LIMITS ON AN EXTENSION OF CARLESON'S $\overline{\partial}$–THEOREM

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Abstract. We study a theorem due essentially to Carleson about solving the $\overline{\partial}$–equation on the unit disk. We show that this theorem generalizes to bordered Riemann surfaces with finitely generated fundamental groups. However, our main result is that the constant appearing in the generalized theorem cannot be taken to be independent of the bordered Riemann surface in question. We exhibit a sequence of (topologically equivalent) Riemann surfaces on which the constant tends to ∞. Since Carleson's $\overline{\partial}$–theorem depends on the notion of a Carleson measure, we also discuss Carleson measures at some length in order to define them appropriately on arbitrary Riemann surfaces.

0. Introduction

The work presented in this paper stems largely from our interested in the well–known

Corona Problem (CP). *Given a bordered Riemann surface Ω and a collection of holomorphic functions $f_1, \dots, f_n : \Omega \to \mathbb{C}$ that satisfy*

(i) $||f_j||_\infty \leq 1$

(ii) $|f_1| + \dots + |f_n| \geq \delta > 0,$

find another collection of holomorphic functions $g_1, \dots, g_n : \Omega \to \mathbb{C}$ such that

(iii) $||g_j||_\infty \leq C = C(n, \delta, \Omega)$

(iv) $f_1 g_1 + \dots + f_n g_n \equiv 1.$

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Among the research surrounding this problem, two lines of development concern us here. The first is the connection, noticed by Hörmander, between CP and theorems about solving the $\overline{\partial}$–equation with L^∞ control on the size of the solution. In the case $\Omega = \Delta =$ "the unit disk," there are now at least three known methods (see [Gar] chapter 8, and [BeRa]) of solving CP that capitalize on Hörmander's insight. In all three cases, the constant that arises in a $\overline{\partial}$–theorem directly influences the size of the $C(n, \delta, \Omega)$ obtained.

The second line of development that we wish to highlight has arisen in the study of CP on domains with complicated topology. Given the solvability of CP when $\Omega = \Delta$, several authors ([Sto], [For], [Gam1], etc.) have demonstrated the solvability of CP when Ω has finite topology—that is, when Ω has finite genus and finitely many boundary components. Gamelin has gone on to prove

Theorem (Gamelin). *Given $\Omega \subset \mathbb{C}$ with $k < \infty$ boundary components, CP is solvable on Ω with a constant $C = C(n, \delta, k)$. That is, the dependence of C on Ω is purely topological for finitely–connected, planar domains. Furthermore, CP is solvable for all planar domains if and only if C for k–connected domains can be chosen to be independent of k.*

Putting these two lines of development together, one might reason heuristically that to prove solvability of CP on Ω with infinitely many boundary componenets, one should search for an appropriate $\overline{\partial}$–theorem whose constants depend only on the genus of Ω. It is this possibility that we explore below. In particular, we study a $\overline{\partial}$–theorem which is essentially due to Carleson in the case $\Omega = \Delta$. We show (Theorem 2.3) that this theorem generalizes naturally to more complicated Ω. But negatively, and more importantly, we show (Theorem 3.3) that the constant appearing in the theorem necessarily depends on more than the genus (in fact, on more than the topology) of Ω. We construct a family of topologically equivalent

Ω for which the constant arising in Carleson's $\overline{\partial}$–theorem cannot possibly have an upper bound.

Remarks. *The Corona Problem as we have stated it here might more accurately be called "the Corona Problem with bounds." As it is usually stated, the condition (iii) is relaxed to require only that the g_j be bounded. C is allowed to depend on more than just Ω, δ, and n.*

Brian Cole [Gam2] *constructed a Riemann surface of infinite genus for which CP cannot be solved. Other authors have have demonstrated the solvability of CP for certain classes of Ω with infinite topology—for example, the so-called Denjoy domains* [JoGa] *and finite-sheeted covers of the unit disk* [HaNa]*. What is lacking is a good description of* **all** *Ω for which CP is solvable.*

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1. The Carleson Norm of a 1–Form

The notion of a Carleson measure has proven very useful in studying bounded analytic functions. In particular, one can use Carleson measures to define a large class of $(0,1)$–forms λ for which $\overline{\partial}b = \lambda$ has a bounded solution. Originally, Carleson defined his measures on the unit disk as follows (see page 238 of [Gar]):

Definition 1.1. *A positive measure λ on Δ is a* **Carleson measure** *if there is a constant $N(\lambda)$, called the* **Carleson norm** *of λ, such that*

$$\lambda(S) \leq N(\lambda)h$$

for every sector $S = \{re^{i\theta} : 1-h \leq r < 1, |\theta - \theta_0| \leq h\}$.

Analogous definitions of a Carleson measure can be formulated for any smoothly bounded planar domain. However, for the purposes that we intend to use Carleson measures, Definition 1.1 has two drawbacks. First of all, Carleson measures as defined above are not obviously conformally invariant objects. Secondly, since we will be dealing with one forms rather than measures, it will be more helpful to us to define the Carleson norm of a one form directly.

To circumvent these two difficulties, we turn to another, equivalent definition of Carleson norm. Note that if λ_1 and λ_2 are two one forms on a Riemann surface, then we can "multiply" them together to get an area measure in the following fashion: write the forms in local coordinates as $\lambda_j = a_j\, dz + b_j\, d\bar{z}$ and set

$$|\lambda_1||\lambda_2| = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}\frac{dz \wedge d\bar{z}}{i}.$$

One can check that this definition does not depend on the choice of local coordinates and that $|dz||d\bar{z}| = dz \wedge d\bar{z}/i$—which is what we should expect. Now we can state

Definition 1.2. *Let λ be a one form on a hyperbolic Riemann surface Ω. Let $\varphi : \Delta \to \Omega$ be a holomorphic universal covering map. Then the* **Carleson norm** *$N(\lambda)$ of λ is*

$$N(\lambda) = \sup_{z_0 \in \Delta} \int_\Delta \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2}\, |\varphi^* \lambda||dz|. \tag{1–1}$$

Since any automorphism of Δ is given by $T(z) = e^{i\theta}\frac{z - z_0}{1 - z\bar{z}_0}$, (1–1) is equivalent to

$$N(\lambda) = \sup_{T \in \mathrm{Aut}\,\Delta} \int_\Delta |(\varphi \circ T)^* \lambda||dz|. \tag{1–2}$$

Lemma VI.3.3 on page 239 of [Gar] shows that Definitions 1.1 and 1.2 are equivalent in the case of the unit disk. (1–2) makes it clear that the Carleson norm is conformally invariant and in particular, that the Carleson norm does not depend on the choice of the covering map φ. We point out in passing that the sort of conformal invariance exhibited by the Carleson norm is more natural to one forms than to area measures. This justifies to some extent our choosing to deal with the former rather than the latter in our definition.

Below we will need to actually estimate some Carleson norms. The following lemma, due essentially to Berndtsson and Ransford [BeRa], will help us obtain the estimates we seek.

Lemma 1.3. *Given λ and a bordered Riemann surface Ω, let $g^\Omega(z,z_0) = g^\Omega_{z_0}(z)$ be the Green's function on Ω with pole at z_0. Choose a meromorphic (0,1)–form ω on Ω, and let $\rho^2_\Omega\,\omega\otimes\bar\omega$ be the Poincaré (constant curvature -4) metric on Ω. Then*

$$N(\lambda) \le 2\sup_{z_0\in\Omega}\int_\Omega g^\Omega_{z_0}(z)\rho_\Omega(z)\,|\lambda||\omega|. \tag{1–3}$$

Proof. Following Berndtsson and Ransford, we first use the explicit forms of Green's function and the Poincaré metric to prove (1–3) on Δ. We then use the transformation properties of the integrand to prove (1–3) for arbitrary Ω. So assume for now that $\Omega = \Delta$.

Since the integral in (1–3) is independent of which ω we choose, set $\omega = dz$. In this case, recall that

$$g^\Delta_{z_0}(z) = \log\left|\frac{1-\bar z_0 z}{z-z_0}\right|,$$

and

$$\rho_\Delta(z) = \frac{1}{1-|z|^2}.$$

This gives

$$\begin{aligned}
2g^\Delta_{z_0}(z)\rho_\Delta(z) &= \frac{1}{1-|z|^2}\log\left|\frac{1-\bar z_0 z}{z-z_0}\right|^2. \\
&= \frac{1}{1-|z|^2}\log\left(\frac{1}{1-\frac{(1-|z_0|^2)(1-|z|^2)}{|1-z\bar z_0|^2}}\right) \\
&\ge \frac{1-|z_0|^2}{|1-z\bar z_0|^2},
\end{aligned}$$

for all $z, z_0 \in \Delta$. To obtain the last inequality, we have used that $\log \frac{1}{1-x} \geq x$ when $x < 1$. Thus, the right side of (1–3) is greater than the right side of (1–1). This finishes the proof for $\Omega = \Delta$.

Now suppose that Ω is any hyperbolic Riemann surface. Then from the above work, and the fact that the Carleson norm of λ on Ω is the same as the Carleson norm of $\varphi^*\lambda$ on Δ, we have

$$N(\lambda) \leq 2 \sup_{w_0 \in \Delta} \int_\Delta g^\Delta_{w_0} \rho_\Delta \, |\varphi^*\lambda||dw|.$$

Suppose that $\Omega_0 \subset \Delta$ is a fundamental polygon for Ω. Recall that the Green's function on Ω may be obtained from

$$g^\Omega_{\varphi(w_0)}(\varphi(w)) = \sum_{T \in G} g^\Delta_{w_0} \circ T(w).$$

Then

$$\begin{aligned} N(\lambda) &\leq 2 \sup_{w_0 \in \Delta} \sum_{T \in G} \int_{T(\Omega_0)} g^\Delta_{w_0} \rho_\Delta \, |dw||\varphi^*\lambda| \\ &= 2 \sup_{w_0 \in \Omega_0} \int_{\Omega_0} \sum_{T \in G} (g^\Delta_{w_0} \circ T) \, |(\varphi \circ T)^*(\rho_\Omega \omega)| \, |(\varphi \circ T)^*\lambda| \\ &= 2 \sup_{w_0 \in \Omega_0} \int_{\Omega_0} \left(\sum_{T \in G} g^\Delta_{w_0} \circ T \right) |\varphi^*(\rho_\Omega \omega)||\varphi^*\lambda| \\ &= 2 \sup_{z_0 \in \Omega} \int_\Omega g^\Omega_{z_0} \rho_\Omega \, |\omega||\lambda|. \end{aligned}$$

In taking the supremum in the second line, we were able to restrict ourselves to $w_0 \in \Omega_0$ because the integral in that line remains unchanged when w_0 is replaced by $T(w_0)$, $T \in G$. This concludes the proof. $\square$

2. Carleson's $\overline{\partial}$–theorem

In his solution of the Corona Problem, Carleson [Car] relied on a theorem about interpolation by bounded analytic functions. Later authors (for instance [Gar] p.

320 Theorem VIII.1.1) following Hörmander's insight have recast the interpolation theorem as a $\overline{\partial}$–theorem. We refer to the result as **Carleson's $\overline{\partial}$–theorem**, and we state it now.

Theorem 2.1. *Suppose λ is a (0,1) form on Δ with Carleson norm $N(\lambda) < \infty$. Then there is a solution $b \in C^\infty(\Delta) \cap C(\overline{\Delta})$ to*

$$\overline{\partial} b = \lambda$$

such that

$$M(b) = \sup_{z \in b\Delta} b(z) \leq CN(\lambda)$$

for some absolute constant C.

We emphasize that this theorem only gives control on b over the boundary of Δ and not necessarily over the interior of Δ. In practical applications to holomorphic functions, the maximum principle can usually be invoked at some point to give any require interior control, but as we shall see below, the a priori lack of interior bounds forces one to exercise a little more care at times.

With the help of a theorem of Forelli, one can easily generalize Theorem 2.1 from the setting of the unit disk to that of a bordered Riemann surface. So let Ω be any bordered Riemann surface with finite genus and finitely many boundary components. Let $\mathrm{H}^\infty(b\Omega) \subset \mathrm{L}^\infty(b\Omega)$ be those functions which are the boundary values of bounded holomorphic functions on Ω. Then Forelli's theorem [For] is

Theorem 2.2. *Given a holomorphic universal cover $\varphi : \Delta \to \Omega$, there is a bounded linear operator $P : \mathrm{L}^\infty(b\Delta) \to \mathrm{L}^\infty(b\Omega)$ satisfying*

$$P : \mathrm{H}^\infty(b\Delta) \to \mathrm{H}^\infty(b\Omega)$$

and

$$P(\varphi^* f \cdot g) = f \cdot P(g)$$

for all $f \in \mathrm{L}^\infty(b\Omega)$.

We call the operator P the **Forelli Projection**. By the maximum principle, P extends to a bounded linear operator from $\mathrm{H}^\infty(\Delta)$ to $\mathrm{H}^\infty(\Omega)$, though not necessarily to an operator between the corresponding L^∞ spaces. Earle and Marden [EaMa] showed that P can also be chosen to preserve continuity on the boundary—i.e. send continuous boundary functions to continuous boundary functions. Now we give the generalization of Carleson's $\overline{\partial}$–theorem.

Theorem 2.3. *Let* λ *be a (0,1) form on* Ω *with Carleson norm* $N(\lambda)$*. Suppose that there is a continuous function* $b : \overline{\Omega} \to \mathbb{C}$ *solving*

$$\overline{\partial} b = \lambda. \tag{2–1}$$

Then b *can be chosen so that*

$$M(b) = \sup_{z \in b\Omega} |b(z)| \leq CN(\lambda), \tag{2–2}$$

where C *is a constant that depends only on* Ω*.*

Proof. Let b_0 be the given bounded solution to (2–1). Let $\tilde{b} : \overline{\Delta} \to \mathbb{C}$ be the solution to $\overline{\partial}\tilde{b} = \varphi^*\lambda$ given by Theorem 2.1. We apply the Forelli projection operator P to obtain a function $b = b_0 + P(\tilde{b} - b_0 \circ \varphi)$. Note that we cannot apply P directly to $\tilde{b}$, except on $b\Delta$. On $b\Omega$, we have $|b| = |b_0 + P(\tilde{b} - b_0 \circ \varphi)| = |P(\tilde{b})| \leq CN(\lambda)$, where C is a constant depending only on the corresponding constant in Theorem 2.1 and the L^∞ norm of P. As noted above, we can arrange that P preserves continuity on the boundary, so we can assume that b is continuous on $b\Omega$. We also have that

$$\overline{\partial} b = \lambda + \overline{\partial}\, P(\tilde{b} - b_0 \circ \varphi) = \lambda,$$

since P preserves holomorphic functions. Thus b satisfies (2–1). □

3. Discussion, Setup, and Statement of the Main Result

Recall from our discussion of the Corona Problem in the introduction that it is useful to have $\overline{\partial}$–theorems like Theorem 2.3 in which the relevant constants are sensitive only to the genus of the Riemann surface Ω. Thus, we might hope that C in (2–2) varies only with genus. In our thesis ([Dil] page 50, Theorem 4.3), we showed that the L^∞ norm of the Forelli Projection depends on more than the topology of Ω, so it is clear that the above proof of Theorem 2.3 will not give us the C we hope for. However, the failure to yield a uniform C is not simply an artifact of the proof. The main goal of the rest of this paper is to exhibit a collection of topologically equivalent Ω for which the set $\{C(\Omega)\}$ of constants appearing Carleson's $\overline{\partial}$–theorem has no upper bound. First we fix some notation.

Let $\hat{\Omega}$ be a compact Riemann surface of genus $g > 0$. We wish to choose a reference (1,0) form on $\hat{\Omega}$. Unfortunately, we cannot choose such a form to be non-vanishing unless $\hat{\Omega}$ is a torus, but we can choose a holomorphic (1,0) form ω on $\hat{\Omega}$, such that ω has finitely many simple zeroes (see Appendix C of [Dil]) for a justification). Let Z be the zero set of ω. Let p be any point in $\hat{\Omega}\backslash Z$. We can choose a neighborhood W of p and coordinates z on W so that $\omega = dz$, p corresponds to $z = 0$, and W corresponds to a z-disk of radius R_0 about p. We will often abuse notation by referring to any point in Ω as 'z', including points outside W. For $R < R_0$ let Ω_R be the bordered Riemann surface obtained from $\hat{\Omega}$ by omitting a z-disk of radius R about p. Let $\rho_R^2\,\omega \otimes \bar{\omega}$ be the Poincaré metric on Ω_R. Notice that ρ_R will have singularities at the zeroes of ω, but that these singularities will at least be integrable. Finally, let $g^R(z, z_0) = g^R_{z_0}(z)$ be the Green's function on Ω_R with pole at $z_0 \in \Omega_R$.

Now we will define a (0,1) form λ, so that we can study the bounded solutions of (2–1) for λ on each Ω_R. Our choice of λ may seem obscure, but it is motivated by work presented in Chapter 3 of [Dil]. Let $\Omega_0 = \hat{\Omega} \setminus \{p\}$. Let $\rho_0^2\, \omega \otimes \bar{\omega}$ be the Poincaré metric on Ω_0. We define λ by

$$\omega \wedge \lambda = \partial\overline{\partial} \log \rho_0. \tag{3–1}$$

The fact that the Poincaré metric has Gauss curvature -4 translates into the partial differential equation $\partial\overline{\partial} \log \rho_0 = \rho_0^2 \omega \wedge \bar{\omega}$ (see [GrHa], page 77). Thus

$$\lambda = \rho_0^2 \bar{\omega}. \tag{3–2}$$

Technically speaking, λ is not a genuine (0,1) form on all of Ω, since ρ_0 has singularities at points in Z. Again however, these singularities are at least integrable. We will say that b is a bounded solution of $\overline{\partial} b = \lambda$ on Ω_R, if b is continuous and bounded on $\overline{\Omega}_R \setminus Z$, and b solves the equation in the classical sense on $\Omega_R \setminus Z$. Let N_R be the Carleson norm of λ on Ω_R, and let M_R be the infimum of the boundary L^∞ norms of all solutions b of $\overline{\partial} b = \lambda$ on Ω_R. We first prove a couple of lemmas that allow us to pass from results about λ to results about genuine (0,1) forms.

Lemma 3.1. *Suppose $M_R < \infty$. Then given any $\epsilon > 0$, there is a C^∞–smooth (0,1) form λ_ϵ on Ω_R with Carleson norm less than $N_R + \epsilon$, such that any solution b of $\overline{\partial} b = \lambda_\epsilon$ that is continuous on $\overline{\Omega}$ satisfies*

$$\max_{z \in b\Omega} |b(z)| \geq M_R - \epsilon.$$

Proof. We will need a sequence of C^∞–smooth cutoff functions χ_j defined as follows. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function satisfying $\chi(x) = 1$ if $|x| > 1$, and $\chi(x) = 0$ if $|x| < 1/2$. Given $z_0 \in Z$, local coordinates z near z_0, and large enough j, we require that $\chi_j(z) = \chi(j|z - z_0|)$ for all z in a neighborhood of z_0. Away from Z, we set $\chi_j \equiv 1$. These functions are designed to cancel the singularities of λ.

Since M_R is finite, we can find a bounded solution b to $\overline{\partial} b = \lambda$. Let $b_j = \chi_j b$. Note that each b_j is smooth and has the same boundary values as b. Let $\lambda_j = \overline{\partial} b_j$. Any solution $\tilde{b}_j$ of $\overline{\partial}\tilde{b}_j = \lambda_j$ differs from b_j by a holomorphic function. If $\tilde{b}_j$ is bounded on $b\Omega_R$, then $\tilde{b}_j - b$ is a bounded holomorphic function. So suppose that for some $\epsilon > 0$ and all j, there is $h_j \in \mathrm{H}^\infty(\Omega_R)$ such that

$$\sup_{z\in b\Omega_R} |b_j(z) + h_j(z)| \leq M_R - \epsilon;$$

Since all the b_j have the same boundary values, the h_j will be uniformly bounded on $b\Omega_R$. By the maximum principle, the h_j will be uniformly bounded on all of $\overline{\Omega}_R$. Let $h \in \mathrm{H}^\infty(\Omega_R)$ be the limit of a convergent subsequence of the h_j. Then we have $\overline{\partial}(b+h) = \lambda$ and

$$\sup_{z\in b\Omega_R} |b(z) + h(z)| \leq M_R - \epsilon.$$

This contradicts the definition of M_R and shows that the h_j cannot exist.

If we can also show that the Carleson norms of the λ_j tend toward the Carleson norm N_R of λ as j goes to ∞, we will be done. Set $\eta_j = \lambda - \lambda_j = (1-\chi_j)\lambda - b\overline{\partial}\chi_j$. Note that the support of η_j is bounded away from $b\Omega_R$ when j is large. Furthermore, the area of the support shrinks to 0 at the rate $1/j^2$. Since b is bounded, and $\overline{\partial}\chi_j$ increases only at the rate j, any L^p norm ($p < 2$) of $b\overline{\partial}\chi_j$ tends to 0 as j goes to ∞. λ has finite L^p norm when $p < 2$, so we conclude that for $p < 2$, the L^p norm of η_j shrinks to 0 as j goes to ∞. It will become clear in the work done to estimate II of (4–6) below, that these conditions on η_j are enough to ensure that the Carleson norm of η_j decreases to 0 as j goes to ∞. $\square$

Lemma 3.2. *There exists a bounded solution of*

$$\overline{\partial} b = \lambda \tag{3–3}$$

on Ω_R*. That is,* M_R *is finite.*

Proof. An unbounded solution of (3–3) is given by

$$b\omega = \partial \log \rho_0. \tag{3–4}$$

Any other solution will arise by adding a holomorphic function to this one. Let z_0 be any zero of ω. Since we have assumed that all such zeroes are simple, in local coordinates z (not the same as the local coordinates used near p), ω will have the form zdz. Consequently, in these coordinates ρ_0 will have the form $A/|z|$ for some bounded, real-analytic, positive function A. Plugging this into (3–4), we see that the unbounded part of b has the form $1/2z^2 + C/z$ near z_0. By the Mittag-Leffler Theorem for open Riemann surfaces, we can construct a meromorphic function h on Ω_0 with singular part exactly equal to the singular part of b near each point in Z. $b - h$ restricted to $\overline{\Omega}_R$ gives the bounded solution we seek. □

We wish to show that the constant C in Theorem 2.3 goes to infinity for Ω_R as R goes to 0. By the last two lemmas, we see that it is enough to show that N_R/M_R goes to 0 with R. As an aside, we note that the last two lemmas were only necessary because our reference form ω has zeroes. We pointed out above that if $\hat{\Omega}$ were a torus, then it would be possible to choose ω without zeroes. However, the need to choose a reference form illustrates an awkwardness in our construction. Namely, our choice of "data" λ is most naturally a (1,1) form (the Poincaré area form), but the "data" in Carleson's $\overline{\partial}$–theorem is a (0,1) form. Only with the help of the reference form can we pass back and forth between these two objects.

Now we give the main result of this paper.

Theorem 3.3.
$$\lim_{R\to 0} \frac{N_R}{M_R} = 0.$$

In particular, the constant C in Theorem 2.3 depends on more than just the topology of Ω.

4. Proof of the Main Result

Theorem 3.3 follows rapidly from the following two lemmas, whose proofs we defer for the moment. The symbol "$\lesssim$" is used to mean "less than a constant times"—where the suppressed constant is independent of R. The symbol "$\gtrsim$" will have the obvious similar meaning.

Lemma 4.1. *For small enough R,*

$$M_R \gtrsim \frac{1}{R}$$

Lemma 4.2. *For small enough R,*

$$N_R \lesssim |\log R| + \int_R^{R_0/2} \left| \frac{\log R}{r^2(\log r)^3} \right| \, dr.$$

Proof of Theorem 3.3, assuming Lemmas 4.1 and 4.2.

$$\begin{aligned}
\lim_{R\to 0} \frac{N_R}{M_R} &\lesssim \lim_{R\to 0} \frac{|\log R| + \int_R^{R_0/2} \left| \frac{\log R}{r^2(\log r)^3} \right| \, dr}{1/R} \\
&= \lim_{R\to 0} \frac{\int_R^{R_0/2} \left| \frac{1}{r^2(\log r)^3} \right| \, dr}{|R\log R|^{-1}} \\
&= \lim_{R\to 0} \left| \frac{R^{-2}(\log R)^{-3}}{-R^{-2}(\log R)^{-1} - R^{-2}(\log R)^{-2}} \right| \qquad \text{(by L'Hôpital's Rule)} \\
&= \lim_{R\to 0} |\log R|^{-2} \\
&= 0. \qquad \square
\end{aligned}$$

Now we prove the two lemmas.

Proof of Lemma 4.1. Let b be any bounded solution to (3–3) on Ω_R. Then

$$b\omega = \partial \log \rho_0 - h\,\omega \tag{4–1}$$

for some holomorphic function h on $\Omega_R \setminus Z$. We can also write

$$b = u\rho_0 \tag{4–2}$$

for some real analytic function u on Ω_R that is continuous on $\overline{\Omega}_R$. Let be V be a small open set containing Z. Then we have

$$\begin{aligned}
\left|\int_{bV} b\,\omega\right| &= \left|\int_{bV} h\,\omega - \partial \log \rho_0\right| \\
&= \left|\int_{b\Omega_R} -b\,\omega + \int_{b\Omega_R} \partial \log \rho_0 - \int_{bV} \partial \log \rho_0\right| \\
&\qquad \text{(by Cauchy's Theorem and (4–1))} \\
&\geq \left|\int_{\Omega_R \setminus \overline{V}} \rho_0^2\,\bar{\omega} \wedge \omega\right| - \left|\int_{b\Omega_R} u\rho_0\,\omega\right| \\
&\qquad \text{(by Stokes' Theorem and (4–2))} \\
&\geq 2A(\Omega_R \setminus \overline{V}) - C_1 L(b\Omega_R),
\end{aligned}$$

where $C_1 = \max_{z \in b\Omega_R} |u(z)|$, and the area A and length L are evaluated with respect to the Poincaré metric on Ω_0. If we let V shrink to Z in an appropriate fashion, then the left hand side of the calculation above goes to 0. This gives us that

$$C_1 \geq \frac{2A(\Omega_R)}{L(b\Omega_R)}. \tag{4–3}$$

We have reduced our problem to studying the behavior of this isoperimetric ratio. First note that the area term increases as R decreases, so we may assume that this term is more than or equal to some constant. In fact, it turns out that $A(\Omega_0)$ is finite, so that this estimate is sharp. Our task becomes studying the behavior of the length term as R decreases.

Suppose Ω_0 is biholomorphic to Δ/G_0 for some group G_0 of Moebius transformations. Let $T \in G_0$ correspond to traveling once around the puncture p (i.e. $z = 0$). Then $\Delta/\{T^j\}_{j=-\infty}^{\infty}$ is biholomorphic to $\Delta \setminus \{0\}$ and forms a covering space of Ω_0. Let $\tau : \Delta \setminus \{0\} \to \Omega_0$ be the corresponding covering map. Taking $\tau(0) = p$ extends τ to a holomorphic map on all of Δ. Traveling once about p corresponds to traveling once about $0 \in \Delta$, so $\tau'(0) \neq 0$. In particular, τ has a local inverse τ^{-1} defined at points $z \in \Omega_0$ near p. One can verify by finding an explicit covering of $\Delta \setminus \{0\}$ by the upper half plane that the Poincaré metric on $\Delta \setminus \{0\}$ is given by $(|w| \log |w|)^{-2}\, dw \otimes d\bar{w}$. Then in local coordinates near p, we see that

$$\begin{aligned} \rho_0(z) &= \left| \frac{(\tau^{-1})'(z)}{\tau^{-1}(z) \log |\tau^{-1}(z)|} \right| \\ &\approx -\frac{1}{|z| \log |z|}, \end{aligned} \tag{4–4}$$

since $\tau^{-1}(z) \approx z(\tau^{-1})'(0)$ near $z = 0$. Thus,

$$L_{Poin}(b\Omega_R) \approx \left| \frac{2\pi}{\log R} \right|. \tag{4–5}$$

From (4–2), (4–3), (4–4) and (4–5) we see that

$$\begin{aligned} M_R &= \max_{z \in b\Omega} |b(z)| \\ &\geq (\max_{z \in b\Omega_R} |u(z)|)(\min_{z \in b\Omega_R} \rho_0(z)) \\ &\gtrsim \log R \frac{1}{R \log R} = \frac{1}{R} \end{aligned}$$

□

The proof of Lemma 4.2 is somewhat longer, involving several more auxiliary lemmas. We need to estimate

$$\begin{aligned} N_R &\lesssim \sup_{z_0 \in \Omega_R} \int_{\Omega_R} g^R_{z_0} \rho_R \rho_0^2 |\omega|^2 \\ &\leq \underbrace{\sup_{z_0 \in \Omega_R} \int_{R<|z|<R_0/2} g^R_{z_0} \rho_R \rho_0^2\, |dz|^2}_{I} + \underbrace{\sup_{z_0 \in \Omega_R} \int_{\Omega_{R_0/2}} g^R_{z_0} \rho_R \rho_0^2\, |\omega|^2}_{II}. \end{aligned} \tag{4–6}$$

In writing I we have taken advantage of the local coordinates near p. We deal with this integral first, using two lemmas. From now on, let $r = |z|$, for z near p.

Lemma 4.3. *On the set* $\{R < |z| < R_0/2\}$, *we have*

$$\rho_0(z) \lesssim \left|\frac{1}{r\log r}\right| \qquad \textit{and} \qquad \rho_R(z) \lesssim \left|\frac{\log R}{r\log(r/R)\log r}\right|.$$

Proof. We use the fact the Poincaré metric is larger on smaller domains to obtain both estimates. First, if $\tilde{\rho}_0^2\, dz \otimes d\bar{z}$ is the Poincaré metric on $\{0 < |z| < R_0\}$, then $\rho_0 \lesssim \tilde{\rho}_0$. As in the proof of Lemma 4.1, one can verify that $\tilde{\rho}_0 = \left|\frac{1}{r\log(r/R_0)}\right|$. Thus, for $r < R_0/2$, $\tilde{\rho}_0$ is comparable to the right side of the first estimate in the lemma.

Second, $\rho_R < \tilde{\rho}_R$ where $\tilde{\rho}_R^2\, dz \otimes d\bar{z}$ is the Poincaré metric on $\{R < |z| < R_0\}$. One can use an explicit covering map from the upper half–plane to $\{R < |z| < R_0\}$ to calculate that

$$\begin{aligned}
\tilde{\rho}_R(z) &= \left|\frac{1}{r\log(R_0/R)\sin\left(\frac{\pi\log(r/R)}{\log(R_0/R)}\right)}\right| \\
&\approx \left|\frac{\log(R_0/R)}{r\log(r/R)\log(R_0/r)}\right| \qquad \text{because } \sin\pi x \approx x(1-x) \\
&\approx \left|\frac{\log R}{r\log(r/R)\log r}\right| \qquad \text{for } R < r < R_0/2,
\end{aligned}$$

which gives the second estimate. $\square$

Lemma 4.4.

$$\int_0^{2\pi} g_{z_0}^R(re^{i\theta})\, d\theta \leq \log\frac{r}{R}, \tag{4–7}$$

where the suppressed constant is independent of z_0.

Proof. Let h be any harmonic function whose domain of definition includes an annulus centered at 0. According to Ahlfors ([Ahl], section 4.6.2),

$$\int_0^{2\pi} h(re^{i\theta})\, d\theta = \alpha\log r + \beta, \tag{4–8}$$

for some constants α and β. Furthermore, let γ be a loop homologous—in the domain of h—to the boundary components of the annulus. Then

$$\alpha = \frac{1}{2\pi} \int_\gamma *dh,$$

where $*dh$ denotes the conjugate differential of h. If $r > |z_0|$ then $\{|z| = r\}$ is homologous to 0 in $\hat{\Omega} \setminus \{z_0\}$. So if $r > |z_0|$, the constant α in (4–8) turns out to be 0, and we are left with

$$\frac{1}{2\pi} \int_0^{2\pi} g^R_{z_0}(re^{i\theta})\, d\theta = \beta,$$

On the other hand, if $|z_0| > r$, the circle of radius r about 0 is homologous to any simple loop about z_0. Abusing notation, we suppose that z is a local coordinate near z_0. Since

$$g^R_{z_0}(z) = \log|z - z_0| + h(z), \qquad h \text{ harmonic},$$

for z near z_0, we see that

$$\begin{aligned} \alpha &= \frac{1}{2\pi} \int_{|z-z_0|=\epsilon} *d \log|z - z_0| \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1. \end{aligned}$$

Combining this with (4–8) and the fact that $g^R_{z_0}(Re^{i\theta}) = 0$, we see that

$$\frac{1}{2\pi} \int_0^{2\pi} g^R_{z_0}(re^{i\theta})\, d\theta = \log \frac{r}{R}$$

when $r < |z_0|$. The integral in (4–7) will at least be continuous across $r = |z_0|$, so

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} g^R_{z_0}(re^{i\theta})\, d\theta &= \begin{cases} \log \dfrac{r}{R}, & r \le |z_0| \\ \log \dfrac{|z_0|}{R}, & |z_0| < r < R_0 \end{cases} \\ &\le \log \frac{r}{R}. \end{aligned}$$

□

The upshot of Lemmas 4.3 and 4.4 is that

$$
\begin{aligned}
|I| &= \left| \sup_{z_0\in\Omega_R} \int_0^{2\pi} \int_R^{R_0/2} g_{z_0}^R \rho_R \rho_0^2 \, r dr d\theta \right| \\
&\lesssim \sup_{z_0\in\Omega_R} \int_R^{R_0/2} \left| \frac{\log R}{r^2 \log(r/R)(\log r)^3} \right| \int_0^{2\pi} g_{z_0}^R (re^{i\theta}) \, d\theta dr \\
&\lesssim \int_R^{R_0/2} \left| \frac{\log R}{r^2 (\log r)^3} \right| \, dr.
\end{aligned} \tag{4–9}
$$

We now turn our attention to II of (4–6), recalling that

$$II = \int_{\Omega_{R_0/2}} g_{z_0}^R \rho_R \rho_0^2 \, |\omega|^2.$$

Let dV be any finite, non-degenerate volume form on $\hat{\Omega}$. $\overline{\Omega}_{R_0/2}$ is compactly contained in Ω_0, so the Poincaré volume element $\rho_0^2|\omega|^2$ will be uniformly comparable to dV on $\Omega_{R_0/2}$. That is,

$$II \approx \int_{\Omega_{R_0/2}} g_{z_0}^R \rho_R \, dV,$$

and this comparability is independent of R. The Poincaré metric on Ω_R shrinks as R decreases, so for small enough R, we get

$$II \lesssim \int_{\Omega_{R_0/2}} g_{z_0}^R \rho_{R_0/4} \, dV.$$

Since $\overline{\Omega}_{R_0/2} \subset\subset \Omega_{R_0/4}$, we can almost say that $\rho_{R_0/4}$ is bounded on $\Omega_{R_0/2}$. However, we must allow for singularities of $\rho_{R_0/4}$ at the zeroes Z of ω. The zeroes of ω are simple, so these singularities will be on the order of simple poles. What we can conclude, therefore, is that $(\rho_{R_0/4})^t$ is integrable for all $t < 2$. Likewise, the singularity of $g_{z_0}^R$ is logarithmic, so $(g_{z_0}^R)^s$ is integrable for all $s < \infty$. By Hölder's inequality

$$II \lesssim ||g_{z_0}^R||_{\mathrm{L}^s} ||\rho_{R_0/4}||_{\mathrm{L}^t} \lesssim ||g_{z_0}^R||_{\mathrm{L}^s},$$

for all $2 < s < \infty$, and $s^{-1} + t^{-1} = 1$. Note that both norms are evaluated over $\Omega_{R_0/2}$ and with respect to dV.

Our aim now is to show that for any $s < \infty$,

$$||g^R_{z_0}||_{\mathrm{L}^s} \lesssim |\log R|, \tag{4–10}$$

with a suppressed constant that is independent of z_0. Once we demonstrate this, our estimates (4–9) and (4–10) will combine to give us Lemma 4.2. We will prove (4–10) by comparing $g^R_{z_0}$ with another function g_{z_0}, harmonic on all of $\hat{\Omega}$—except for a positive logarithmic singularity at z_0 and a negative logarithmic singularity at p.

Another way of describing g_{z_0} is to say that g_{z_0} has a Laplacian which is zero everywhere except for a negative point mass at z_0 and a positive point mass at p; i.e. g_{z_0} solves the equation

$$4\,\partial\overline{\partial} g_{z_0} = -\delta_{z_0} + \delta_p, \tag{4–11}$$

where δ_z is the point mass at z. We note that this equation is at least consistent in the sense that both sides have the same degree as currents on a Riemann surface. That is, both sides belong naturally to the dual space of C^∞-functions—the right side by definition, and the left side as a (1,1) form which can be multiplied by a function and integrated over $\hat{\Omega}$. The advantages to finding a solution to (4–11) are that the solution will be independent of R, and that we can arrange that the solution will vary continuously with z_0 in an appropriate sense. The difference between g_{z_0} and $g^R_{z_0}$ will be harmonic on Ω_R (even at z_0). Therefore, it will be easier to study than $g^R_{z_0}$ alone.

Lemma 4.5. *We can solve (4–11) on $\hat{\Omega}$ in such a way that*

(i) $z_0 \mapsto g_{z_0}(\cdot)$ *is continuous as a map from $\hat{\Omega}$ to $L^p(\hat{\Omega})$, $p < \infty$.*

(ii) *If $U \subset \hat{\Omega}$ is open and $z_0, p \notin \overline{U}$, then g_{z_1} converges uniformly to g_{z_0} on U as $z_1 \to z_0$.*

(iii) *If the set U in (ii) contains z_0, and z is a local coordinate on U, then $g_{z_1}(z)+\log|z-z_1|$ converges uniformly to $g_{z_0}+\log|z-z_0|$ on U as $z_1 \to z_0$. Similar statements hold if p, or both p and z_0, are in U.*

Proof. The total mass of the right hand side of (4–11) is zero. We also have that $-\delta_{z_0}+\delta_p$ is in the Sobelev space $\mathcal{W}^2_{-1-\epsilon}$ for all $\epsilon > 0$ (see [Tay] page 20). Thus, according to the Hodge Theory (see [GrHa], for example), one can find a linear operator solving (4–11) such that

$$||g_{z_0}||_{\mathcal{W}^2_{1-\epsilon}} \lesssim ||-\delta_{z_0}+\delta_p||_{\mathcal{W}^2_{-1-\epsilon}},$$

The kernel of the Laplacian on a compact Riemann surface consists only of constants, so we may suppose that the solution operator is independent of ϵ. A refined version of the Sobolev embedding theorem (page 20 of [Tay] again) then gives us

$$||g_{z_0}||_{\mathrm{L}^s} \lesssim ||-\delta_{z_0}+\delta_p||_{\mathcal{W}^2_{-1-\epsilon}},$$

for any $s < s(\epsilon)$, where $s(\epsilon) \to \infty$ as $\epsilon \to 0$. Since $||-\delta_{z_0}+\delta_p||_{\mathcal{W}^2_{-1-\epsilon}}$ depends continuously on z_0, we obtain statement (i) of the lemma. As a special case of (i), we get the L^2 continuity of g_{z_0} as a function of z_0. (ii) and (iii) follow fairly rapidly from the observation that L^2 control of harmonic functions implies pointwise control on compact sets [Hor]. We omit the details. □

Let g_{z_0} be the solution to (4–11) guaranteed by the last lemma. Since g_{z_0} and $g^R_{z_0}$ have the same singular part at z_0, we can write

$$g^R_{z_0} = g_{z_0} + h^R_{z_0}, \tag{4–12}$$

where $h^R_{z_0}$ is a harmonic function of z. Suppose that $z_0 \notin \{|z| < R_0\} \subset \hat{\Omega}$. Then by Lemma 4.5, there is another function h_{z_0}, harmonic on $\{|z| < R_0\}$, continuous on $\{z_0 \in \overline{\Omega}_{R_0}\}$, and satisfying

$$g_{z_0}(z) = \log|z| + h_{z_0}(z).$$

Using the fact that $g^R_{z_0}(z) = 0$ for $|z| = R$, we see that

$$|h^R_{z_0}(z)| = |\log|z| + h_{z_0}(z)| \lesssim |\log R| \tag{4–13}$$

for $|z| = R$. The suppressed constant is independent of z_0 because $h_{z_0}(z)$ is continuous in both z_0 and z and will take a maximum at some $(z_0, z) \in \overline{\Omega}_{R_0} \times \{|z| = R\}$. Applying the maximum principle to $h^R_{z_0}$, we see that (4–13) holds for all $z \in \Omega_R$.

We still need to estimate $h^R_{z_0}$ when $R < |z_0| < R_0$. To accomplish this, we note that the difference $g^{R/2}_{z_0} - g^R_{z_0}$ is harmonic on Ω_R and strictly positive on $b\Omega_R$. Then for small R, the maximum principle implies that

$$|h^R_{z_0}| = |g^R_{z_0} - g_{z_0}| < |g^{R/2}_{z_0} - g_{z_0}| = |h^{R/2}_{z_0}|. \tag{4–14}$$

If $R < |z| < R_0$, then Lemma 5.13 gives us a function h_{z_0} that is harmonic for $|z| < R_0$ and continuous for $|z_0| \le R_0$, and satisfies

$$g_{z_0}(z) = \log|z| - \log|z - z_0| + h_{z_0}(z).$$

We obtain from this equation and the boundary condition on $g^{R/2}_{z_0}$, that

(4–15)

$$\begin{aligned} |h^{R/2}_{z_0}(z)| &= |\log|z| - \log|z - z_0| + h_{z_0}(z)| \\ &\le |\log(R/2)| + |\log(R/2)| + k_1, \qquad k_1 \text{ independent of } z_0 \text{ and } z \\ &\lesssim |\log R|, \end{aligned}$$

when $|z| = R/2$ and R is small. The maximum principle, (4–13), (4–14), and (4–15) together tell us that $|h^R_{z_0}(z)| \lesssim |\log R|$ for all $z, z_0 \in \Omega_R$. We combine this fact with (4–12) to get

$$|g^R_{z_0}(z)| \lesssim |g_{z_0}(z)| + |\log R|, \qquad z, z_0 \in \Omega_R.$$

Consequently,

$$||g^R_{z_0}||_{\mathrm{L}^s} \lesssim ||g_{z_0}||_{\mathrm{L}^s} + |\log R|.$$

But $||g_{z_0}||_{\mathrm{L}^s}$ is a continuous function of z_0 whenever $s < \infty$. In particular, it takes a maximum for some $z_0 \in \hat{\Omega}$. Thus, for $s < \infty$ and small R

$$||g^R_{z_0}||_{\mathrm{L}^s} \lesssim |\log R|.$$

This is the inequality (4–10) that we wanted to prove.

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