A CANONICAL $\overline{\partial}$ PROBLEM FOR BORDERED RIEMANN SURFACES

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1. Introduction

Any Riemann surface uniformized by the unit disk admits a constant curvature -1 metric called the *Poincaré* metric. Those surfaces which have infinite area in the Poicaré metric are said to be of *infinite type.* Our aim in this paper is to prove

Theorem 1.1. *Let Y be a Riemann surface of infinite type and finitely generated fundamental group. Let dA be the area form induced by the Poincaré metric on Y. Then there is a (1,0) form η on Y satisfying*

$$\overline{\partial}\eta = dA, \tag{1.1}$$

and

$$\langle \eta(z) \rangle \leq C, \qquad \text{for all } z \in Y \tag{1.2}$$

where $\langle \eta(z) \rangle$ denotes the Poincaré length of η at z, and C is a constant depending only on the Euler characteristic of Y and the length of the shortest closed geodesic on Y.

The $\overline{\partial}$ problem addressed by this theorem arises in one form or another in several of our previous papers. It is connected with averaging problems for multiple–valued holomorphic functions (see [Di1] and [Di2]), with theorems about solving the $\overline{\partial}$ equation with L^∞ control (see [Di3]), and with contraction properties of the "Poincaré Series Operator" (see [BD1]). In all of these papers, we exploited *negative* results about solvability of 1.1 to shed light on other problems. The positive result that we prove here lends some completeness to this work.

In a sequel [BD2] to this paper, we will apply Theorem 1.1 to further study the Poincaré series operator. We briefly describe the main result of the sequel here. Let $\pi : Y \to X$ be a holomorphic covering of one Riemann surface by another, and let $Q(Y)$, $Q(X)$ denote the spaces of L^1 holomorphic quadratic differentials on Y and X, respectively. Then the Poincaré series operator $\theta : Q(Y) \to Q(X)$ is

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the natural pushforward operator obtained by summing over fibers of π. Our main result in the sequel is that θ has norm less than one whenever X is a hyperbolic surface of finite type, and Y satisfies the hypothesis of the interior of Theorem 1.1. We obtain explicit lower bounds on $1 - ||\theta||$ in terms of the length of the shortest closed geodesic on X and the Euler characteristics of X and Y. In particular, we recover a large part of the theorem due to Curt McMullen [Mc] which says that $||\theta||$ is less than 1 whenever π is "non-amenable." Our proof differs markedly from McMullen's. Theorem 1.1 enters the proof when we use Stokes' Theorem to bound certain length integrals in terms of area integrals.

Section 2 of this paper describes the relationships among our problem, an isoperimetric inequality, and a related, but apparently simpler differential equation. The other sections of the paper concern the proof of Theorem 1.1.

The broad strategy of our proof goes as follows: we first give explicit formulas for the solution of 1.1 in the case where Y is a disk, a punctured disk, or an annulus. Using these, we can solve 1.1 near the boundary of any bordered surface Y. This allows us to reduce the $\overline{\partial}$ problem to one with compactly supported data. Then we use the Green's function on Y to produce a function u whose Laplacian equals the compactly supported data. Taking the holomorphic derivative of u solves the problem. The details of this strategy are contained in sections 3 and 6.

Practically speaking, most of the space in the paper is occupied with relating our solution of 1.1 to the topology and geometry of Y. To this end in section 4, we prove a sort of diameter estimate for the convex core of Y. Then in section 5 we establish a variety of results that give us control on the size of the Green's function for Y.

2. An Isoperimetric Inequality and a Related Differential Equation

In this section, we suppose that Y satisfies the hypothesis of Theorem 1.1 and that in addition, Y has no finite volume ends (i.e. no isolated points in the ideal boundary) in the Poincaré metric on Y. Then the length ℓ of the shortest closed geodesic on Y is twice the *injectivity radius* of Y—i.e. the largest number I such that all metric disks of radius I in Y are topological disks.

Consider the following problem, similar to the one solved by Theorem 1.1 except that the $\overline{\partial}$ operator is replaced by the ordinary exterior derivative:

Problem. *Find a one form η on Y satisfying*

$$d\eta = dA \tag{2.1}$$

and

$$\langle \eta(z) \rangle \leq C \tag{2.2}$$

at each point $z \in Y$.

A solution to this problem automatically gives a linear isoperimetric inequality for relatively compact subdomains Ω of Y. Namely, suppose that $b\Omega$ is smooth and that η solves the problem. Using ds to denote the Poincaré length density, we obtain from Stokes Theoerem that

$$\text{Area}(\Omega) = \int_\Omega dA = \int_{b\Omega} \eta \leq C \int_{b\Omega} ds = C\,\text{Length}(b\Omega). \tag{2.3}$$

On the other hand, as we now outline, a linear isoperimetric inequality on Y implies the existence of a bounded solution to 2.1. By theorem 6.2 of [Ch], 2.3 is equivalent to

$$||f||_1 \leq C||df||_1 \tag{2.4}$$

for all test functions $f \in C_0^\infty(Y)$. Let E denote the image under the exterior derivative d of C_0^∞. Since Y is non–compact, the kernel of d acting on C_0^∞ is trivial. So d has a set–theoretic inverse $d^{-1} : E \to C_0^\infty(Y)$. By 2.4, d^{-1} is L^1–bounded, so it extends to a continuous linear operator acting on the L^1 closure of E. By the Hahn–Banach Theorem, d^{-1} extends to an operator acting on the closure of the set of all L^1–bounded one forms on Y. This gives a linear functional

$$\ell(\lambda) = \int_Y d^{-1}\lambda \wedge dA$$

acting on the same set of one forms. Clearly $||\ell|| \leq C$. By the Riesz representation theorem there exists a bounded one form η such that

$$\ell(\lambda) = \int_Y \lambda \wedge \eta \qquad \text{and} \qquad ||\eta||_\infty \leq C.$$

In other words, η is a bounded, weak solution of 2.1. Standard elliptic theory implies that η is actually a smooth solution to the problem.

The isoperimetric inequality 2.3 is well–known to hold on the Riemann surfaces Y that we are considering. Hence, the simple duality argument above suffices to solve 2.1 boundedly.

Now we turn to the problem addressed by Theorem 1.1. Given the theorem, an application of Stokes' formula again implies 2.3. So large subdomains with small boundary (e.g. domains bounded by short, closed geodesics) prevent us from picking too small a bound C for η. On the other hand, it is not apparent to us that the same duality argument will obtain the theorem from the isoperimetric inequality. As a rule of thumb, solutions to the $\overline{\partial}$ equation are harder to come by than solutions of the exterior derivative equation. For the applications that we give in [BD2], a solution to the harder problem is absolutely essential.

3. Preliminaries and Definitions

Throughout the rest of this paper, Y will denote a connected, hyperbolic Riemann surface of infinite type and finitely generated fundamental group. We refer to the constant curvature -1 metric that Y inherits from the unit disk as the *Poincaré metric* on Y, denoting the induced area form by dA, and the induced length element by ds. If v is any tensor on Y, then we denote its length at $p \in Y$ in the Poincaré metric by $\langle v(p)\rangle$.

There exists a bordered Riemann surface $\overline{Y}$ containing Y such that $\overline{Y} \setminus Y = b\overline{Y} \cup P$, where P is a discrete set. We refer to points in P as punctures of Y. These points are metrically distinguished by the fact that they compactify finite volume ends of Y—i.e. given $p \in P$, there exists a neighborhood $U \subset \overline{Y}$ of p such that $\mathrm{Area}(U \cap Y) < \infty$ in the Poincaré metric on Y. We let g be the genus of Y (= the genus of $\overline{Y}$), m be the number of punctures of Y, and n be the number of components of bY. The hypothesis that the fundamental group of Y is finitely generated implies that $g, m, n < \infty$. The hypothesis that Y is of infinite type implies that $n > 0$. We label the components of $b\overline{Y}$ by $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$.

On the unit disk, $\Delta = \{|z| < 1\}$, the Poincaré metric has the infinitesimal form

$$ds = \frac{2|dz|}{1 - |z|^2}, \tag{3.1}$$

on the upper half plane $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$ it has the form

$$\frac{|dz|}{\mathrm{Im}\, z}, \tag{3.2}$$

on the punctured disk $\Delta^* = \{0 < |z| < 1\}$, it has the form

$$-\frac{|dz|}{|z| \log |z|} \tag{3.3}.$$

and on the annulus $A = A_R = \{e^{-R} < |z| < e^R\}$ it has the form

$$\frac{\pi |dz|}{2R|z| \cos(\pi \frac{\log |z|}{2R})}. \tag{3.4}$$

The curve $\{|z| = 1\} \subset A$ is a closed geodesic in the Poincaré metric of length π^2/R. These facts are all fairly standard. We mention them here for the reader's convenience, because in what follows, we will occasionally omit brute computation done in the Poincaré metric.

Given any non–vanishing holomorphic $(1,0)$ form λ on Y, we can write

$$ds = \rho\, |\lambda|, \qquad dA = \frac{\rho^2}{2i}\, \bar{\lambda} \wedge \lambda$$

for some smooth, positive function ρ. The curvature condition on the metric translates to (see [GrHa] page 77)

$$\overline{\partial} \partial \log \rho = \frac{\rho^2}{4}\, \bar{\lambda} \wedge \lambda = \frac{i\, dA}{2}.$$

Thus $-2i\partial \log \rho$ solves 1.1. This observation suggests that one way to prove Theorem 1.1 is to find a "good" holomorphic (1,0) form. In fact, this is what we do if Y is a disk or an annulus.

Lemma 3.1. *1.1 is solvable with $\langle\eta\rangle \leq 1$ if Y is the disk, the punctured disk, or an annulus.*

Proof. Taking $\lambda = dz$ on the unit disk, we obtain $\rho = 2(1-|z|^2)^{-1}$ and

$$\eta = \frac{-2i\bar{z}\,dz}{1-|z|^2}.$$

Also, $\langle\eta\rangle = |z| \leq 1$. On the punctured disk, we take $\lambda = \frac{dz}{z}$. From 3.3, we obtain $\rho = -(\log|z|)^{-1}$ and

$$\eta = \frac{i\,dz}{z\log|z|} \tag{3.5}$$

This time, $\langle\eta\rangle \equiv 1$. On the annulus, we again take $\lambda = \frac{dz}{z}$. We apply 3.4 and see that

$$\rho = \frac{\pi}{2R\cos(\frac{\pi\log|z|}{2R})}$$

and

$$\eta = \frac{-i\pi}{2Rz}\tan(\frac{\pi\log|z|}{2R})\,dz. \tag{3.6}$$

Once again, $\langle\eta\rangle = \sin(\frac{\pi\log|z|}{2R}) \leq 1$. $\square$

We conclude this section by fixing some more notation. Given a point $p \in Y$, we refer to the uniformization map $\pi_p : \Delta \to Y$ such that $\pi_p(0) = p$ as *standard coordinates* near p. Since π_p is a local isometry and injective in some neighborhood of 0, computations in the Poincaré metric near p can be lifted to computations in the unit disk.

One can also define standard coordinates near a puncture as follows: given $p \in P$, let $T \in \operatorname{Aut}\Delta$ be the deck transformation corresponding to traveling once about p in Y. Then $\Delta/\{T^n\}$ is biholomorphic to Δ^*. By dividing by the rest of the group of deck transformations of Y, one obtains a holomorphic cover $\pi_p : \Delta^* \to Y$ that extends to a map of Δ into $\overline{Y}$ such that $\pi_p(0) = p$ and π_p is injective near the origin. Similarly, given a simple, closed geodesic γ of length ℓ_γ. We obtain a holomorphic covering $\pi_\gamma : A_R \to Y$ such that π_γ, where $R = \pi^2/\ell_\gamma$, and π_γ maps $\{|z| = 1\}$ bijectively onto γ. It is well–known that for every component $\hat{\gamma}_j$ of $b\overline{Y}$, there exists a simple closed geodesic γ_j homotopic (in $\overline{Y} \setminus P$) to $\hat{\gamma}$. If $\gamma = \gamma_j$ for some j, we can assume that π_γ maps $\{e^{-R} < |z| < 1\}$ bijectively onto the annulus between $\hat{\gamma}_j$ and γ_j.

4. A distance estimate for the core of Y.

Since Y is non-compact, its diameter in the Poincaré metric is infinite. However, we will show in this section that if one removes appropriate neighborhoods of each puncture in P and each component of $b\overline{Y}$, then the remainder, or "core," of Y has a diameter that can be estimated in terms of the topology of Y. In what follows, various constants that depend only on the numbers g, m, and n will arise. Rather than keep track of them, we will denote them by C_1, C_2, etc, not meaning to imply that C_j has the same meaning on one line that it does on the next.

As a preliminary, we need to discuss the extent to which the standard coordinate maps described in the last section are injective. First of all, given $p \in Y$, we define the injectivity radius $I(p)$ of Y at p to be the largest number I such that the metric disk of radius I at p is a topological disk. If $Y \neq \Delta$, $I(p)$ is always finite. A local coordinate computation reveals that π_p maps the disk $\{|z| < \tanh(I(p)/2)\}$ onto the disk of radius $I(p)$ about p. By the definition of $I(p)$, we see that in fact π_p does this injectively.

Given a puncture p, we set $\tilde{\mathcal{C}}_b(0) = \{0 < |z| < e^{-\pi}\}$ and define the *cusp* $\mathcal{C}_b(p)$ about p to be the set $\pi_p(\tilde{\mathcal{C}}_b(0))$. Likewise, given a simple closed geodesic γ of length ℓ_γ on Y, we choose $R' < R$ to satisfy

$$\tan \frac{\ell R'}{2\pi} = \frac{1}{\sinh(\ell_\gamma)/2},$$

set $\tilde{\mathcal{C}}(\gamma) = \pi_p(\{e^{-R'} < |z| < e^{R'}\})$, and define the *collar* $\mathcal{C}(\gamma)$ about γ to be the set

$$\pi_\gamma(\tilde{\mathcal{C}}(\gamma)) = \left\{ p \in Y : \operatorname{dist}(p,\gamma) < \sinh^{-1} \frac{1}{\sinh(\ell_\gamma/2)} \right\}.$$

Finally, if $\gamma = \gamma_j$ is homotopic to a component of $b\overline{Y}$, we set $\tilde{\mathcal{C}}_b(\gamma) = A_R \cap \{|z| < 1\}$ and call the annulus $\mathcal{C}_b(\gamma) = \pi_\gamma(\tilde{\mathcal{C}}_b(\gamma))$ between γ and $b\overline{Y}$ the *boundary collar* about γ. We pointed out in the last section π_γ maps $\tilde{\mathcal{C}}_b(\gamma)$ injectively onto $\mathcal{C}_b(\gamma)$. It is a remarkable fact that a similar theorem holds for the other collars we have defined here. By a *short* geodesic in the next theorem, we mean a closed geodesic whose length is less than $2\sinh^{-1} 1$.

Collar Theorem. *The following statements are true for Y.*

1. *Given a puncture p of Y, π_p maps $\tilde{\mathcal{C}}(0)$ bijectively onto the cusp $\mathcal{C}(p)$. Likewise, given a simple, closed geodesic γ on Y, π_γ maps $\tilde{\mathcal{C}}(\gamma)$ bijectively onto the collar $\mathcal{C}(\gamma)$.*
2. *All short geodesics on Y are simple.*
3. *The possible number of short geodesics is bounded above in terms of g, m, and n.*
4. *Cusps, collars about short geodesics, and boundary collars of geodesics are all mutually disjoint.*
5. *A point p lies in a cusp or the collar about a short geodesic if and only if $I(p) < \sinh^{-1} 1$.*

To paraphrase the theorem, short closed geodesics and punctures in Y are surrounded by large, mutually disjoint embedded annuli; and the injectivity radius of X is never small outside of these annuli. A good reference for the Collar Theorem is Buser's book [Bu] (see Chapter 4). There is another fact about the collars of short geodesics that will be useful below. Namely, one can show by direct computation that there is an upper bound on Length($b\mathcal{C}()$) that is independent of the core geodesic γ. The same upper bound holds for the length of the boundary of a cusp (which one can think of as the limiting case that one obtains by letting the length of a short geodesic tend to 0). We omit the details of the computation.

We refer to the set $Y_{thick} \subset Y$ obtained by removing cusps and collars of short geodesics as the *thick part* of Y, and the set $Y_0 \subset Y$ obtained by removing all cusps and boundary collars from Y as the *core* of Y. We let ℓ denote the length of the shortest closed geodesic on Y. Our goal in this section is to prove

Theorem 4.1. *There are constants C_1, C_2 depending only on g, m, n with the following property. Given any point $p \in Y_0$ there is a boundary geodesic γ_j such that*

$$\mathrm{dist}(p, \gamma_j) < C_1 + C_2 \log \frac{1}{\ell}$$

Furthermore, there is an absolute constant C_3 such that if γ is a length-minimizing geodesic from p to the nearest point in γ_j, then $I(q) \geq \min\{C_3, \ell/2\}$ for every point $q \in \gamma$.

To prove the theorem, we fix a point $p_0 \in Y_0$ and let $\gamma \subset Y$ be a length minimizing geodesic connecting p_0 to the nearest point in a boundary geodesic. We can divide γ up into those points which lie in cusps, those points which lie in collars of short geodesics, and those points which lie in $Y_{thick} \cap Y_0$. The injectivity radius of Y is bounded below by $\ell/2$ in Y_0, so we need only prove the estimate on injectivity radius along parts of γ that pass through cusps.

Cusps will not disconnect Y, so given any puncture $p \in P$ we can replace $\gamma \cap \mathcal{C}(p)$ with a portion of $b\mathcal{C}(p)$, adding at most a universally bounded amount of length to γ. This observation suffices to establish two things. First of all, the part of γ that intersects any given cusps stays with some universally bounded distance of the edge of the cusp. Thus the injectivity radius at points in the intersection admits a universal lower bound C_3. Secondly, the intersection between γ and the union of all cusps of Y has length bounded by a constant times n.

Now consider the portion of γ which intersects a collar $\mathcal{C}(\gamma')$ of a short geodesic. In this case, we replace $\gamma \cap \mathcal{C}(\gamma')$ by a segment of one of the boundary components of $\mathcal{C}(\gamma')$ and possibly a path connecting this segment to the other component of $b\mathcal{C}(\gamma')$. Direct computation shows that the replacement path has length bounded above by

$$A + B \log 1\mathrm{Length}(\gamma') \leq A + B \log \frac{1}{\ell}.$$

Since the number of short geodesics is controlled by g, m, n we see that the portion of γ lying in collars of short geodesics satisfies the conclusion of the Theorem 4.1. The following lemma completes the proof of the theorem.

Lemma 4.2. *There is a constant C depending only on m, g, and n such that any connected component of $\gamma \cap Y_{thick}$ has length bounded above by C.*

Proof. Let γ' be a connected component of $\gamma \cap Y_{thick}$, and denote the endpoints of γ' by p_1 and p_2. Let $\pi : \mathbb{H} \to Y$ be a universal covering map such that γ' lifts to the vertical segment $\tilde{\gamma}' = \{z = x + iy : x = 0, 1 \le y \le e^D\}$. One can check that $D = \text{Length}(\gamma')$ and that the region

$$U = \{z = re^{i\theta} : 1 \le R \le e^D, |\cos\theta| \le 2^{-1/2}\}$$

is the hyperbolic tube of radius $(\sinh^{-1} 1)/3$ about $\tilde{\gamma}'$. We claim that U injects into Y under π. If it does not inject into Y, choose points z_1 and z_2 such that $\pi(z_1) = \pi(z_2)$. Let $\hat{z}_1$ and $\hat{z}_2$ be the corresponding closest points on $\tilde{\gamma}'$. Note that the hyperbolic distance between $\pi(\hat{z}_1)$ and $\pi(\hat{z}_2)$ (and hence the length of the segment of γ' which they bound) is less than $2(\sinh^{-1} 1)/3$, since both points are within $(\sinh^{-1} 1)/3$ of the same point in Y. However, this implies that both z_1 and z_2 lie in the hyperbolic disk of radius $\sinh^{-1} 1$ about $\hat{z}_1$. But $\pi(\hat{z}_1)$ lies in Y_{thick}, so this disk injects into Y—a contradiction.

Now it is possible that $\pi(U)$ does not lie entirely in Y_0 even though γ' does. In particular, $\pi(U)$ might intersect a boundary collar. However, it is clear from the definition of U and that fact that γ' lies on a geodesic which minimizes distance to bY_0, that any intersection between $\pi(U)$ and a boundary collar lies within a disk of radius $\sinh^{-1}(1)$ about the endpoint $\gamma \cap bY_0$. The rest of $\pi(U)$ lies in either Y_0 or one of the cusps of Y. Hence, we estimate

$$\begin{aligned}\text{Area}(Y_0) + m\,\text{Area}\,\tilde{\mathcal{C}}_b(0) + \text{Area}\{p \in Y : \text{dist}(p, \gamma \cap bY_0) < \sinh^{-1} 1\}&\\ \ge \text{Area}\,U = 2D\sinh(\frac{1}{3}\sinh^{-1} 1).&\end{aligned}$$

The Gauss–Bonnet Theorem allows us to compute the sum of the first two terms on the right side in terms of g, m, n; the third term is clearly bounded by a universal constant. This gives us the desired upper bound for D. □

5. Green's function

In this section we prove several lemmas about the size of the Green's function $G(p,q)$ with pole at q on $\overline{Y}$, beginning with some lemmas about more general functions. We adopt the non-standard convention that $G(p,q) > 0$, since it will make upper bounds on the size of G easier to state. Our first lemma and its corollary, to be used in section 5, is a derivative estimate for functions with bounded Laplacian.

Lemma 5.1. *Let $u : \Delta \to \mathbb{R}$ be a smooth function such that $|u| < M_1$ and $|\Delta u| < M_2$. Then*

$$\left|\frac{\partial u}{\partial z}(0)\right| < M_1 + \frac{2\pi}{3}M_2$$

Proof. By working on slightly smaller disks and passing to a limit, we may assume that u is smooth across $b\Delta$. Then we have that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, re^{i\theta})u(e^{i\theta})\,d\theta + \frac{1}{2i}\int_\Delta G(w,z)\Delta u(w)\,d\bar{w}\wedge dw,$$

where $G(\cdot, z)$ is the Green's function with pole at z and P is the Poisson kernel on the unit disk. These functions may be written down explicitly and differentiated with respect to z. After doing so and evaluating at $z = 0$, we obtain

$$\frac{\partial u}{\partial z}(0) = -\int_0^{2\pi} e^{-i\theta}u(e^{i\theta})\,d\theta + \frac{1}{2i}\int_\Delta \frac{1-|w|^2}{w}\Delta u(w)\,d\bar{w}\wedge dw.$$

Taking absolute values of each integrand and integrating finishes the proof. □

Corollary 5.2. *Given a smooth function $u : \Delta \to \mathbb{R}$ such that $|u| \le M_1$ and $\langle\partial\overline{\partial}u\rangle \le M_2$, we have*

$$\langle\partial u\rangle \le C_1M_1 + C_2M_2$$

for some constants C_1 and C_2.

Proof. The hypotheses and conclusion of the corollary are invariant under automorphisms of Δ, so it suffices to obtain the estimate at the origin. Writing out $\langle\partial\overline{\partial}u\rangle$ explicitly gives us that $|\Delta u| \le C_1M_1(1-|z|^2)^{-2}$. For $|z| \le 1/2$, this becomes $|\Delta u| \le C_1M_1$. Making a change of coordinates to scale the disk $\{|z| < 1/2\}$ by a factor of 2 and applying Lemma 5.1 proves the corollary. □

We will also need an invariant version of Harnack's Inequality for harmonic functions. This is

Lemma 5.3 (Invariant Harnack Inequality). *Suppose that $U \subset Y$ is open and $h : U \to \mathbb{R}^+$ is harmonic. For each $p \in U$, let $r(p)$ denote the distance from p to bU. Then we have*

$$\langle d\log h(p)\rangle \le \frac{1}{\tanh(r/2)}.$$

In particular, if p_1, p_2 are joined by a path $\gamma \in U$, then

$$\frac{h(p_1)}{h(p_2)} \le e^{\int_\gamma \frac{ds}{\tanh r(p)/2}}.$$

Proof. Fix p, and let $\pi_p : \Delta \to Y$ be standard coordinates about p. Then $h \circ \pi_p$ is positive and harmonic on the open disk $\{|z| < \tanh(r(p)/2)$. Thus if $|w| < \tanh(r(p)/2)$, the usual Harnack inequality tells us that

$$|\log h \circ \pi_p(w) - \log h \circ \pi_p(0)| \leq \log \frac{\tanh(r(p)/2) + |w|}{\tanh(r(p)/2) - |w|}.$$

If we let w tend to 0 and use the fact that the Poincaré and Euclidean metrics differ by a factor of two at 0, we obtain

$$\langle d \log h \circ \pi_p(0) \rangle \leq \frac{1}{\tanh r(p)/2}$$

in the Poincaré metric on Δ. Since π_p is a local isometry, the result descends to

$$\langle d \log h(p) \rangle \leq \frac{1}{\tanh r(p)/2}$$

in the Poincaré metric on Y. $\square$

The next lemma is specifically about the Green's function. It is almost identical to Lemma 4.4 of [Di3]. From now on we will rely heavily on the *standard coordinate* notation defined at the end of Section 3. In particular, where standard coordinates $\pi_\gamma : A_R \to Y$ about a closed geodesic γ are in use, the notation $G(z, w)$ (where $z, w \in A_R$), will be local coordinate shorthand for $G(\pi_\gamma(z), \pi_\gamma(w))$; the analogous convention will hold when we work in standard coordinates about a point.

Lemma 5.4. *Let $\gamma = \gamma_j$ be a boundary geodesic of Y and $\pi_\gamma : A_R \to Y$ be standard coordinates about γ. Choose $w \in A_R$ to have minimal modulus in the set $\pi_\gamma^{-1}(\pi_\gamma(w))$. Then for all $r \leq |w|$ we have that*

$$\frac{1}{2\pi} \int_0^{2\pi} G(re^{i\theta}, w)\, d\theta = \alpha_j(\log r + R) \tag{5.1}$$

for some $0 < \alpha_j \leq 1$. If $|w| < 1$, then we also have that

$$\frac{1}{2\pi} \int_0^{2\pi} G(re^{i\theta}, w)\, d\theta = (\alpha_j - 1)\log r + \log|w| + \alpha_j R \tag{5.2}$$

for all r between $|w|$ and 1. Finally, $\alpha_1 + \cdots + \alpha_n = 1$.

Proof. We recall that any harmonic function h on an annulus satisfies (see [Ah] Chapter 4, Theorem 20)

$$\frac{1}{2\pi} \int_0^{2\pi} h(re^{i\theta})\, d\theta = \alpha \log r + \beta$$

for some constants α and β. Furthermore α is given by

$$\alpha = \frac{1}{2\pi}\int_\gamma *dh,$$

where γ is any loop homologous (in $\overline{Y}$) to $\{|z| = r\}$ and $*dh$ is the conjugate differential to dh.

G is harmonic in $z = re^{i\theta}$ for $e^{-R} < r < |w|$ and zero for $|r| = e^{-R}$. Consequently $\beta = \alpha R$ for r in this range. Furthermore, the fact that G becomes positive away from $r = e^{-R}$ forces $\alpha > 0$. So we get a positive α_j for each boundary component of Y. The union of all the boundary components is homologous to arbitrarily small loops about $\pi_\gamma(w)$ in $\overline{Y}$, so the sum of the α_j plus the integral of $*dh$ about one of these small loops is 0. A limit computation capitalizing on the fact that G grows like $\log|z-w|$ near w shows that the sum of the α_j must be 1.

Now the integral on the lefthand side of 5.1 must at least be continuous as a function of r—even across $r = |w|$. If $w \in A_j$ (i.e. $|w| < 1$), then another homology argument shows that the constant α will drop by exactly 1 as r passes $|w|$. Furthermore, π_γ is injective for $|z| < 1$, so there are no other preimages of w in A_R with modulus less than 1. Therefore, for $|w| < r < 1$, we have $\alpha = \alpha_j - 1$ and $\beta = \alpha_j R + \log|w|$. $\square$

Through a pair of lemmas, we will spend the rest of this section proving

Theorem 5.5. *There exist positive constants C_1, C_2, k depending only on g, m, n that make the following statement true: given $q \in \overline{Y}$, and $p \in Y_0$ such that* $\operatorname{dist}(p,q) \geq C_1\ell$, *we have*

$$G(p,q) \leq \frac{C_2}{\ell^k}$$

Corollary 5.6. *Suppose that $p \in Y_0$, and* $\operatorname{dist}(p,q) \leq \ell$. *Then*

$$G(p,q) \leq \frac{C_1}{\ell^k} - C_2 \log \operatorname{dist}(p,q).$$

Proof. This follows immediately from Theorem 5.5 and the fact that $G(\cdot, q)$ has a logarithmic pole at q. $\square$

Before beginning the proof of Theorem 5.5, we recall the well known fact that $I(p) > \ell/2$ for points $p \in Y$ which do not lie in cusps. If we relax our requirement on p slightly to allow for points which lie in cusps but within a given fixed distance of Y_0, then we obtain a similar lower bound $I(p) > C_1\ell$, for some absolute constant C_1. We point out this fact, because in what follows we will consider points in cusps that lie within unit distance of Y_0.

Lemma 5.7. *Theorem 5.5 is true if q lies well inside a cusp—more precisely, if q lies in a cusp $\mathcal{C}_b(p)$ about some puncture $p \in P$, and* $\operatorname{dist}(q, Y_0) \geq 1$.

Proof. Let γ be the boundary geodesic nearest to $\mathcal{C}_b(p)$, and let $\pi_\gamma : A_R \to Y$ be standard coordinates about γ. Choose $w \in A_R$ such that $|w|$ is minimal among points in $\pi_\gamma^{-1}(q)$, and similarly choose $z \in A_R$ such that $|z|$ is minimal in $\pi_\gamma^{-1}(\overline{\mathcal{C}_b(p)})$. Note that by hypothesis z is at least one unit closer to the unit circle than w in the Poincaré metric on A_R.

We set $M = \max_\theta G(|z|e^{i\theta}, w)$ and rotate A_R so that the maximum is achieved when $\theta = 0$. We apply Theorem 5.3 to obtain

$$G(|z|e^{i\theta}, w) \geq Me^{-C|\theta|L}$$

where L is the Poincaré length of the circle of Euclidean radius $|z|$, C is a positive constant, and $|\theta| \leq \pi$. Lemma 5.4 then gives us that

$$\int_{-\pi}^{\pi} Me^{-C|\theta|L}\, d\theta \leq \alpha_j(\log|z| + R) < 2R.$$

We evaluate the integral in this inequality, solve for M, and compute L in terms of $|z|$ to arrive at

$$G(z, w) \leq M \leq \frac{C_1 RL}{1 - e^{-C_2 L}} \leq R(C_1 L + C_2) \leq R\left(C_1 \sec\frac{\pi\log|z|}{2R} + C_2\right)$$

By Theorem 4.1, we know that in the Poincaré metric on A_R, the distance between z and the unit circle is no greater than $C_1 + C_2 \log\frac{1}{\ell}$. After a brief computation, we see that this bound translates into

$$\sec\frac{\pi\log|z|}{2R} \leq \cosh\left(C_1 + C_2\log\frac{1}{\ell}\right) \leq \frac{C}{\ell^k}.$$

Hence,

$$G(z, w) \leq R\frac{C}{\ell^k} + C' \leq \frac{C_2}{\ell^k}$$

since $R = \pi^2/\ell_j \leq \pi^2/\ell$ and $\ell \leq 1$. This gives us the desired upper bound at one point in the boundary of the cusp. The fact that the pole q of G is at least one unit inside the cusp combined with Theorem 5.3 gives us a similar bound at all points on the boundary of the cusp. Finally, we invoke the maximum principle to obtain the bound everywhere outside the cusp. □

Lemma 5.8. *Theorem 5.5 is true if q lies in a boundary collar, or* $\operatorname{dist}(q, Y_0) \leq 1$.

Proof. Let γ be the closest boundary geodesic to q, and let $\pi_\gamma : A_R \to Y$ be standard coordinates about γ. As before, choose $w \in \pi_\gamma^{-1}(q)$ to have minimal modulus. After rotation we can assume that $w = |w|$. If q lies in a boundary collar, then $|w| < 1$. Otherwise, a computation in the Poincaré metric on A_R and Theorem 4.1 imply that

$$\sec \frac{\pi \log |w|}{2R} \leq \frac{C}{\ell^k}.$$

For any $\theta_0 < \pi$, let $m(\theta_0) = \min_{|\theta|<\theta_0} G(|w|e^{i\theta}, w)$. Then by Lemma 5.4,

$$\begin{aligned} m(\theta_0) &\leq \frac{1}{2\theta_0} \int_{-\theta_0}^{\theta_0} G(|w|e^{i\theta}, w)\, d\theta \\ &\leq \frac{\pi}{\theta_0}(\log |w| + R) \qquad \text{by 5.1} \\ &\leq \frac{2R}{\theta_0}. \end{aligned}$$

We want to choose θ_0 so that all points within angular distance θ_0 of w also lie within Poincaré distance $C_1\ell$ of w. One can check by integrating 3.4 along the circle of radius $|w|$ that this will be true if we set

$$\theta_0 = \frac{2R\ell}{\pi} \cos \frac{\pi \log |w|}{2R}.$$

For this value of θ_0 we have

$$m(\theta_0) = \frac{\pi}{\ell} \sec \frac{\pi \log |w|}{2R} \leq \frac{C}{\ell^k}.$$

By the minimum principle and the fact G tends to ∞ near q, the same upper bound holds for $G(p, q)$ for some p such that $\operatorname{dist}(p, q) = \ell$. As we noted above, the hypothesis of this lemma implies that $I(p)$ is at least $C_1\ell$. Thus the set $\{p' \in Y : \operatorname{dist}(p', q) = C_1\ell\}$ constitutes a circle of perimeter less than a constant times ℓ, and we can apply Theorem 5.3 to conclude that

$$G(p', q) \leq \frac{C}{\ell^k}$$

for all points in the circle. By the maximum principle, the upper bound holds everywhere in Y outside the circle. $\square$

6. Proof of Theorem 1.1

Lemma 3.1 proves Theorem 1.1 when Y is a disk or an annulus. We now return to the problem of solving 1.1 for more complicated Y.

To begin with, let $\gamma = \gamma_j$ be a boundary geodesic with standard coordinates $\pi_\gamma : A_R \to Y$ about γ. Lemma 3.1 gives a solution (whose specific form we will use later) to 1.1 on A_R. We use injectivity of π_γ on the inner half of A_R to project this down to a solution η_j of 1.1 on the boundary collar $\mathcal{C}_b(()\gamma)$. We then choose cutoff functions χ_j to patch together the solutions η_j for different values of j. By choosing arbitrarily close smooth approximations, we may assume that χ_j is a piecewise linear function of $\log|z|$ (in standard coordinates). Namely, for $1 \leq j \leq n$ let

$$\chi_j(z) = \begin{cases} 1 \quad \text{if} \quad \log|z| < -R_j/2 \\ \dfrac{-2\log|z|}{R_j} \quad \text{if} \quad -R_j/2 < \log|z| < 0 \\ 0 \quad \text{if} \quad |z| \geq 1. \end{cases} \tag{6.1}$$

We also define local solutions of 1.1 on cusps. Namely, label the cusps $p_1, \dots, p_m$ and consider standard coordinates $\pi_{p_j} : \Delta^* \to Y$ about p_j. Lemma 3.1 again gives us a solution to 1.1 on Δ^*. Since π_{p_j} is injective on $\{|z| < e^{-\pi}\}$, we can project this solution down to a solution η_{j+n} on the cusp $\mathcal{C}_b(p_j)$. In standard coordinates on the cusp, we define a cutoff function

$$\chi_j(z) = \begin{cases} 1 \quad \text{if} \quad |z| < e^{-2\pi} \\ -2\pi\log|z| - 1 \quad \text{if} \quad e^{-2\pi} < |z| < e^{-\pi} \\ 0 \quad \text{if} \quad |z| \geq e^{-\pi}. \end{cases} \tag{6.2}$$

Define

$$\tilde{\eta} = \sum_{j=1}^{n+m} \chi_j \eta_j.$$

Since boundary collars and cusps do not overlap, $\langle\tilde{\eta}\rangle \leq 1$. Let

$$dA_0 = dA - \overline{\partial}\tilde{\eta} = \left(1 - \sum \chi_j\right) dA + \sum \overline{\partial}\chi_j \wedge \eta_j.$$

Then dA_0 is compactly supported in Y. Using the Green's function $G = G(p,q)$ on $\overline{Y}$, we can get a solution η_0 to $\overline{\partial}\eta_0 = dA_0$ as follows: Define a function $h : Y \to \mathbb{R}$ by

$$h(w) = -\int_Y G(p,q)\, dA_0(z). \tag{6.3}$$

Since dA_0 is compactly supported, h is bounded. Furthermore, $\partial\overline{h} = dA_0$, so $\eta_0 = \partial h$ is the form we seek. That is, $\eta = \tilde{\eta} + \eta_0$ satisfies 1.1. Now we only need to obtain a good upper bound on

$$\langle\eta\rangle \leq \langle\tilde{\eta}\rangle + \langle\partial h\rangle.$$

The first term on the righthand side is bounded by 1. We can estimate the second term by using a uniformization map to pull h back to Δ and then applying Corollary 5.2. The result is that

$$\langle\eta\rangle \leq 1 + C_1||h||_\infty + C_2\langle dA_0\rangle, \tag{6.4}$$

We will be done once we have appropriately estimated each of the last two terms.

Lemma 6.1. *The form dA_0 in 6.4 satisfies*

$$\langle dA_0\rangle \leq C$$

for some absolute constant C.

Proof. We have

$$\langle dA_0\rangle \leq \langle dA\rangle + \langle \sum \overline{\partial}\chi_j \wedge \eta_j\rangle = 1 + \langle \sum \overline{\partial}\chi_j \wedge \eta_j\rangle.$$

The terms in the sum contribute to $\langle dA_0\rangle$ only inside cusps and boundary collars where the functions η_j are non–constant. Since cusps and boundary collars are mutually disjoint, at most one term in the sum will be non–zero at any given point. If $1 \leq j \leq n$, we compute in standard coordinates about γ_j that

$$\overline{\partial}\chi_j \wedge \eta_j = \frac{-i}{2R^2|z|^2}\tan\left(\pi\frac{\log|z|}{2R}\right) d\bar{z}\wedge dz, \tag{6.5}$$

for $-R_j/2 < \log|z| < 0$, and $\overline{\partial}\chi_j \wedge \eta_j = 0$ otherwise. From this and 3.4, we see that

$$\langle \overline{\partial}\chi_j \wedge \eta_j\rangle \leq 4.$$

If $n+1 \leq j \leq m$, we compute in standard coordinates about $p_{j-n} \in P$ that

$$\overline{\partial}\chi_j \wedge \eta_j = \frac{-\pi i}{|z|^2\log|z|}\, d\bar{z}\wedge dz.$$

for $-2\pi < \log|z| < -\pi$, $\overline{\partial}\chi_j \wedge \eta_j = 0$ otherwise. We apply 3.3 to obtain that

$$\langle \overline{\partial}\chi_j \wedge \eta_j\rangle \leq 2.$$

Putting these estimates together gives $\langle dA_0\rangle \leq 5$ everywhere on Y. $\square$

To estimate $||h||_\infty$, we break the integral in 6.3 up into several parts. That is,

$$\begin{aligned} |h(q)| \leq & \int_{Y_0} G(p,q)\, dA(p) \\ & + \sum_{j=1}^{n}\left(\left|\int_{\mathcal{C}_b(\gamma_j)} G(p,q)(1-\chi_j)\, dA\right| + \left|\int_{\mathcal{C}_b(\gamma_j)} G(p,q)\,\overline{\partial}\chi_j \wedge \eta_j\right|\right). \\ & + \sum_{j=n+1}^{n+m}\left(\left|\int_{\mathcal{C}_b(p_j)} G(p,q)(1-\chi_j)\, dA\right| + \left|\int_{\mathcal{C}_b(p_j)} G(p,q)\,\overline{\partial}\chi_j \wedge \eta_j\right|\right). \end{aligned} \tag{(6.6)}$$

The first term on the right side can be estimated using Theorem 5.5 and Corollary 5.6.

$$\begin{aligned} \int_{Y_0} G(p,q)\, dA(p) &\leq \int_{Y_0} \frac{C_1}{\ell}^k\, dA + \int_{\mathrm{dist}(p,q)\leq C_2\ell} C_3 \log \mathrm{dist}(p,q) \\ &\leq \frac{C_1 \,\mathrm{Area}\, Y_0}{\ell^k} + C_2 \leq \frac{C}{\ell^k}, \end{aligned}$$

where C and k are constants depending only on m, n, g.

Lemma 6.2. *There is an absolute upper bound for the second line of equation (6.6).*

Proof. The estimates on the terms inside the sum on the second line of (6.6) are standard coordinate computations. We give the details necessary to estimate the second of these terms.

$$\begin{aligned}
\left|\int_{A_R} G(z,w)\,\overline{\partial}\chi_j \wedge \eta_j\right| &= \left|\int_{A_R} G(z,w)\frac{-i\pi}{2R^2|z|^2}\tan\left(\frac{\pi\log|z|}{2R}\right)\,d\bar{z}\wedge dz\right| \\
&= \left|\int_{e^{-R/2}}^{1}\int_0^{2\pi}\frac{\pi G(re^{i\theta},w)}{2R^2 r}\tan\left(\frac{\pi\log r}{2R}\right)\,drd\theta\right| \\
&\le \left|\int_{e^{-R/2}}^{1}\frac{2\pi^2\alpha_j}{R^2 r}(\log r + R)\tan\left(\frac{\pi\log r}{2R}\right)\,dr\right| \\
&= \left|4\alpha_j\int_{-\pi/4}^{0}(2u+\pi)\tan u\,du\right| \qquad \text{(where } u = \pi\log r/2R) \\
&\le 4\pi\alpha_j.
\end{aligned}$$

The inequality on the third line follows from Lemma 5.4. A similar computation shows that the first term inside the sum in (6.6) is also dominated by a constant times α_j. Since the α_j's sum to 1, we see that the contribution made by the seconed line of (6.6) is no greater than a constant. $\square$

Lemma 6.3. *The third line of equation (6.6) is bounded by C/ℓ^k for some constant C depending on g, m, n.*

Proof. Fix a puncture p_j, and in standard coordinates about p_j set

$$A(r) = \int_0^{2\pi} G(re^{i\theta}, w)\,dw,$$

where w is as usual a preimage of the pole of G with minimal modulus. By Theorem 5.5 and Corollary it is not hard to see that $A(e^{-\pi}) \le C/\ell^k$. Arguing as in the proof of Lemma 5.4, we see that either $A(r)$ remains constant as r decreases from $e^{-\pi}$ (i.e. the pole q of G lies outside the cusp about p_j), or $A(r)$ grows like $-\log r$ as r decreases from $e^{-\pi}$ (i.e. the pole lies inside the cusp). Either way, we have

$$A(r) \le \frac{C}{\ell^k} - \log r + \pi$$

for all $r \le e^{-\pi}$. In both integrals on the third line of (6.6) the part of the integrand complementary to G has a size which depends only on r in local coordinates. Furthermore, this size is non–zero only for $e^{-\pi} \le r \le e^{-2\pi}$, and it is bounded above absolutely even on this interval. A simple polar coordinate calculation now finishes the proof. $\square$

Inserting the conclusions of the previous two lemmas into equation (6.6) shows that

$$|h(q)| \leq \frac{C}{\ell^k}$$

for all $q \in Y$. This estimate, along with equation 6.4 and Lemma 6.1 combine to show that our solution η to equation 1.1 on Y satisfies the bound

$$\langle \eta \rangle \leq \frac{C}{\ell^k},$$

where C and k depend only on g, m, n—i.e. only on the topology of Y. This concludes the proof of Theorem 1.1.

7. Conclusion

One might pursue several useful generalizations of Theorem 1.1. An interesting possibility would be to try to solve 1.1 only on the part Y_ϵ of Y where the injectivity radius is greater than some fixed number ϵ. We would hope to find solutions whose bounds depend on ϵ rather than on the length of the shortest closed geodesic on Y.

We mentioned in the introduction to this paper that equation 1.1 is related to averaging problems for multiple–valued holomorphic functions. We would very much like to know whether small solution of 1.1 implies corresponding good solutions to averaging problems? Chapter 4 of [Di1] discusses the converse of this question at length.

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