A NEW CONSTRUCTION OF RIEMANN SURFACES WITH CORONA

David E. Barrett and Jeffrey Diller

University of Michigan
Indiana University

1. Introduction

An open Riemann surface X is said to *satisfy the corona theorem* if for every collection $f_1, \dots, f_n$ of holomorphic functions on X such that

$$\delta^2 < |f_1|^2 + \cdots + |f_n|^2 < \delta^{-2} \text{ for some } \delta > 0$$

there are bounded holomorphic functions $g_1, \dots, g_n$ on X such that

$$f_1 g_1 + \cdots + f_n g_n \equiv 1;$$

equivalently [Gar,VIII.2], the *corona* $\mathcal{M}(X) \setminus \overline{\iota(X)}$ is empty. (Here $\mathcal{M}(X)$ is the maximal ideal space of the algebra $H^\infty(X)$ of bounded holomorphic functions on X and ι is the natural inclusion $X \hookrightarrow \mathcal{M}(X)$.) If X does not satisfy the corona theorem then X may be said to *have corona.*

Riemann surfaces known to satisfy the corona theorem include the unit disk [Car], bordered Riemann surfaces [All] [Sto], and various classes of planar domains [GaJo] [Moo]. The question of whether general planar domains satisfy the corona theorem is open.

The first construction of a Riemann surface with corona is due to Cole [Gam].

The goal of this paper is to prove the following.

Theorem 1.1. *Let K be a compact subset of the Riemann sphere $\hat{\mathbb{C}}$ with positive logarithmic capacity and zero length. Then the homology cover $\widetilde{(\hat{\mathbb{C}} \setminus K)}_{\mathrm{Hom}}$ has corona.*

Here by *length* we mean one-dimensional Hausdorff measure. Also, the *homology cover* $\tilde{X}_{\mathrm{Hom}}$ of a Riemann surface X is the essentially unique covering space

---

1991 *Mathematics Subject Classification.* Primary: 30H50; secondary 30F45, 32H50.

First author supported in part by a grant from the National Science Foundation.

whose fundamental group is the commutator subgroup $\pi_1'(X)$ of $\pi_1(X)$. Since the commutator subgroup is normal, $\tilde{X}_{\mathrm{Hom}}$ is a regular covering of X with deck group naturally isomorphic to $\pi_1(X)/\pi_1'(X) = H_1(X)$.

$\tilde{X}_{\mathrm{Hom}}$ is the smallest covering space of X on which every harmonic function on X has a well-defined harmonic conjugate [For, 28.6]

We will denote by ρ_X the projection $\tilde{X}_{\mathrm{Hom}} \to X$.

Any compact $K \subset \mathbb{C}$ with Hausdorff dimension strictly between 0 and 1 will satisfy the hypotheses of Theorem 1.1 [Tsu, Thm. III.19].

Theorem 1.1 is proved in Section 2. In Section 3 we show that the unsolvable corona problem used in the proof of Theorem 1.1 arises naturally in the study of multivariate dynamics. Indeed, the map F defined in Section 2 was inspired by contemplation of the paper [HuPa] of Hubbard and Papadopol.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 will be based on the following lemmata.

Lemma 2.1. *Let $F = (f_1, f_2)$ be a holomorphic map from a Riemann surface X into the shell*

$$S_\delta \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C}^2 : \delta < \|z\| < \delta^{-1}\}.$$

Let $\mathcal{W}_F$ denote the Wronskian form

$$f_2\, df_1 - f_1\, df_2$$

and let

$$\omega_F = 2i \frac{\mathcal{W}_F \wedge \overline{\mathcal{W}_F}}{\|F\|^4}.$$

Then bounded holomorphic functions g_1, g_2 solving

$$f_1 g_1 + f_2 g_2 \equiv 1 \tag{2.1}$$

exist on X if and only if there is a $(1,0)$-form η on X solving the $\overline{\partial}$ problem

$$\overline{\partial}\eta = \omega_F \tag{2.2}$$

and satisfying a bound

$$|\eta \wedge \overline{\eta}| \le 2C^2 |\omega_F| \tag{2.3}$$

for some positive constant C.

Note that F induces a map $X \to \hat{\mathbb{C}} = \mathbb{CP}^1$ given by the meromorphic function f_1/f_2 or (in homogeneous projective coordinates) by $[f_1 : f_2]$. The form ω_F is simply the area form corresponding to the pullback $F^*(ds_{\mathrm{sphere}})$ to X of the spherical metric

$$ds_{\mathrm{sphere}} = 2\frac{|z_2\,dz_1 - z_1\,dz_2|}{|z_1|^2+|z_2|^2}$$

on $\hat{\mathbb{C}}$. The bound (2.3) simply says that η is bounded relative to the (possibly degenerate) metric $F^*(ds_{\mathrm{sphere}})$; we will rewrite (2.3) as

$$|\eta| \le CF^*(ds_{\mathrm{sphere}}). \tag{2.4}$$

For study and application of the corresponding $\overline{\partial}$ problem for hyperbolic metrics see [Dil1], [BaDi], [Dil2].

Lemma 2.2. *Let ds be a (possibly degenerate) conformal metric on a Riemann surface X and let $\omega = ds^2$ be the corresponding area form. Suppose that for some constant $C>0$ there is a $(1,0)$-form η on X solving*

$$\overline{\partial}\eta = \omega$$

and satisfying

$$|\eta| \le C\,ds.$$

Then every relatively compact subdomain $\Omega \subset X$ with piecewise smooth boundary satisfies the linear isoperimetric estimate

$$\mathrm{area}(\Omega) \le C\,\mathrm{length}(\mathrm{b}\,\Omega). \tag{2.5}$$

Lemma 2.3. *Let ds be a (possibly degenerate) conformal metric on a Riemann surface X and let $\omega = ds^2$ be the corresponding area form. Suppose that for some constant $C>0$ there is a $(1,0)$-form $\tilde{\eta}$ on $\tilde{X}_{\mathrm{Hom}}$ solving*

$$\overline{\partial}\tilde{\eta} = \rho_X^*\omega$$

and satisfying

$$|\tilde{\eta}| \le C\rho_X^* ds.$$

Then there is a $(1,0)$-form η on X solving

$$\overline{\partial}\eta = \omega$$

and satisfying

$$|\eta| \le C\,ds.$$

Proof of Theorem 1.1. Since K has positive logarithmic capacity, there is a harmonic function h on $\mathbb{C}\setminus K$ such that h is bounded off of a neighborhood of

∞ and $h + \log|\zeta|$ has a removable singularity at ∞ [Tsu, Thm. III.12]. Set $\delta = \exp(-\|h(\zeta) + \frac{1}{2}\log(|\zeta|^2+1)\|_{L^\infty(\mathbb{C}\setminus K)})$.

In a neighborhood of each point in $\hat{\mathbb{C}}\setminus K$ (including ∞) we can find a holomorphic function with absolute value e^h. Continuation of such a function along a loop γ will multiply the function by a unimodular constant λ_γ. In particular, continuation along a commutator loop $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ is trivial; hence we can find a well-defined holomorphic function H on $\widetilde{(\hat{\mathbb{C}}\setminus K)}_{\mathrm{Hom}}$ with $\log|H| = h \circ \rho_{\hat{\mathbb{C}}\setminus K}$. ($H$ will have simple zeros at points lying over ∞.)

We define the map

$$\begin{aligned} F = (f_1, f_2) : \widetilde{(\hat{\mathbb{C}}\setminus K)}_{\mathrm{Hom}} &\to S_\delta, \\ \zeta &\mapsto (\rho_{\hat{\mathbb{C}}\setminus K}(\zeta)H(\zeta), H(\zeta)). \end{aligned}$$

(F is holomorphic since the poles of $\rho_{\hat{\mathbb{C}}\setminus K}$ are canceled by the zeros of H.) Note that the induced map $\widetilde{(\hat{\mathbb{C}}\setminus K)}_{\mathrm{Hom}} \to \hat{\mathbb{C}}$ is just the projection map $\rho_{\hat{\mathbb{C}}\setminus K}$.

Suppose that $\widetilde{(\hat{\mathbb{C}}\setminus K)}_{\mathrm{Hom}}$ satisfies the corona theorem. Then by Lemma 2.1 there is a $(1,0)$-form $\tilde{\eta}$ on $\widetilde{(\hat{\mathbb{C}}\setminus K)}_{\mathrm{Hom}}$ bounded relative to $F^*(ds_{\mathrm{sphere}})$ and solving $\overline{\partial}\tilde{\eta} = \omega_F = (\rho^*_{\hat{\mathbb{C}}\setminus K} ds_{\mathrm{sphere}})^2$. By Lemma 2.3 we may push down to get a $(1,0)$-form η on $\hat{\mathbb{C}}\setminus K$ bounded relative to the spherical metric and solving $\overline{\partial}\eta = ds^2_{\mathrm{sphere}}$. Thus by Lemma 2.2 the linear isoperimetric inequality (2.5) is valid for the spherical metric on suitable subdomains of $\hat{\mathbb{C}}\setminus K$. But K can be covered by an open set V with arbitrarily small area and arbitrarily short piecewise smooth boundary, so setting $\Omega = \hat{\mathbb{C}}\setminus\overline{V}$ we arrive at a contradiction. $\square$

Proof of Lemma 2.1. Suppose that g_1, g_2 are bounded holomorphic functions solving (2.1) and and set $C = \delta^{-1}\|\,|g_1|^2 + |g_2|^2\|^{1/2}_{L^\infty(X)}$.

Let

$$u = \frac{g_2\overline{f_1} - g_1\overline{f_2}}{\|F\|^2}$$

and

$$\eta = -2iu\mathcal{W}_F.$$

Then direct computation shows that

$$\overline{\partial}u = -\frac{\overline{\mathcal{W}_F}}{\|F\|^4}$$

so that η solves (2.2), and moreover

$$|\eta| = 2|u||\mathcal{W}_F| \le \frac{2C\delta}{\|F\|}|\mathcal{W}_F| \le CF^*(ds_{\mathrm{sphere}})$$

so that η satisfies (2.4) and hence (2.3).

For the converse first note that if $\mathcal{W}_F \equiv 0$ then without loss of generality f_2 is a constant multiple of f_1 and we may take $g_1 = f_1^{-1}, g_2 = 0$.

In the general case let Z denote the zero set of $\mathcal{W}_F$ and set

$$u = \frac{i\eta}{2\mathcal{W}_F}$$

on $X \setminus Z$. Then (2.3) implies that

$$|u| = \frac{|\eta|}{2|\mathcal{W}_F|} \leq \frac{C}{\|F\|^2}.$$

Set

$$\begin{aligned} g_1 &= \frac{\overline{f_1}}{\|F\|^2} + uf_2, \\ g_2 &= \frac{\overline{f_2}}{\|F\|^2} - uf_1 \end{aligned}$$

on $X \setminus Z$. Then direct computation shows that

$$\overline{\partial} g_1 \equiv 0 \equiv \overline{\partial} g_2,$$

$$|g_1|^2 + |g_2|^2 = \|F\|^{-2} + u^2\|F\|^2 \leq \frac{1}{\delta^2} + (\frac{C}{\delta})^2,$$

and

$$f_1 g_1 + f_2 g_2 \equiv 1$$

on $X \setminus Z$. Thus the singularities of g_1, g_2 at points of Z are removable, and g_1, g_2 extend to bounded holomorphic functions solving (2.1) on all of X. $\square$

Proof of Lemma 2.2.

$$\begin{aligned} \text{area}(\Omega) &= \int_\Omega \omega \\ &\overset{\text{(Stokes)}}{=} \int_{\mathrm{b}\,\Omega} \eta \\ &\leq \int_{\mathrm{b}\,\Omega} C\, ds \\ &= C \,\text{length}(\mathrm{b}\,\Omega). \quad \square \end{aligned}$$

Proof of Lemma 2.3. This is a standard "amenability" argument. (Compare [Sto, Thm. 2.1] [McM].)

Let $\tau_1, \tau_2, \dots \in \mathrm{Aut}(\tilde{X}_{\mathrm{Hom}})$ be a list of generators for the deck group ($\simeq H_1(X)$). Let

$$\tilde{\eta}_k = (2k+1)^{-k} \sum_{|m_j| \le k} (\tau_1^{m_1})^* \dots (\tau_k^{m_k})^* \tilde{\eta}.$$

Since

$$(\tau_1^{m_1})^* \dots (\tau_k^{m_k})^* \rho_X^* \omega = \rho_X^* \omega$$

we have

$$\overline{\partial} \tilde{\eta}_k = \rho_X^* \omega$$

and

$$|\tilde{\eta}_k| \le C \rho_X^* ds_{\mathrm{sphere}};$$

moreover, since the τ_j commute, we also have

$$|\tau_j^* \tilde{\eta}_k - \tilde{\eta}_k| \le \frac{2C}{2k+1}$$

for $j = 1, \dots, k$.

Generalized Cauchy estimates [Hör, Thm. 1.2.4] show that derivatives of the $\tilde{\eta}_k$ are uniformly bounded on compact subsets. Hence, by a normal families argument some subsequence of the $\tilde{\eta}_k$ converges to a limit $\tilde{\eta}_\infty$ which satisfies

$$\overline{\partial} \tilde{\eta}_\infty = \rho_X^* \omega,$$

$$|\tilde{\eta}_\infty| \le C \rho_X^* ds_{\mathrm{sphere}},$$

and

$$\tau_j^* \tilde{\eta}_\infty = \tilde{\eta}_\infty$$

for all j and thus induces the desired form η on X. $\square$

Remark: In the proof of Theorem 1.1, Lemma 2.1 together with the corona theorem for the universal cover $\tilde{X}_{\mathrm{univ}} \simeq \Delta$ of $X = \hat{\mathbb{C}} \setminus K$ implies that our basic $\overline{\partial}$ problem does admit a bounded solution on $\tilde{X}_{\mathrm{univ}}$ despite the fact that no bounded solution exists on X.

3. Connection with multivariate dynamics

Let $p_1(z_1, z_2), p_2(z_1, z_2)$ be homogeneous polynomials of degree $k \ge 2$ such that

$$\{(z_1, z_2) : p_1(z_1, z_2) = p_2(z_1, z_2) = 0\} = \{0\}.$$

Then the map

$$\begin{aligned} \Phi &: \mathbb{C}^2 \to \mathbb{C}^2, \\ (z_1, z_2) &\mapsto (p_1(z_1, z_2), p_2(z_1, z_2)) \end{aligned}$$

induces a self-map ϕ of $\mathbb{CP}^1 = \hat{\mathbb{C}}$ given by $[z_1 : z_2] \mapsto [p_1(z_1, z_2) : p_2(z_1, z_2)]$ or $z \mapsto \dfrac{p_1(z,1)}{p_2(z,1)}$.

The origin is a superattractive fixed point for Φ; let U denote the corresponding basin of attraction. Also, let $\mathcal{F} \subset \hat{\mathbb{C}}$ denote the Fatou set of ϕ and let η denote the quotient map $\mathbb{C}^2 \setminus \{0\} \to \hat{\mathbb{C}}$.

Then we have the following (see [HuPa]);

(1) There is $\delta > 0$ such that $B(0, \delta) \subset U \subset B(0, \delta^{-1})$.
(2) U is a complete circular domain (i.e., $z \in U, |\lambda| \le 1$ implies $\lambda z \in U$).
(3) $\mathrm{b}\, U \cap \eta^{-1}(\mathcal{F})$ is a smooth hypersurface foliated by Riemann surfaces.
(4) For each leaf L of this foliation the restriction η_L of η to l is a covering map onto $\mathcal{F}$.
(5) Continuation of a branch of η_L^{-1} along a loop γ multiplies the branch by a unimodular scalar λ_γ.

In particular, continuation of a branch of η_L^{-1} along a commutator loop is trivial and thus the covering $\rho_L : L \to \mathcal{F}$ is subordinate to the homology covering of $\mathcal{F}$.

Suppose now that the Julia set $\hat{\mathbb{C}} \setminus \mathcal{F}$ has length zero: this happens in particular when Φ is given by

$$(z_1, z_2) \mapsto (z_1^2 + c z_2^2, z_2^2),$$

$c \notin$ the Mandelbrot set [Ran, Ex. 2]. Then the proof of Theorem 1.1 now shows that L has corona; in particular, the coordinate functions z_1, z_2 satisfy $\delta^2 < |z_1|^2 + |z_2|^2 < \delta^{-2}$ on L but fail to generate the algebra of bounded holomorphic functions on L.

Addendum. Bo Berndtsson has suggested the following alternate proof of Theorem 1.1.

Construct f_1 and f_2 as before and suppose that there are bounded holomorphic g_1 and g_2 on $(\hat{\mathbb{C}} \setminus K)_{\mathrm{Hom}}$ solving $f_1 g_1 + f_2 g_2 \equiv 1$.

Averaging $f_1 g_1$ and $f_2 g_2$ over fibers as in Lemma 2.3 we obtain bounded holomorphic functions h_1 and h_2 on $\hat{\mathbb{C}} \setminus K$ solving $h_1 + h_2 \equiv 1$. By Painlevé's Theorem h_1 and h_2 are constant. Now $f_2 g_2$ vanishes at points lying over ∞, so $h_2(\infty) = 0$. Assuming, as we may, that $0 \notin K$ we find similarly that $h_1(0) = 0$. But now $h_1 \equiv 0 \equiv h_2$, a contradiction.

References

[All] N. Alling, *A proof of the corona conjecture for finite open Riemann surfaces*, Bull. Amer. Math. Soc. **70** (1964), 110-112.

[BaDi] D. Barrett and J. Diller, *Contraction properties of the Poincaré series operator* (to appear).

[Car] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Annals of Math. **76** (1962), 547-559.

[Dil1] J. Diller, *A canonical $\overline{\partial}$ problem for bordered Riemann surfaces* (to appear).

[Dil2] J. Diller, *Limits on an extension of Carleson's $\overline{\partial}$ theorem*, J. Geometric Analysis (to appear).

[For] O. Forster, *Lectures on Riemann surfaces*, Springer-Verlag, 1981.

[Gam] T. Gamelin, *Uniform algebras and Jensen measures*, London Math. Soc. Lecture Note Series # 32, 1978.

[Gar] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[GaJo] J. B. Garnett and P. W. Jones, *The corona theorem for Denjoy domains*, Acta Math. **155** (1985), 27-40.

[Hör] L. Hörmander, *An introduction to complex analysis in several variables (3rd ed.)*, North-Holland, 1990.

[HuPa] J. H. Hubbard and P. Papadopol, *Superattractive fixed points in* $\mathbb{C}^n$, Indiana Univ. Math. J. **43** (1994), 321-365.

[McM] C. McMullen, *Amenable coverings of complex manifolds and holomorphic probability measures*, Invent. Math. **110** (1992), 29-37.

[Moo] C. Moore, *The corona theorem for domains whose boundary lies in a smooth curve*, Proc. Amer. Math. Soc. **100** (1987), 266-270.

[Ran] T. J. Ransford, *Variation of Hausdorff dimension of Julia sets*, Ergodic Theorey and Dynamical Systems **13** (1993), 167-179.

[Sto] E. L. Stout, *Bounded holomorphic functions on finite Riemann surfaces*, Trans. Amer. Math. Soc. **120** (1965), 255-285.

[Tsu] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, 1959.

Ann Arbor, MI 48105 USA
E-mail address: barrett@math.lsa.umich.edu

Bloomington, IN 47405 USA
E-mail address: jdiller@ucs.indiana.edu