\pi^*\pi_*)\theta_+ = \pi_*f^*\theta_+ = \lambda\pi_*\theta_+,$$

so that $\pi_*\theta_+(f) = \theta_+(\check{f})$. Therefore, either $\mathcal{E}(f) = \emptyset$ and f is an automorphism, or we can repeat the above argument to contract another curve. Since each contraction reduces $h^{1,1}(\check{X})$ by 1, the map $\check{f}$ will become an automorphism after finitely many contractions. $\qquad\square$

Theorem 5.4. *Let $f : X \circlearrowleft$ and $g : Y \circlearrowleft$ be AS bimeromorphic maps conjugate via a proper modification $\pi : X \to Y$. Assume $\lambda_1(f) > 1$ (or equivalently, $\lambda_1(g) > 1$). Then $\theta_+(f)^2 = 0$ if and only if $\theta_+(g)^2 = 0$.*

We point out that if f and g are conjugate by a map $\pi : X \to Y$ which is merely bimeromorphic, then as asserted in Theorem 0.4, the conclusion of Theorem 5.4 continues to hold. Indeed, both maps lift to the same map $h : \Gamma \circlearrowleft$ on a desingularization of the graph of h. After further blowing up Γ, we can arrange for h to be AS and then apply Theorem 5.4 to both pairs h, f and h, g.

Proof. For the sake of simplicity we let $\theta_f := \theta_+(f)$ and $\theta_g := \theta_+(g)$. We claim $\theta_g = \pi_*\theta_f$ (up to normalization). To see this, apply Equation (12) to π^{-1}. For any $k \geq 0$, we get

$$\begin{aligned}
\lambda_1^{-k}(g^k)^*(\pi_*\theta_f) &= \lambda_1^{-k}(\pi \circ f^k \circ \pi^{-1})^*(\pi_*\theta_f) \\
&= \lambda_1^{-k}\pi_*f^{k*}(\pi^*\pi_*\theta_f) = \pi_*\theta_f + \pi_*\lambda_1^{-k}f^{k*}D,
\end{aligned}$$

for some effective divisor D supported on $\mathcal{E}(\pi)$. Let k tend to infinity. The left side tends to a multiple of θ_g by Theorem 5.1, and the second term on the right tends to a non-negative multiple of $\pi_*\theta_f$. This justifies our claim.

Corollary 3.4 now gives $(\theta_g, \theta_g) = (\pi_*\theta_f, \pi_*\theta_f) \geq (\theta_f, \theta_f) \geq 0$. Hence $\theta_g^2 = 0$ implies $\theta_f^2 = 0$. Conversely assume $\theta_f^2 = 0$. We claim $(\theta_f, W) = 0$ for every irreducible component W of $\mathcal{E}(\pi)$. Given this, Corollary 3.4 yields $\theta_g^2 = (\pi_*\theta_f)^2 = 0$. To prove the claim, there are two cases to consider.

Case 1: there exists an integer $N \geq 0$ so that $\operatorname{Supp} f_*^N[W] \not\subset \mathcal{E}(\pi)$.

Let $\pi(W) = p$. Then $p \in I(g^N)$. As g is AS, this implies that for all $k \geq 1$

$$\operatorname{Supp} f^{k*}[W] \subset \mathcal{E}(\pi) \cup I(f).$$

That is, $f^{k*}[W]$ is a non-negative *integer* combination of irreducible components of $\mathcal{E}(\pi)$. Suppose first that $(f^{k*}[W], \theta_f) > 0$ for every k. Let $V \subset \mathcal{E}(\pi)$ be the irreducible component for which (V, θ_f) is non-zero but otherwise as small as possible. Since θ_f is nef,

$$0 < (V, \theta_f) \leq (f^{k*}[W], \theta_f) = (W, f_*^k\theta_f) = \lambda_1^{-k}(W, \theta_f),$$

where the last equality is proven as in Theorem 5.3. This is absurd for large k, so we must have instead that for k large enough,

$$0 = (f^{k*}[W], \theta_f) = \lambda_1^{-k}(W, \theta_f),$$

which justifies the claim in this case.

Case 2: for any $k \geq 0$, Supp $f_*^k[W] \subset \mathcal{E}(\pi)$.

If $(W, \theta_f) > 0$, then we can apply the fifth conclusion of Theorem 5.1 to f^{-1} and obtain

$$\lambda_1^{-k} f_*^k[W] \to c\, \theta_-(f)$$

for some $c > 0$. In particular, $\theta_-(f) = \sum c_j \{V_j\}$ where V_j are the irreducible components of $\mathcal{E}(\pi)$. But this means that $\pi_* \theta_-(f) = 0$, which cannot be because $\theta_-(f)$ is nef and Proposition 1.13 implies that $\pi^* \pi_* \theta_-(f) \geq \theta_-(f)$. Hence $(W, \theta_f) = 0$ as desired. $\square$

6. The Green current.

To any AS rational map $f : \mathbb{P}^n \circlearrowleft$ with algebraic degree $d > 1$, one can associate a positive closed $(1,1)$ current T on $\mathbb{P}^n$ satisfying $f^*T = dT$ (see [Sib]). This *Green current* is a fundamental tool for obtaining a measure theoretic understanding of the dynamics of a rational map. Here we use the results from the previous section to construct an invariant current for an AS bimeromorphic map of any compact Kähler surface.

Proof of Theorem 0.5. We follow ideas of [Gue] and the presentation in [Fav3], relying on the following volume estimate. Let $\mathcal{SA}(f)$ denote the (finitely many) attracting periodic points that lie in $\mathcal{E}(f)$.

Lemma 6.1 (see [Dil], [Fav3]). *Fix a small neighborhood Ω of $\mathcal{SA}(f)$. For any $\lambda > 1$, there exist constants $C_1, C_2 > 0$ s.t.*

$$(15) \qquad\qquad\qquad Vol\, f^k(E) \geq (C_1\, Vol\, (E))^{C_2 \lambda^k}$$

for all k and any Borel set $E \subset X \setminus f^{-k}\Omega$.

Now take a real smooth closed $(1,1)$ form α_+ such that $\{\alpha_+\} = \theta_+$. We do not require α_+ to be positive. In order to demonstrate convergence of the sequence of currents $\lambda_1^{-k}(f^k)^*\alpha_+$, write $f^*\alpha_+ = \lambda_1 \alpha_+ + dd^c h$ for some function $h \in L^1(X)$. The form $f^*\alpha_+$ is smooth outside the indeterminacy set $I(f)$, so h is also smooth outside $I(f)$. In particular, h is locally bounded in a neighborhood of $\mathcal{SA}(f)$.

We have

$$(16) \qquad\qquad \lambda_1^{-k}(f^k)^*\alpha_+ = \alpha_+ + dd^c \left(\sum_{i=1}^{k} \lambda_1^{-i} h \circ f^{i-1} \right).$$

To show that $\lambda_1^{-k}(f^k)^*\alpha_+$ converges, it is enough to prove that the series $\sum_i \lambda_1^{-i} h \circ f^{i-1}$ converges in $L^1(X)$. We estimate $\|h \circ f^{i-1}\|_{L^1(X)}$ using the the following lemma.

Lemma 6.2 (see [Fav3]). *Assume $f^*\alpha = \beta + dd^c h$, where α, β are real smooth closed $(1,1)$ forms and $h \in L^1(X)$. Then there exist constants $B, C > 0$ such that*

$$(17) \qquad\qquad\qquad Vol\, \{|h| > t\} \leq B \exp(-Ct)$$

for all $t \geq 0$.

We postpone the proof of this lemma to the end of the section.

Fix a small neighborhood Ω of $\mathcal{S}A(f)$ and let $A := \sup_{f^{-1}\Omega} |h| < +\infty$. We have

$$
\begin{aligned}
\int_X \lambda_1^{-i}|h \circ f^{i-1}|\, dV \;&=\; \int_{f^{-i}\Omega} \lambda_1^{-i}|h \circ f^{i-1}|\, dV + \int_{X\setminus f^{-i}\Omega} \lambda_1^{-i}|h \circ f^{i-1}|\, dV \\
&\leq\; \lambda_1^{-i}A + \lambda_1^{-i}\int_{t\geq 0} \mathrm{Vol}\,\left(\{h \circ f^{i-1} \geq t\} \cap X \setminus f^{-i}\Omega\right)\, dt \\
(\text{Lemma 6.1}) \quad &\leq\; \lambda_1^{-i}A + \lambda_1^{-i}\int_{t\geq 0} C_1^{-1}\mathrm{Vol}\,\{|h| \geq t\}^{1/C_2\lambda^i}\, dt \\
(\text{Lemma 6.2}) \quad &\leq\; \lambda_1^{-i}A + \lambda_1^{-i}\int_{t\geq 0} C_4 \exp(-C_3 t/\lambda^i)\, dt \leq \lambda_1^{-i}(A + C_5\lambda^i)\ ,
\end{aligned}
$$

which defines a summable sequence if we choose $\lambda < \lambda_1$. Hence $\sum \lambda_1^{-i}h \circ f^{i-1}$ converges in $L^1(X)$ to a function G and $\lim_{k\to\infty} \lambda_1^{-k}(f^k)^*\alpha_+ = \alpha_+ + dd^c G$.

The proof of convergence (1) when $\{\alpha\} \neq \theta_+$ is an arbitrary class in $H^{1,1}_{\mathbb{R}}(X)$ is similar. In particular, we have $\lambda_1^{-k}(f^k)^*\omega \to (\omega, \theta_+)T_+$ for any Kähler form. As $(\omega, \theta_+) \geq 0$, we infer that T_+ is a positive current. Since f^* acts continuously on positive currents, we also conclude that $f^*T_+ = \lambda_1 T_+$. $\qquad\square$

Proof of Lemma 6.2. If ω_X is the Kähler form on X, then $C\omega_X - (C\omega_X - \alpha)$ expresses α as a difference between two positive closed $(1,1)$ currents when C is large enough. Hence we can assume α is positive. But under this assumption, h is a quasi-plurisubharmonic function and (17) follows then from standard capacity arguments (see e.g. [Kis] Theorem 3.1). $\qquad\square$

The Green current T_+ has many other dynamically important properties. We conclude this section by stating several of these without proof. The interested reader will find full details presented in [Fav3].

Proposition 6.3. *Under the hypothesis of Theorem 0.5, we have:*

1. *$\mathrm{Supp}\,T_+ \subset J(f) := X \setminus \{p \in X : \{f^n\} \text{ is a normal family near } p\}$.*
2. *The current T_+ is an extremal current.*
3. *The Lelong number $\nu(x, T_+) > 0$ if and only if $f^n(x) \in I(f)$ for some $n \geq 0$. In particular, T_+ does not charge any complex curves.*

Remark 6.4. *When f is not AS, one can still show that the sequence $(f^*)^k\alpha/\rho(f^*)^k$ converges to a positive closed current. This current is supported on the countable union of curves that eventually map onto the indeterminacy set.*

When f is AS, one can show that the Green current represents the distribution of many positive closed $(1,1)$ currents. The following theorem makes this precise and completely describes (see [FG] or [Fav3]) the structure of the cone of positive closed $(1,1)$ currents satisfying $f^*T = \lambda_1 T$.

Theorem 6.5. *There exists an exceptional set $\mathcal{E}$ of finitely many critical periodic orbits $\mathcal{E} = \{\mathcal{O}(p_1), \cdots, \mathcal{O}(p_k)\}$ such that*

1. *for any (not necessarily smooth) $T \in \mathcal{C}_1^+(X)$, Equation (1) holds if and only if $\nu(x, T) = 0$ for all $x \in \mathcal{E}$.*
2. *to any point $p_i \in \mathcal{E}$, there is a unique analytic cycle V_i with positive coefficients and support $|V_i| \subset f^{-1}\mathcal{O}(p_i)$, such that $(V_i, \omega) = 1$ and $f^*[V_i] = \lambda_1[V_i]$;*
3. *we have $\{T \in \mathcal{C}_1^+(X) : f^*T = \lambda_1 T\} = \mathbb{R}_+ T_+ + \sum_i \mathbb{R}_+[V_i]$.*

7. Bimeromorphic maps and classification of surfaces.

In this section, we briefly indicate which surfaces can support the various types (i.e. classes [1] through [5] in Table 1) of bimeromorphic self-maps discussed in this paper. The case of automorphisms has been treated thoroughly by Cantat. We first state one of his results.

Proposition 7.1 (see [Can1]). *Let X be a compact Kähler surface. Assume $f \in \text{Aut}(X)$ satisfies $\lambda_1(f) > 1$. Then X is a Torus, a K3 surface, an Enriques surface or a non minimal rational surface.*

We remark that there are known examples of automorphisms with $\lambda_1 > 1$ on each of these types of surface (see [Can1]).

Bimeromorphic maps of class [5] are handled by

Theorem 7.2. *Let X be a compact Kähler surface. Assume $f : X \circlearrowleft$ is a bimeromorphic map with $\lambda_1(f) > 1$ and $\theta_+^2 > 0$. Then X is a rational surface.*

We postpone proving this fact until the end of this section, proceeding now to classes [2] and [3].

Remark 7.3. *Any ruled surface is birational to a product $C \times \mathbb{P}^1$ for some compact curve C. Let $g : C \to \mathbb{P}^1$ be any non-constant meromorphic map. Then $f(c, z) := (c, g(c)z) : C \times \mathbb{P}^1 \circlearrowleft$ belongs to Class [2]. Hence any ruled surface admits a birational map of Class [2].*

Remark 7.4. *There exists an elliptic surface not admitting any automorphism in Class [3]. Let $X := \mathbb{T} \times \mathbb{P}^1$ and pick $f \in \text{Aut}(X)$. The surface X admits a unique rational fibration which is hence preserved. By Lemma 4.5, $\|f^{n*}\|$ is bounded. This shows that f belongs to Class [1].*

The proof of Theorem 7.2 is based on the following classical proposition (see [IS] p.180).

Proposition 7.5. *Let X be a minimal compact complex surface with $\text{kod}(X) \geq 0$. Any bimeromorphic map $f : X \circlearrowleft$ is in fact a biholomorphism.*

Proof. Let Γ be a minimal desingularization of the graph of f and $\pi_1, \pi_2 : \Gamma \to X$ the projections onto first and second factors. Factor $\pi_1 = \varpi_1 \circ \cdots \circ \varpi_n$ into a sequence of point blow-ups. Let V_i be the exceptional component of ϖ_i and set $E_i := (\varpi_{i+1} \circ \cdots \circ \varpi_n)^*[V_i]$. We first note that the canonical class K_Γ of Γ can be related to K_X by the following formula (see [BPV] p.28):

$$(18) \qquad\qquad K_\Gamma = \pi_1^* K_X + \sum_i E_i .$$

Now fix an exceptional curve $C \subset \Gamma$. By the genus formula we get $(C, K_\Gamma) = 2g(C) - 2 - (C, C) = -3 < 0$. As $\text{kod}(X) \geq 0$ and X is minimal, the canonical class K_X is nef (see [BPV] p.73). Equation (18) shows then $\sum_i (E_i, C) = -(K_X, \pi_{1*}C) + (K_\Gamma, C) < 0$. This implies that C lies in the support of some E_i and is therefore blown down by π_1. The same argument shows that C is also blown down by π_2.

Since Γ was chosen to be a minimal desingularization, we conclude that no exceptional curve $C \subset \Gamma$ exists. Hence, π_1, π_2 are biholomorphisms, and we are done. $\square$

Proof of Theorem 7.2. Let $f : X \circlearrowleft$ be a bimeromorphic map with $\lambda_1 > 1$ and $\theta_+^2 > 0$. Theorem 0.4 and the preceding proposition show that $\text{kod}(X) = -\infty$. By the Enriques-Kodaira classification, X is either rational or ruled on a non rational base. In the former case we are done. In the latter, the surface admits a unique rational fibration. The map f preserves it and induces an automorphism of the base curve. We infer $\lambda_1 = 1$ which contradicts our assumption. $\square$

8. Periodic points of birational maps.

In this section, we show that λ_1 controls the growth of periodic orbits. The main idea comes from [Fav1] where a proof was given in the case $X = \mathbb{P}^2$. By Theorem 7.2, we can limit ourselves to the rational case.

Proof of Theorem 0.6. Pick an integer $k \geq 0$. Let $\Delta \subset X \times X$ denote the diagonal and $\Gamma_k \subset X \times X$ the graph of f^k. Let f_i^* denote pull-back on $H^i(X, \mathbb{R})$. The Lefschetz fixed point formula reads (see [Ful] p.314)

$$\{\Delta\} \cdot \{\Gamma_k\} = \sum_{i=0}^{2} (-1)^i \mathrm{Tr}((f^k)_i^*) \ .$$

As X is a rational surface we have $H_{\mathbb{R}}^{p,q}(X) = 0$ except for $p = q \in \{0, 1, 2\}$. On $H_{\mathbb{R}}^0(X) = \mathbb{R}$ we have $(f^k)_0^* = 1$, and on $H_{\mathbb{R}}^4(X) = \mathbb{R}$, we have $(f^k)_4^* = 1$ since f is birational. Hence Theorem 5.1 implies

$$\{\Delta\} \cdot \{\Gamma_k\} = 2 + \mathrm{Tr}((f^k)_2^*) = 2 + \mathrm{Tr}\left((f^k)^*|_{H_{\mathbb{R}}^{1,1}(X)}\right) = \lambda_1^k + O(1)$$

with $|O(1)| \leq h^{1,1}(X) + 1$.

We now follow the arguments of [Fav1] to estimate $\{\Delta\} \cdot \{\Gamma_k\}$. By assumption f^k admits no curves of periodic points, so the intersection $\Gamma_k \cap \Delta$ is finite and we have:

$$\{\Delta\} \cdot \{\Gamma_k\} = \mathrm{Per}_k + \sum_{p \in I(f^k)} \mu((p,p), \Gamma_k \cap \Delta)$$

where the last term denotes the multiplicity of intersection between Γ_k and Δ. The theorem follows now from

Lemma 8.1 ([Fav1])**.** *There exists a constant $D' > 0$ s.t.*

$$\sum_{p \in I(f^k)} \mu((p,p), \Gamma_k \cap \Delta) \leq D'$$

for any $k \geq 0$.

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

9. A class of examples.

In this section, we study a class of examples from [AABHM1]. By applying the main results of the present article, we are able to explain some experimental results presented in that paper. We will see that all cases described in Table 1, excepting Case [4], appear in this family (see the table below). We freely use the notation from previous sections. For the sake of brevity, we omit most proofs and computations, presenting sufficient detail for the reader interested in verifying them.

Given $\varepsilon \in \mathbb{C}$ and affine coordinates (x, y), the maps

$$f_\varepsilon(z, w) := \left(w + 1 - \varepsilon, \ z\frac{w - \varepsilon}{w + 1}\right) \qquad \text{and} \qquad f_\varepsilon^{-1}(z, w) := \left(w\frac{z + \varepsilon}{z - 1}, \ z - 1 + \varepsilon\right)$$

extend to $\mathbb{P}^1 \times \mathbb{P}^1$ as a birational self-map and its inverse. When $\epsilon = -1$, the map f_ϵ is an automorphism with $f_\epsilon^2 = \mathrm{Id}$. Otherwise the indeterminacy and exceptional sets of f_ϵ are given by

$$I(f) = \{(0, -1); (\infty, \varepsilon)\} \ , \ \mathcal{E}(f) = \{V_1, V_2\},$$

respectively, with $V_1 = \{w = \varepsilon\}$ and $V_2 = \{w = -1\}$.

To check whether f is AS, one has to compute the orbit of $\mathcal{E}(f)$. This is easily done using the fact that the line $\{z - w = 1\}$ is invariant under f, while the lines $\{z = \infty\}$, $\{w = \infty\}$ are interchanged.

Lemma 9.1. *For all $k \geq 0$, $f_\varepsilon^{k+1}(V_1) = (1 - k\varepsilon, -k\varepsilon)$. For all $k \geq 1$, $f_\varepsilon^{2k-1}(V_2) = (-1 + k(1 - \varepsilon), \infty)$; and $f_\varepsilon^{2k}(V_2) = (\infty, -1 + k(1 - \varepsilon))$.*

The map f_ε is AS if and only if $f_\varepsilon^k(V_i) \notin I(f)$ for all $k \geq 1$ and $i = 1, 2$. Hence

Lemma 9.2. *The map f_ε is AS in $\mathbb{P}^1 \times \mathbb{P}^1$ if and only if $\varepsilon \notin \mathcal{S}_1 \cup \mathcal{S}_2$ where*

$$\mathcal{S}_1 := \{1/k, \ k \in \mathbb{N}^*\},$$
$$\mathcal{S}_2 := \{k/k + 2, \ k \in \mathbb{N}\}.$$

The action of f^* is easily computed.

Proposition 9.3. *Assume $\varepsilon \neq -1$. In the natural basis $\{z = 0\}, \{w = 0\}$ of $H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$, f^* has matrix*

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

Thus if (in addition) $\varepsilon \notin \mathcal{S}_1 \cup \mathcal{S}_2$, we have $\lambda_1(f_\varepsilon) = (1 + \sqrt{5})/2$. One can take $\theta_+ = (2, 1 + \sqrt{5})$. In particular, $\theta_+^2 > 0$, and f_ε belongs to Class [5].

Note that since λ_1 is irrational, the map f_ϵ will almost never be AS as a self-map of $\mathbb{P}^2$. This is our principal reason for treating f_ϵ as a map of $\mathbb{P}^1 \times \mathbb{P}^1$ instead.

In the sequel, we consider non-generic parameters $\varepsilon \in \mathcal{S}_1 \cup \mathcal{S}_2$. Table 9 summarizes our results ($\varphi := (1 + \sqrt{5})/2$ denotes the golden ratio).

$\varepsilon = -1$	Aut $(\mathbb{P}^1 \times \mathbb{P}^1)$	Class [1]
$\varepsilon = 0, \ 1$	Rational fibration preserved	Class [2]
$\varepsilon = 1/2, \ 1/3$	Elliptic fibration preserved	Class [3]
$\varepsilon \notin \{1/k, \ k/(k+2), k \geq 0\}$	$\lambda_1 = \varphi, \ \theta_+^2 > 0$	Class [5]
$\varepsilon = 1/k, \ k \geq 4$	$\lambda_1^{k+1} = \lambda_1^{k-1} + \cdots + 1$ $1 < \lambda_1 < \varphi, \ \theta_+^2 > 0$	Class [5]
$\varepsilon = k/(k+2), \ k \geq 3$	$\lambda_1^{2k} = \lambda_1^{2k-2} + \cdots + 1$ $1 < \lambda_1 < \varphi, \ \theta_+^2 > 0$	Class [5]

The following remark will be useful in our analysis.

Lemma 9.4. *Let $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $\{(0,-1),(\infty,\varepsilon)\} = I(f_\varepsilon)$. Then for $\epsilon \neq 0$ the map $\tilde{f}_\varepsilon := f_\varepsilon \circ \pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic.*

Proof. One has to check that $\tilde{f}_\varepsilon$ contains no points of indeterminacy on the two exceptional divisors of π. We leave it to the reader. $\square$

If, consequently, X_ϵ is the surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ along each orbit segment $f(V_i), f^2(V_i), \ldots, f^j(V_i) \in I(f)$ that begins with the image of an exceptional curve V_i and ends with a point of indeterminacy, then f_ϵ lifts to an AS map of X_ϵ. For convenience, we introduce some additional notation. For $k \geq 0$ we let $E_k \subset X_\epsilon$ denote those exceptional components above points $f^{k+1}(V_1)$, and for $k \geq 1$, we let F_k denote the exceptional components above points $f^k(V_2)$. Finally, we let $L_z, L_w \in H^{1,1}(X_\varepsilon)$ denote the classes induced by the pullbacks $L_z := \pi^*\{z = 0\}$, $L_w := \pi^*\{w = 0\}$.

Let us first consider parameters $\varepsilon \in \mathcal{S}_1 \cap \mathcal{S}_2 = \{1/2, 1/3\}$.

Proposition 9.5. *Assume $\varepsilon \in \mathcal{S}_1 \cap \mathcal{S}_2$. Then $f_\varepsilon \in \text{Aut}(X_\varepsilon)$, $\lambda_1(f) = 1$, and $\|f_\varepsilon^{n*}\|$ is unbounded. In particular, there exists an invariant elliptic fibration and f_ε belongs to Class [3].*
* *For $\varepsilon = 1/2$, one has $h^{1,1}(X_\varepsilon) = 2 + 9$ and the fibration is given by*

$$d\left[\frac{(2zw + w + 1)(2zw - z + 1)(2zw + z - w - 1)}{(z - w - 1)^2}\right] = 0 \ .$$

With respect to the basis, $L_z, L_w, E_0, E_1, E_2, F_1, \cdots, F_6$, the action f^* has matrix

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \ ,
$$

and $\ker(f_{1/2}^* - Id) = \{(\alpha, \alpha, \beta, \beta, \beta, \gamma, \gamma, \gamma, \gamma, \gamma, \gamma) \text{ with } \alpha + \beta + \gamma = 0\}$. For a class $v \in \ker(f_{1/2}^* - Id)$, $v^2 \leq 0$ and $v^2 = 0$ if and only if $\alpha = -3\gamma$, $\beta = 2\gamma$.

* *For $\varepsilon = 1/3$, one has $h^{1,1}(X_\varepsilon) = 2 + 8$ and the fibration is given by*

$$d\left[\frac{(3zw + z + 1 + w)(3zw - z - w + 1)}{z - w - 1}\right] = 0 \ .$$

With respect to the basis, $L_z, L_w, E_0, E_1, E_2, E_3, F_1, F_2, F_3, F_4$, the action f^* has matrix

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} ,$$

and $\ker(f_{1/3}^* - Id) = \{(\alpha, \alpha, \beta, \beta, \beta, \beta, \gamma, \gamma, \gamma, \gamma)$ *with* $\alpha + \beta + \gamma = 0\}$*. For a class* $v \in \ker(f_{1/3}^* - Id)$*,* $v^2 \le 0$ *and* $v^2 = 0$ *if and only if* $\alpha = -2\beta = -2\gamma$*.*

There are other values for which f_ε is integrable.

Proposition 9.6.
- *For* $\varepsilon = -1$*,* $f(z, w) = (w + 2, z) \in$ *Aut* $(\mathbb{P}^1 \times \mathbb{P}^1)$ *belongs to Class [1].*
- *For* $\varepsilon = 0$*,* $f_0 = (w + 1, zw/(w + 1))$ *has unbounded degrees and preserves the rational fibration* $d(zw) = 0$*. It belongs to Class [2]. With respect to the basis* L_z, L_w, F_1, F_2*, the matrix of* f^* *is*

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0
\end{bmatrix} ,$$

  and $\det(f_0^* - tId) = t(t + 1)(t - 1)^2$*. Moreover* $\ker(f_0^* - Id) = (\alpha, \alpha, -\alpha, -\alpha)$*.*
- *For* $\varepsilon = 1$*,* $f_1 = (w, z(w - 1)/(w + 1))$ *has unbounded degrees and preserves the rational fibration* $d(zw/(z - w - 1)) = 0$*. It belongs to Class [2]. With respect to the basis* L_z, L_w, E_0, E_1*, the matrix of* f^* *is*

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0
\end{bmatrix} ,$$

  and $\det(f_1^* - tId) = t(t + 1)(t - 1)^2$*. Moreover* $\ker(f_1^* - Id) = (\alpha, \alpha, -\alpha, -\alpha)$*.*

We conclude by considering the cases $\varepsilon \in \mathcal{S}_1 \setminus \mathcal{S}_2$ and $\varepsilon \in \mathcal{S}_2 \setminus \mathcal{S}_1$.

Proposition 9.7. *Assume* $\varepsilon = 1/k$ *with* $k \ge 4$*. With respect to the basis* $L_z, L_w, E_0, \cdots, E_k$*, the matrix of* f^* *is*

$$\begin{bmatrix}
0 & 1 & 0 & & & & \\
1 & 1 & 1 & & & 0 & \\
0 & 0 & 0 & 1 & & & \\
& & & 0 & \ddots & & \\
& 0 & & & \ddots & 1 \\
& -1 & & & & 0
\end{bmatrix} ,$$

and $\det(f_\varepsilon^* - tId) = (-1)^{k+1} t(t - 1)(t^{k+1} - \sum_{j=0}^{k-1} t^j)$*. The first dynamical degree is an algebraic integer which is the unique real number of modulus greater than 1 such that* $\lambda^{k+1} = \sum_{j=0}^{k-1} \lambda^j$ *. One*

has $1 < \lambda_1(f_\varepsilon) < (1 + \sqrt{5})/2$ and one can take

$$\theta_+(f_\varepsilon) = (\lambda_1^k, \lambda_1^{k+1}, -1, -\lambda_1, \cdots, -\lambda_1^k) \ .$$

In particular $\theta_+^2 > 0$ and f_ε belongs to Class [5].

Proposition 9.8. *Assume $\varepsilon = k/(k+2)$ with $k \geq 3$. With respect to the basis $L_z, L_w, F_1, \cdots, F_{2k}$, the matrix of f^* is*

$$\begin{bmatrix} 0 & 1 & 0 & & & & \\ 1 & 1 & 1 & & 0 & & \\ 0 & 0 & 0 & 1 & & & \\ & & & 0 & \ddots & & \\ & 0 & & & \ddots & 1 \\ & -1 & & & & 0 \end{bmatrix} ,$$

and $\det(f_\varepsilon^* - tId) = t(t-1)(t^{2k} - \sum_{j=0}^{2k-2} t^j)$. The first dynamical degree is an algebraic integer which is the unique real number of modulus greater than 1 such that $\lambda^{2k} = \sum_{j=0}^{2k-2} \lambda^j$. One has $1 < \lambda_1(f_\varepsilon) < (1 + \sqrt{5})/2$ and one can take

$$\theta_+(f_\varepsilon) = (\lambda_1^{2k}, \lambda_1^{2k+1}, -1, -\lambda_1, \cdots, -\lambda_1^{2k}) \ .$$

Moreover $\theta_+^2 > 0$ and f_ε belongs to Class [5].

Remark 9.9. *If one takes $k = 5$, then λ_1 is a root of $P(t) = t^5 - t^3 - t^2 - t - 1$. One can check that P is irreducible in $\mathbb{Z}[t]$. Thus $f_{5/7}$ is an example of a birational map of $\mathbb{P}^2$ whose first dynamical degree is not a quadratic integer and is therefore not AS on any minimal rational surface.*

APPENDIX: AUTOMORPHISMS WITH UNBOUNDED DEGREES AND $\lambda_1 = 1$.

In this section, we give detailed proofs of Theorem 4.3 and Proposition 4.4. Let X be a compact Kähler surface, let $h^i(\mathcal{F}) := \dim_{\mathbb{C}} H^i(X, \mathcal{F})$ for any sheaf $\mathcal{F}$ of complex vector spaces on X and $i \geq 0$, and let $\chi(\mathcal{F})$ denote the Euler characteristic of $\mathcal{F}$. Given $f \in \mathrm{Aut}(X)$ such that $\|(f^n)^*\|$ is unbounded and $\lambda_1(f) = 1$, we first want to show that f preserves an elliptic fibration. Note that the intersection form is preserved by f^*.

Step 1: Up to positive multiple, there is a unique nef class $\theta \in H^{1,1}(X)$ such that $f^*\theta = \theta$. Moreover $\theta^2 = 0$, $(\theta, K_X) = 0$, and we can assume that $\theta \in H^2(X, \mathbb{Z})$.

Proof. Fix a Kähler class ω and let θ be a limit point for the sequence $\|(f^n)^*\omega\|^{-1}(f^n)^*\omega \longrightarrow \theta$. By construction, the class θ is nef. Since the nef cone is strict and $\|f^{n*}\omega\| \to \infty$ sub-exponentially, we see that $f^*\theta = \theta$.

Let $\alpha \in H^{1,1}(X)$ be another class (e.g. $\alpha = K_X$, the canonical class) satisfying $f^*\alpha = \alpha$. Then

$$0 = \lim_{n \to \infty} \frac{1}{\|(f^n)^*\omega\|}(\omega, \alpha) = \lim_{n \to \infty} \left(\frac{(f^n)^*\omega}{\|(f^n)^*\omega\|}, \alpha \right) = (\theta, \alpha).$$

By Hodge index theorem, we infer that either $\alpha = k\theta$ or $\alpha^2 < 0$. This shows that $H_{nef}^{1,1}(X) \cap \ker(f^* - \mathrm{Id})$ (in fact, $\{f^*\alpha = \alpha : \alpha^2 \geq 0\}$) is generated by θ.

To see that $\theta \in H^2(X, \mathbb{Z})$, we choose a basis $\{e_i\} \in H^2(X, \mathbb{Q})$ for which $f^* : H^2(X, \mathbb{Z}) \circlearrowleft$ is in Jordan form. As $\|(f^n)^*\omega\|^{-1}(f^n)^*\omega \longrightarrow \theta$ for any Kähler class, and $\|(f^n)^*\alpha\|$ is bounded for any class $\alpha \in H^{2,0}(X) \oplus H^{0,2}(X)$ (see the proof of Lemma 4.1), the set of classes attracted to θ under pullback is open. Hence θ is a multiple of the basis vector e_i corresponding to the 'top' of the

(unique) largest Jordan block for the eigenvalue one.   □

In particular, by Lefschetz' theorem $\theta = c_1(L)$ for some line bundle $L \in H^1(X, \mathcal{O}^*)$. We apply the Enriques-Kodaira classification of compact complex surfaces (see [BPV] p.188) to limit the possible candidates for X. If $\mathrm{kod}(X) = 2$, then $\mathrm{Aut}(X)$ is finite, which contradicts unboundedness of $\|f^{n*}\|$. If $\mathrm{kod}(X) = 1$, the Iitaka fibration defines an elliptic fibration which is $\mathrm{Aut}(X)$-invariant, and we are done. If $\mathrm{kod}(X) = 0$, then up to a finite cover one can assume X is a K3 surface or X is hyperelliptic. In the latter case, X admits a canonical elliptic fibration which is $\mathrm{Aut}(X)$-invariant. In the former case, $\chi(\mathcal{O}_X) = 2$. If $\mathrm{kod}(X) = -\infty$, X is ruled. If the base C is not rational, then any automorphism of X preserves the ruling and induces an automorphism of C, so f^* acts trivially on $H^{1,1}(X)$, contradicting our assumption. Hence X is rational and $\chi(\mathcal{O}_X) = 1$. To summarize we are left with two cases:

- either X is a K3 surface: K_X is trivial and $\chi(\mathcal{O}_X) = 2$;
- or X is rational: $\{K_X\}$ is not psef and $\chi(\mathcal{O}_X) = 1$.

Step 2: For any integer n large enough, $h^0(nL) \geq 2$.
Proof. The Riemann-Roch theorem and Serre duality yield

$$(19) \qquad h^0(L) + h^0(K_X - L) = \chi(\mathcal{O}_X) + h^1(L).$$

Assume first X is a K3 surface. If $h^0(K_X - L) \geq 1$, one infers $-\theta = \{-L\} = \{K_X - L\} \geq 0$ which is absurd. As $\chi(\mathcal{O}_X) = 2$, one concludes $h^0(L) \geq 2$.

Assume now that X is rational. For any integer $n \geq 0$, one has

$$(20) \qquad h^0(K_X - nL) = 0$$

as $\{K_X\}$ is not psef. Hence Equation (19) gives $h^0(nL) \geq 1$ for all $n \geq 0$.

If $h^0(L) \geq 2$ we are done. Assume $h^0(L) = 1$. The zeroes of the unique section of L define a curve C which satisfies $f^*C = C$. Note that C is not necessarily reduced or connected. The Riemann-Roch theorem and Serre duality for singular curves (see [BPV] p.51, 55 and 67) gives the genus formula

$$(21) \qquad 1 + \frac{1}{2}(C, C + K_X) = h^1(\mathcal{O}_C)$$

hence $h^1(\mathcal{O}_C) = 1$. For any $n \geq 0$ from the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(nC) \longrightarrow \mathcal{O}_X((n+1)C) \longrightarrow \mathcal{O}_C \longrightarrow 0 \ ,$$

we get the exact sequence

$$H^1((n+1)L) \longrightarrow H^1(\mathcal{O}_C) \longrightarrow H^2(nL) \ .$$

But $H^2(nL) \cong H^0(K_X - nL) = 0$ by (20) and $H^1(\mathcal{O}_C) \cong \mathbb{C}$. Hence $h^1(nL) \geq 1$ for all $n \geq 1$ and $h^0(2L) \geq 1 + 1 = 2$. We are done.   □

Step 3: Construction of the invariant elliptic fibration.

We can assume $h^0(L) \geq 2$. The space $H^0(L)$ induces a surjective meromorphic map $\pi : X \dashrightarrow \mathbb{P}^1$. Write D for the fixed part of the linear system $|L|$ and decompose a fiber $\pi^{-1}(c) = F_c + D$. As $L^2 = 0$ and L is nef, $(F_c, L) = 0$. But $F_c^2 \geq 0$ hence F_c is proportional to L and $F_c^2 = 0$. By removing D, we obtain a linear system without fixed part that we again denote by $|L|$. This system has no base points because $L^2 = 0$ and is f-invariant. We therefore obtain a surjective holomorphic map $\pi : X \to \mathbb{P}^1$ which defines a fibration. Using Stein factorization, we further produce a fibration

$\widetilde{\pi} : X \to C$ over some curve C with generically irreducible fibers. The genus formula implies that the fibration is elliptic, and since $f^*L = L$ the fibration is necessarily invariant.

Proof of Proposition 4.4.

Consider the restriction $f^*|_{\theta^\perp}$. Recall from Step 1 above that $\theta \in \theta^\perp$. Since $f^*\theta = \theta$, we obtain an induced map $g : \theta^\perp/\theta :\circlearrowleft$. The intersection form $(,)$ projects down to $\theta^\perp/\theta$ as a negative definite quadratic form $(,)|_{\theta^\perp/\theta}$. The action g preserves this form and the lattice $H^2(X, \mathbb{Z}) \cap \theta^\perp/\theta$. Hence $g^N = \mathrm{Id}$ for some N. Replacing f by f^N, we can assume $N = 1$. In the sequel we denote by $\varpi : \theta^\perp \to \theta^\perp/\theta$ the natural projection.

Observe that $x \mapsto \varpi(f^*x - x)$ defines a linear map $\pi : H^{1,1}(X) \to \theta^\perp/\theta$. We claim that $\pi \not\equiv 0$. If $\pi \equiv 0$, then $f^*x = x + l(x)\theta$ for some linear form $l \in (H^{1,1}(X))^*$. By Step 1 above, $\ker l \subset \theta^\perp$. Hence the previous paragraph gives that $\ker l = \theta^\perp$. One can therefore write $f^*x = x + c \cdot (x, \theta)\theta$ for some $c \in \mathbb{C}$. In fact, $(x, y) = (f^*x, f^*y) = (x, y) + 2c \cdot (x, \theta) \cdot (y, \theta)$ shows that $c = 0$. But $f^* \neq \mathrm{Id}$ so this is absurd. We conclude that $\pi \not\equiv 0$.

Pick a Kähler class ω for which $\pi(\omega) \neq 0$ and write $f^*\omega = \omega + h$ with $h \in \theta^\perp$. We have $f^*\omega = \omega + h_n$, where $h_n := \sum_{k=0}^{n-1} (f^k)^*h$. Since $f|_{\theta^\perp/\theta} = \mathrm{Id}$, there is $c \in \mathbb{C}$ such that $f^*h = h + c\theta$, so the sequence $\|(f^k)^*h\|$ grows at most linearly. Hence $\|h_n\|$ and (therefore as well) $\|f^{n*}\omega\|$ grow at most quadratically.

On the other hand, $\varpi(h_n) = n \, \varpi(h)$. So from

$$\frac{(f^n)^*\omega}{\|(f^n)^*\omega\|} = \frac{\omega}{\|(f^n)^*\omega\|} + \frac{h_n}{\|(f^n)^*\omega\|} \longrightarrow \theta$$

we deduce that $n\varpi(h)/\|(f^n)^*\omega\| \to 0$. Since $\varpi(h) \neq 0$ we infer $\|(f^n)^*\omega\|^{-1} = o(1/n)$. That is, $\|(f^n)^*\omega\|$ grows at least quadratically, too. $\qquad\square$

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