Dynamics of Birational Maps II: Limits of Non-Closed Currents and the Invariant Measure

Jeffrey Diller *
Cornell University
Ithaca, NY 14853

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1 Introduction

Anyone who seeks to understand the dynamics of a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ faces an immediate problem: birational maps are not generally maps. That is, except when f_+ has degree one, there exists a finite non-empty set I^+ of points where f_+ cannot be defined continuously. In a precise sense, f_+ "blows up" each of these points of indeterminacy to an entire algebraic curve. Nevertheless, in this paper and its predecessor [Dil], we present an approach to the dynamics of birational maps of $\mathbf{P}^2$ which, as far as possible, pretends to deal with smooth dynamical systems.

There are two reasons for our hope in this approach. One is that we have pluripotential theory at our disposal. Brolin [Bro] showed three decades ago that one could produce a very natural subharmonic "Green's" function by iterating a polynomial map of $\mathbf{C}$. More recently, Hubbard and Papadopol [HP] presented a generalization of Brolin's construction that applies at least formally to any rational map of $\mathbf{P}^n$ in any dimension n. We showed in [Dil]

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that for birational maps of $\mathbf{P}^2$ the construction turns out to be effective, giving us a Green's function to use as a starting point for understanding dynamics. The other reason for our hope is that one class of birational maps has been studied with much success recently. Bedford, Smillie et all (see [FM], [BS1], [BS2], [BS3], [FS1], [HO1], [HO2]) have obtained a detailed picture of the dynamics of polynomial diffeomorphisms of $\mathbf{C}^2$. Since these maps extend birationally to $\mathbf{P}^2$, we are able to conduct our own more general investigation with a reliable model and proven techniques to guide us. The reader familiar with work done for polynomial diffeomorphisms will recognize several of the theorems in this paper. In places, we are even able to appropriate proofs with only minor changes. Nonetheless, as we hope will become evident, it is our contribution here and in [Dil] to develop the pluripotential theoretic framework to the point where such proofs apply.

Before proceeding further, we note that others have taken different approaches to dealing with indeterminacy in rational maps. Papers by Friedland [Fri2] [Fri1], Hubbard and Oberste-Vorth [HO1] [HO2], Hubbard, Papadopol and Veselov [HPV], have proceeded by replacing $\mathbf{P}^2$ with a more complicated space. The idea is that the rational map will lift to a completely well-defined map on the new space. A recent paper of Russakovski and Shiffman [RS] adopts a distribution theoretic point of view toward dynamics of rational maps. Our use of the graph of f_+ in Section 5 below is inspired by that paper.

After presenting some background on birational maps and on currents in Section 2, we discuss the pushforward and pullback of a positive closed $(1,1)$ current T by a birational map f_+. These notions are usually only defined for smooth maps, but the algebraic nature of a birational map and the close relationship between positive closed currents and algebraic varieties allows us to reasonably extend the usual definitions here. In algebraic terms we define the pushforward of T by f_+ to be the natural generalization of the proper transform of an algebraic curve. Likewise, we define the pullback of T by the inverse map f_- to be the natural generalization of the total transform of an algebraic curve by f_+. The main result of the section is Theorem 2.3, which says that these two operations agree for a positive closed $(1,1)$ current T if and only if the Lelong number of T vanishes at each point of I^+.

Section 3 provides the connection between Theorem 2.3 and the rest of the paper. Given that the (algebraic) degrees of iterates of f_+ grow properly, i.e. that $\deg f_+^n = (\deg f_+)^n$ for all n, one can construct a canonical current

$$\mu^+ = \lim_{n\to\infty} \frac{1}{\deg f_+^n} f_+^* \Theta,$$

where Θ is the usual Fubini-Study Kähler form on $\mathbf{P}^2$. Corollary 3.7, a consequence of Theorem 2.3, states that μ^+ is an extreme point in the cone of closed, positive (1,1) currents. One can view this corollary as a sort of strong ergodicity property of μ^+, and we make essential use of it to prove a uniqueness result in Section 5.

In Section 4, we show that a certain natural measure associated with f_+ is invariant. The current μ^+ satisfies the simple transformation property $f_{+*}\mu^+ = \mu^+/\deg f_+$. The corresponding current μ^- associated with the inverse map f_- satisfies $f_{+*}\mu^- = (\deg f_+) \cdot \mu^-$. Therefore, one might expect the measure $\mu = \mu^+ \wedge \mu^-$ to be invariant under pushforward. However, this heuristic expectation relies on the assumptions that the wedge product between μ^+ and μ^- admits a reasonable definition and that if so, f_{+*} will act distributively across the wedge product. Since local potentials for μ^+ and μ^- can be quite singular, neither of these assumptions are obviously true. Nevertheless, if the *extended indeterminacy set* (i.e. the closure of the backward orbit of I^+) of f_+ is disjoint from the extended indeterminacy set of f_-, we are able to justify both of them.

In section 5, we turn our attention to positive $(1,1)$ currents which are not closed—more precisely, we consider positive and locally closed $(1,1)$ currents which have been "truncated" by multiplying with smooth functions. The definition of pushforward given in Section 2 does not easily apply in this setting. Therefore, we take inspiration from [RS], and define pushforward using a desingularization of the graph of f_+. Our main results (Theorem 5.4 and 5.6) state that iterated pushforwards of a truncated (1,1) current tend to converge to μ^- after normalization. The theorems are direct generalizations of the results presented in Section 1 of [BS3].

Sections 6 and 7 present some implications of section 5 for the dynamics of f_+. Little is known at present about the topological properties of $\operatorname{supp}\mu^+$, so we show in section 6 that this set tends to be nowhere dense. For instance, if $\operatorname{supp}\mu^-$ excludes a single point of $\operatorname{supp}\mu^+$, then $\operatorname{supp}\mu^+$ is nowhere dense. In particular, if f_+ has an attracting periodic point and satisfies the hypothesis on the extended indeterminacy set employed in section 4, then $\operatorname{supp}\mu^+$ is nowhere dense. We also show that if f_+ is *completely separating* (see section 5 below), then $\operatorname{supp}\mu^-$ is equal to the closure of the unstable manifold of any saddle periodic point. In section 7 we prove three ergodicity-type results. Most significantly, we show that f_+ is mixing with respect to the measure $\mu = \mu^+ \wedge \mu^-$ defined in Section 4.

2 Birational Maps and Positive Closed (1,1) Currents

Let $\pi : \mathbf{C}^3 \setminus \{\mathbf{0}\} \to \mathbf{P}^2$ be the canonical projection giving homogeneous coordinates on $\mathbf{P}^2$. Any rational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ can be regarded as the natural relation induced by a homogeneous polynomial map $\tilde{f} : \mathbf{C}^3 \to \mathbf{C}^3$. Clearly, f does not change if we multiply each of the coordinates of $\tilde{f}$ by the same homogeneous polynomial. Therefore, we will assume that $\tilde{f}$ is a *minimal* representative for f in the sense that the coordinate functions of f have lowest possible degree. Under this assumption, we define the (algebraic) degree $\deg f$ to be the degree of $\tilde{f}$.

The critical set $\mathcal{C}$ of f is an algebraic curve equal to the image under π of the critical set of $\tilde{f}$. It can happen that $\tilde{f}^{-1}(\mathbf{0})$ is non-trivial even when $\tilde{f}$ is minimal. In this case $f(\pi(\tilde{p}))$ is ill-defined whenever $\tilde{f}(\tilde{p}) = \mathbf{0}$. The set $I = \pi(\tilde{f}^{-1}(\mathbf{0})) \subset \mathbf{P}^2$ of all such *points of indeterminacy* is always finite, and we will persist in writing $f : \mathbf{P}^2 \to \mathbf{P}^2$, as if f were well-defined everywhere.

A rational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is *birational* if there exists another rational map $f_- : \mathbf{P}^2 \to \mathbf{P}^2$ and an algebraic curve V such that $f_+ \circ f_- = f_- \circ f_+ = \mathrm{id}$ on $\mathbf{P}^2 \setminus V$. The use of $+/-$ superscripts to distinguish a birational map from its rational inverse emphasizes the fact that f_+ and f_- are not, strictly speaking, set theoretic inverses. We will use $+/-$ subscripts and superscripts in all of what follows to distinguish objects corresponding to f_+ from objects corresponding to f_-. For instance, I^- denotes the indeterminacy set for f_-. The following proposition (see [Dil] for a proof) describes the relationship between indeterminacy and critical sets for a birational map.

Proposition 2.1 *The following statements are true for any birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$.*

1. *$I^+ \subset \mathcal{C}^+$, and every irreducible component of $\mathcal{C}^+$ contains a point of I^+.*

2. *Given any irreducible curve $V \subset \mathcal{C}^+$, $f_+(V)$ is a single point in I^-; likewise, given any $p^- \in I^-$, $f_+^{-1}(p^-)$ is a component of $\mathcal{C}^+$. In particular*

$\mathcal{C}^+$ coincides with the exceptional set of f_+.

3. $f_+ : \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ is a biholomorphism.

Note that we make an important technical distinction between the image of a closed set K under f_+ and its preimage under f_-. We declare that $f_+(K) = \overline{f_+(K \setminus I^+)}$ and $f_-^{-1}(K) = \overline{\{p \in \mathbf{P}^2 \setminus I^- : f_-(p) \in K\}}$. In general, $f_+(K) \subset f_-^{-1}(K)$, but the inclusion can be strict if equality holds if and only if $K \cap I^+ \neq \emptyset$.

Degree one birational maps of $\mathbf{P}^2$ are dynamically rather simple, so we assume in what follows that all birational maps under consideration have degree greater than one. Such maps will necessarily have non-empty critical sets and therefore, by Proposition 2.1, non-empty indeterminacy sets as well. Therefore, one must be rather careful when using a birational map to transform an analytic object such as a form or a current.

We want specifically to consider actions of birational maps on positive closed $(1,1)$ currents. Before doing so, however, we fix some notation and recall a couple of facts about positive currents on $\mathbf{P}^2$. For more thorough background on positive currents, we refer the reader to the book [Kli] by Klimek and survey articles by Demailly [Dem] and Skoda [Sko]. The *mass* of a positive current T on a set $K \subset \mathbf{P}^2$ is

$$\mathrm{M}_K[T] = \sup\{T(\varphi) : |\varphi| \leq 1, \operatorname{supp}\varphi \subset K\}.$$

Of course this definition implies the choice of an Hermitian metric on a neighborhood of $\overline{K}$, but for any two such choices, the resulting mass norms are comparable. Where we do not indicate otherwise, we imply the use of the Fubini Study metric on $\mathbf{P}^2$, letting Θ denote the associated Kähler form. It turns out that

$$||T|| \stackrel{\text{def}}{=} \mathrm{M}_{\mathbf{P}^2}[T] = \int_{\mathbf{P}^2} T \wedge \Theta.$$

If T is positive and closed on $\mathbf{P}^2$, then T can be written locally as $dd^c u$ for some plurisubharmonic function u. Fornæss and Sibony [FS2] have shown more strongly that there exists a global "homogeneous" potential $\tilde{u} : \mathbf{C}^3 \to \mathbf{R} \cup \{-\infty\}$ for T. That is,

$$\begin{aligned} \pi^* T &= dd^c \tilde{u} \\ \tilde{u}(\lambda\tilde{p}) &= \tilde{u}(\tilde{p}) + c \log|\lambda|. \end{aligned}$$

for every $\lambda \in \mathbf{C}$, every $\tilde{p} \in \mathbf{C}^3$, and $c = ||T||$. Given T, the potential $\tilde{u}$ is unique up to additive constants. Moreover, any $\tilde{u}$ satisfying the homogeneity condition for some $c > 0$ induces a positive closed $(1,1)$ current T on $\mathbf{P}^2$ as follows: If $U \subset \mathbf{P}^2$ and $\pi^{-1} : U \to \mathbf{C}^3$ is a holomorphic section, then $T|_U$ is given by $dd^c\tilde{u} \circ \pi^{-1}$. Homogeneity guarantees that this definition does not depend on the choice of π^{-1}. We will abuse notation by writing $T = \pi_* dd^c \tilde{u}$.

Now we recall two actions of a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ on a positive closed $(1,1)$ current $T = \pi_* dd^c\tilde{u}$. First of all, we define the "pullback" by $f_+^* T = \pi_* dd^c(\tilde{u} \circ \tilde{f})$. Besides its consistency with notation used in related papers (e.g. [HP] and [FS2]), this definition of f_+^*T has the advantage that mass transforms predictably according to the formula $||f_+^*T|| = \deg f_+ ||T||$. As we hope will emerge below, f_+^*T is in some sense the largest reasonable notion of the preimage of T, generalizing the notion of the total transform of an algebraic curve by a rational map.

Next we define the pushforward of T by f_+. Taking advantage of the fact that $f_+ : \mathbf{P}^2 \backslash \mathcal{C}^+ \to \mathbf{P}^2 \backslash \mathcal{C}^-$ is a biholomorphism, we first push the restriction $T|_{\mathbf{P}^2 \backslash \mathcal{C}^+}$ forward to a positive closed $(1,1)$ current on $\mathbf{P}^2 \backslash \mathcal{C}^-$. We then extend T by zero across $\mathcal{C}^-$. Thanks to an extension theorem of Harvey and Polking [HP], the result is a well-defined positive closed (1,1) current on $\mathbf{P}^2$. We denote this current by $f_{+*}T$. It should be clear that $f_{+*}T$ is the smallest reasonable notion of the image of T, analogous to the proper transform of a curve by a rational map. Before stating the next proposition, we recall that T is *extremal* among positive closed $(1,1)$ currents if every decomposition $T = T_1 + T_2$ into a sum of positive closed currents is trivial—i.e. $T_j = c_j T$.

Proposition 2.2 *If T is extremal, then so is $f_{+*}T$. If $T|_{\mathcal{C}^+} = 0$ and $f_{+*}T$ is extremal, then so is T.*

Proof. First assume that T is extremal. Let $f_{+*}T = S_1 + S_2$ be a decomposition. By adding and subtracting a multiple of $f_{+*}T$ to the right side, we can assume that S_1 dominates no positive multiple of $f_{+*}T$. It is clear that pushforward acts linearly and preserves positivity, so item (5) of Proposition 4.7 in [Dil] gives that $T \geq f_{-*}f_{+*}T \geq f_{-*}S_1$. That is, $T = f_{-*}S_1 + (T - f_{-*}S_1)$, so $f_{-*}S_1 = cT$. Using the same fact from [Dil], we then conclude that $S_1 \geq cf_{+*}T$ and that therefore, $c = 0$. The only non-trivial elements in the kernel of f_{-*} are supported on $\mathcal{C}^-$, and by definition $f_{+*}T$ has no support on this set. Hence $S_1 = 0$.

Now assume that $f_{+*}T$ is extremal and T has no mass concentrated on $\mathcal{C}^+$. Let $T = T_1 + T_2$ be a decomposition, and note that $f_{+*}T = f_{+*}T_1 + f_{+*}T_2$.

Hence $f_{+*}T_1 = cf_{+*}T$. From the above-mentioned fact in [Dil], we have $T_1 = cT$. $\square$

As with images and preimages of closed sets, it is not always the case that $f_{+*}T = f_-^*T$. The main result of this section is a necessary and sufficient condition for equality. To state it, we recall that the *Lelong number* of a positive closed current T at $p \in \mathbf{P}^2$ is given in local coordinates z centered at p by

$$\nu(T,p) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{||z||<r} T \wedge \theta,$$

where $\theta = dd^c||z||^2$. If $T = dd^c u$ near p, then the Lelong number can be computed from u by (see [Dem], equation (5.5e))

$$\nu(T,p) = \sup\{\gamma \geq 0 : u(q') \leq \gamma \log \operatorname{dist}(p,q) + O(1)\}. \tag{1}$$

Theorem 2.3 *Suppose that T is a positive closed $(1,1)$ current on $\mathbf{P}^2$, and that $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is birational. Then $f_-^*T - f_{+*}T$ is a non-negative linear combination of currents of integration over components of $\mathcal{C}^-$. Furthermore, $f_{+*}T = f_-^*T$ if and only if $\nu(T,p) = 0$ for every $p \in I^+$.*

This theorem is a consequence of the following result about Lelong numbers.

Theorem 2.4 *Suppose that T is a positive closed $(1,1)$ current on $\mathbf{P}^2$. Then $\nu(f_-^*T, p) \neq 0$ if and only if either $p \in I^-$ or $\nu(T, f_-(p)) \neq 0$.*

Proof of Theorem 2.3 Since $f_+ : \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ is a biholomorphism, $f_{+*}T$ and f_-^*T coincide with the usual notions of pushforward and pullback on $\mathbf{P}^2 \setminus \mathcal{C}^-$. In particular, they coincide with each other on this set. Hence $f_-^*T - f_{+*}T$ is supported on $\mathcal{C}^-$. The restriction of $f_{+*}T$ to $\mathcal{C}^-$ is trivial by definition, so $f_-^*T - f_{+*}T$ is positive. A well-known theorem of Siu [Siu] implies that $f_-^*T - f_{+*}T = \sum_{V \subset \mathcal{C}^-} c_V[V]$, where $V \subset \mathcal{C}^-$ is an irreducible component and $c_V \geq 0$. If $c_V > 0$ then $\nu(T,p) > 0$ for every $p \in V$. Therefore, we can apply Theorem 2.4 to any $p \in V \setminus I^-$ and conclude that $\nu(T, f_-(p)) > 0$. Since $f_-(p) \in I^+$, this proves the 'only if' portion of the corollary.

If on the other hand $f_-^*T - f_{+*}T = 0$, then it follows that the restriction of f_-^*T to $\mathcal{C}^-$ is trivial. Thus by Siu's results again, $\nu(f_-^*T, p) = 0$ for every

$p \in \mathcal{C}^-$ outside a countable subset. Each $p \in I^+$ is the f_--image of some non-trivial algebraic curve in $\mathcal{C}^-$ by Proposition 2.1—in particular, $p = f_-(q)$ for some q such that $\nu(f_-^*T, q) = 0$. Therefore, Theorem 2.4 implies that $\nu(T, p) = 0$ as well. $\square$

Proof of Theorem 2.4 Let $\tilde{u} : \mathbf{C}^3 \to \mathbf{R} \cup \{-\infty\}$ be a homogeneous potential for T, and let $\tilde{f}_-$ be a homogeneous representative for f_-. Choose a holomorphic section $\pi^{-1} : W \to \mathbf{C}^3 \setminus \{0\}$ of π on some neighborhood $W \ni p$. Then

$$u = \tilde{u} \circ \tilde{f}_- \circ \pi^{-1}$$

is a local potential for f_-^*T on W. For any $q \in W$,

$$u(q) = \tilde{u}\left(\frac{\tilde{f}_- \circ \pi^{-1}(q)}{||\tilde{f}_- \circ \pi^{-1}(q)||}\right) + \log||\tilde{f}_- \circ \pi^{-1}(q)|| \leq M + \log||\tilde{f}_- \circ \pi^{-1}(q)||,$$

where M is the maximum value of $\tilde{u}$ on the unit sphere in $\mathbf{C}^3$. Suppose first that $p \in I^-$—i.e. that $\tilde{f}_- \circ \pi^{-1}(p) = \mathbf{0}$. Hence, $||\tilde{f}_- \circ \pi^{-1}||$ tends to zero at a polynomial rate near p, giving

$$u(q) \leq M + \log A\mathrm{dist}(q, p)^k = A + B \log \mathrm{dist}(q, p)$$

for some constants $A, B > 0$. In combination with equation (1) this implies that $\nu(f_-^*T, p) > 0$.

Now suppose that $p \notin I^-$. Choose a holomorphic section π_V^{-1} of π on a neighborhood V of $f_-(p)$. We can assume that the neighborhood $W \ni p$ is small enough that $f_-(W) \subset V$. We choose $v = \tilde{u} \circ \pi_V^{-1}$ as a local potential for T near $f_-(p)$. From the relation $\pi \circ \tilde{f}_- = f_- \circ \pi$, we see that $\tilde{f}_- \circ \pi^{-1}(q) = \lambda(q)\,[\pi_V^{-1} \circ f_-](q)$ for all $q \in W$ and some non-vanishing holomorphic function λ. Therefore,

$$(v \circ f_-) - u = \tilde{u} \circ \pi_V^{-1} \circ f_- - \tilde{u} \circ \tilde{f}_- \circ \pi^{-1} = \log|\lambda|$$

is a harmonic function. We conclude that $v \circ f_-$ is a potential for f_-^*T on W.

Suppose that $\nu(T, f_-(p)) \neq 0$. That is, there exist constants $A, B > 0$ such that

$$v(q) \leq A \log \mathrm{dist}(q, f_-(p)) + B. \tag{2}$$

for all $q \in V$. From equation (2) and the fact that f_- is uniformly Lipschitz on small neighborhoods of p, we obtain

$$v \circ f_-(q) \leq A \log \mathrm{dist}(f_-(q), f_-(p)) + B \leq A \log \mathrm{dist}(q, p) + B'$$

on W. Therefore, $\nu(f_-^*T, p) \neq 0$.

Finally, suppose that $p \notin I^-$ and $\nu(f_-^*T, p) \neq 0$. If $p \notin \mathcal{C}^-$, then we can argue as in the previous case. That is, by equation 1 there exist constants $A, B > 0$ such that

$$u(q) \leq A \log \operatorname{dist}(q, p) + B$$

for all q near p. Since $p \notin I^-$, we have that f_+ is Lipschitz at $f_-(p)$ and therefore that

$$u \circ f_+(q) \leq A \log \operatorname{dist}(f_+(q), p) + B \leq A \log \operatorname{dist}(q, f_-(p)) + B'$$

for all q near $f_-(p)$. But $T = dd^c(u \circ f_+)$ near $f_-(p)$. Therefore, $\nu(T, f_-(p)) \neq 0$.

If $p \in \mathcal{C}^-$, we must argue more carefully. The problem is that f_+ is not Lipschitz near $f_-(p)$. The f_+-image of a small neighborhood of $f_-(p)$ will always contain an entire component of $\mathcal{C}^-$. After passing to local coordinates, we can assume that $W = V = \Delta \times \Delta$ and that $p = f_-(p) = (0, 0)$. We can also assume that the set $\Delta \times \{0\}$ intersects $\mathcal{C}^-$ only at $(0, 0)$. Let $\Delta(r)$ denote the set $\{(x, 0) : |x| < r\}$ and let $D(r, x, y)$ denote the translate of $f_-(\Delta(r))$ by (x, y). We rely on the following technical lemma to complete the proof.

Lemma 2.5 *There exist constants $C, k > 0$ such that the following holds: if $r < 1/2$ and $||(x, y)|| < Cr^k$, then $bD(r, x, y) \subset f_-(W)$ and, more specifically,* $\operatorname{dist}(f_+(bD(r, x, y)), b\Delta(r)) < r/2$.

By assumption and equation (1), we have

$$v \circ f_-(x, y) \leq A \log ||(x, y)|| + B$$

for all $(x, y) \in \Delta^2$ and constants $A, B > 0$. In particular, if $r < 1/2$ and $\operatorname{dist}((x, y), b\Delta(r)) < r/2$, we have

$$v \circ f_-(x, y) \leq A \log r + B'.$$

Applying the lemma, we see that

$$v \leq A \log r + B'$$

on $bD(r, x, y)$ whenever $||(x, y)|| < Cr^k$. By the maximum principle, the estimate carries over to all of $D(r, x, y)$. Note that the union of these $D(r, x, y)$ contains $B_{\mathbf{0}}(Cr^k)$. Setting $s = Cr^k$, we obtain

$$v \leq A' \log s + B''$$

Figure 1: Situation covered by Lemma 2.5

on $B_{\mathbf{0}}(s)$ for some constants $A', B'' > 0$. Letting r (and thus s) vary, we conclude that

$$v(x,y) \leq A' \log ||(x,y)|| + B''$$

for (x, y) close enough to $(0, 0)$. Hence $\nu(T, f_-(p)) \neq 0$ as desired. $\square$

Proof of Lemma 2.5 Since $\Delta(1) \cap \mathcal{C}^- = \{(0,0)\}$, we see that f_- is non-constant (in fact, injective) on $\Delta(1)$. By considering the component functions of f_-, one of which must be non-constant, we see that

$$||f_-(x,0)|| \geq A|x|^j \tag{3}$$

for some $A, j > 0$ and $|x|$ small enough. Since f_- is injective on $\Delta(1)$, compactness allows us to extend this estimate with possibly smaller constants to $x \in \Delta(1/2)$. The Euclidean metric in local coordinates is comparable on compact sets of $\Delta \times \Delta$ to the Fubini-Study metric on $\mathbf{P}^2$. Consequently, the coordinate estimate (3) translates into the estimate

$$\mathrm{dist}(f_-(q), f_-(p)) \geq Ar^j \tag{4}$$

for all $q \in b\Delta(r)$ and constants $A, j > 0$. We will use Euclidean distance and Fubini-Study distance interchangeably throughout the rest of the proof.

In [Dil], we defined the expansion function $\gamma : \mathbf{P}^2 \to \mathbf{R} \cup \{-\infty\}$ associated with f_+ (note that we capitalized γ in [Dil]; in this paper, we will make different use of the capital letter Γ). For our purposes, the two key properties of γ are Propositions 2.6 and 2.7 in [Dil]. That is, there exist constants $A, B, C > 0$ such that for all $q, q' \in \mathbf{P}^2$

$$\gamma(q) \geq A \log \mathrm{dist}(q, I^+) - B$$

and

$$\mathrm{dist}(f_+(q), f_+(q')) \leq Ce^{-\gamma},$$

where $\gamma = \min\{\gamma(q), \gamma(q')\}$. Assuming that $\mathrm{dist}(q, q') < \frac{1}{2}\mathrm{dist}(q, I)$, we can combine these two equations and obtain

$$\mathrm{dist}(f_+(q), f_+(q')) \leq A \frac{\mathrm{dist}(q, q')}{[\mathrm{dist}(q, I^-)]^k} \tag{5}$$

for constants $A, k > 0$.

In the present context, if $q \in f_-(b\Delta(r)) = bD(r, 0, 0)$ and $q' \in bD(r, x, y)$ for $||(x, y)||$ small enough (e.g. one half of the right side of equation (3) will do), then equation (5) implies that

$$\mathrm{dist}(f_+(q), f_+(q')) \leq A \frac{||(x, y)||}{r^k}$$

for constants $A, k > 0$. From this it is evident that to achieve

$$\mathrm{dist}(f_+(q), f_+(q')) \leq r/2$$

it is enough to require

$$||(x, y)|| \leq Cr^k$$

for appropriate positive constants C and k. □

3 Escape Currents are Extremal

Hubbard and Papadopol [HP] showed that one can construct a positive closed $(1, 1)$ current on $\mathbf{P}^2$ by iterating a holomorphic map $f : \mathbf{P}^2 \to \mathbf{P}^2$. Their construction applies at least formally to a birational map f_+ of $\mathbf{P}^2$. Given a minimal representative $\tilde{f}_+$, one defines a homogeneous potential

$$\tilde{G}^+(\tilde{p}) = \lim_{n\to\infty} \frac{1}{\deg \tilde{f}_+^n} \log ||\tilde{f}_+^n(\tilde{p})||, \tag{6}$$

which induces a positive closed (1,1) current T on $\mathbf{P}^2$. The obvious difficulty in this construction is to ensure that the limit defining $\tilde{G}^+$ converges. A less obvious but dynamically more fundamental difficulty is that $\tilde{f}_+^n$ need not be a minimal representative for f_+^n—i.e. $\deg f_+^n$ can be strictly less than $\deg \tilde{f}_+^n$.

Let $I_n^+ = \bigcup_{j=0}^{n-1} f_+^{-j}(I^+)$ be the indeterminacy set of of f^n. It is a simple consequence of Proposition 2.1 that degree growth of the iterates of $\tilde{f}_+$ is related to the dynamical interaction between I^+ and I^-. We observed in [Dil] that

Proposition 3.1 *The following statements are equivalent for a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ with inverse f_-:*

1. $\deg(f_+^n) = (\deg f_+)^n$ *for all n;*
2. $I^+ \cap f_+^n(I^-) = \emptyset$ *for all n.*
3. $f_-^n(I^+) \cap f_+^m(I^-) = \emptyset$ *for all $n, m \geq 0$;*

Fornæss and Sibony [FS2] call rational maps *generic* if they satisfy the first condition. Since in our narrower context, this condition is equivalent to conditions requiring indeterminacy orbits to avoid each other, we refer to a birational map satisfying condition 1 as *minimally separating*. Condition 3 of proposition 3.1 implies that f_+ is minimally separating if and only if f_- is. Condition 2 applied to f_- shows that the indeterminacy set of f_+^n is $I_n^+ = \bigcup_{j=0}^{n-1} f_-^j(I^+)$. For convenience, we allow $n = \infty$ in the definition of I_n^+. We showed in [Dil] that the construction of Hubbard and Papadopol succeeds for minimally separating birational maps (see also [Fav]; Sibony [Sib] has recently given an elementary proof of this result for arbitrary generic rational maps).

Theorem 3.2 *If f_+ is minimally separating, then the limit in (6) converges pointwise and in L^1_{loc} to a plurisubharmonic function $\tilde{G}^+$ satisfying*

1. $\tilde{G}^+ \circ \tilde{f}_+(\tilde{p}) = (\deg f_+) \cdot \tilde{G}^+(\tilde{p})$*;*

2. $\tilde{G}^+(\lambda\tilde{p}) = \tilde{G}^+(\tilde{p}) + \log|\lambda|$;

for all $\tilde{p} \in \mathbf{C}^3$ *and all* $\lambda \in \mathbf{C}$.

We refer to $\tilde{G}^+$ as the *escape function* for f_+ and note that f_+ determines $\tilde{G}^+$ up to an additive constant. It is important to note that by replacing $\tilde{f}_+$ with a small multiple of $\tilde{f}_+$ (if necessary), one can arrange that the sequence defining $\tilde{G}^+$ is actually decreasing. We refer to the unique induced current $\mu^+ = \pi_* dd^c\tilde{G}^+$ as the *escape current* for f_+. We showed in our previous paper that μ^+ transforms well under f_+.

Theorem 3.3 *The escape current μ^+ for a minimally separating birational map has the following properties:*

1. *μ^+ has no support concentrated on any algebraic curve (see [FS2]);*
2. *$f_+^*\mu^+ = (\deg f_+) \cdot \mu^+$;*
3. *$f_{-*}\mu^+ = (\deg f_+) \cdot \mu^+$;*
4. *$f_{+*}\mu^+ = \mu^+/\deg f_+$.*

Remark 3.4 *The assertion that is conspicuously missing in the conclusion of Theorem 3.3—namely, that $f_-^*\mu^+ = \mu^+/\deg f_+$—cannot be true since $||f_-^*\mu^+|| = (\deg f_+) \cdot ||\mu^+|| \neq ||\mu^+/\deg f_+||$. In particular, $f_-^*\mu^+ \neq f_{+*}\mu^+$. This fact can also be seen rather readily from Theorem 2.3, equation (6), and equation (1).*

An immediate consequence of Theorems 2.3 and 3.3 is

Corollary 3.5 *If $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is minimally separating, then $\nu(\mu^+, p) = 0$ for each $p \in I_\infty^-$.*

Another consequence of Theorem 2.3 for escape currents depends on a convergence theorem from [Dil].

Theorem 3.6 *Suppose that $W \subset \mathbf{P}^2$ is a (possibly empty) open set containing all superattracting periodic points of a minimally separating birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$. Suppose that $\{T_n\}$ is a sequence of positive closed $(1,1)$ currents such that $\operatorname{supp} T_n \cap W = \emptyset$ and that $||T_n|| = c$ is constant with respect to n. Then*

$$\lim_{n\to\infty} \frac{1}{\deg f_+^n} f_+^{n*} T_n = c\mu^+.$$

This convergence theorem implies an "ergodic-like" property for μ^+.

Corollary 3.7 *If $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is minimally separating, then μ^+ is extremal in the cone of positive closed $(1,1)$ currents. That is, if $T \leq \mu^+$ for some positive closed $(1,1)$ current T, then $T = c\mu^+$ for some $c > 0$.*

This corollary is proven for Hénon maps in Section VII.3 of [FS3]. Our proof is a generalization of the one given there.

Proof. We can more or less repeat the proof of Theorem 5.5 from [Dil], using Theorem 2.3 to cope with the weaker hypothesis that we employ here. Note that according to our definition, birational pushforward acts linearly on currents and preserves positivity. It follows from this observation and Theorem 3.3 that both $f^n_{+*}T$ and f^{n*}_+T are dominated by multiples of μ^+ for all $n \geq 0$. Consequently, Corollary 3.5 implies that $\nu(f^n_{+*}T, p) = 0$ at every point in I^-_∞.

We showed in [Dil] that $T - f^n_{-*}f^n_{+*}T$ is positive and concentrated on an algebraic curve. Since μ^+ concentrates no support on any algebraic curve, we must actually have $T = f^n_{-*}f^n_{+*}T$ for all n. Theorem 2.3 implies further that $T = f^{n*}_+ f^n_{+*}T$. In particular,

$$||T|| = ||f^{n*}_+ f^n_{+*}T|| = (\deg f^n_+) \cdot ||f^n_{+*}T||.$$

Hubbard and Papadopol [HP] showed that if iterates of f form a normal family on an open set W, then $\operatorname{supp}\mu^+ \cap W = \emptyset$. Therefore, there is a neighborhood W of any superattracting cycle such that

$$(\operatorname{supp} f^n_{+*}T) \cap W \subset (\operatorname{supp}\mu^+) \cap W = \emptyset$$

for all n. We can now apply Theorem 3.6 to the sequence $T_n = (\deg f^n_+) f^n_{+*}T$ to conclude that

$$||T||\mu^+ = \lim_{n\to\infty} \frac{1}{\deg f^n_+} f^{n*}_+(d^n f^n_{+*}T) = T.$$

□

4 Wedge Products and the Invariant Measure

A formal construction suggests that pluripotential theory ought to yield an invariant measure for minimally separating birational maps. Namely, since f_+ is minimally separating if and only if f_- is, we can apply Theorem 3.2 to construct currents μ^+ and μ^- associated with f_+ and f_-, respectively. Then we set $\mu = \mu^+ \wedge \mu^-$. It seems reasonable that

$$f_{+*}\mu = f_{+*}\mu^+ \wedge f_{+*}\mu^- = \frac{\mu^+}{\deg f_+} \wedge (\deg f_+)\mu^- = \mu^+ \wedge \mu^- = \mu. \tag{7}$$

However, for the same reason that one cannot always multiply a pair of distributions together, it is not generally possible to form the wedge product of two currents. Furthermore, even if one can make sense of the wedge product, it remains to determine whether pushforward by f_+ will distribute across the product as is assumed in (7). Our goal in this section is to overcome these difficulties and show that with a stronger hypothesis on f_+, the construction of an invariant measure from μ^+ and μ^- succeeds.

Bedford and Taylor (see [Kli]) originated an integration by parts method for taking the wedge product of positive closed currents with locally bounded potentials. If $W \subset \mathbf{C}^2$ is open, $u : W \to \mathbf{R}$ is locally bounded and plurisubharmonic, and T is a positive closed $(1,1)$ current on W, then the action of the measure $dd^c u \wedge T$ on a test form φ is given by

$$\langle dd^c u \wedge T, \varphi\rangle = \langle T, u\, dd^c \varphi\rangle.$$

It turns out that this defines $dd^c u \wedge T$ as a positive measure. This can be seen from the following theorem of Bedford and Taylor (see [Kli]).

Theorem 4.1 *Suppose that $u_j, v_j : W \to \mathbf{R}$ are decreasing sequences of plurisubharmonic functions converging pointwise to locally bounded plurisubharmonic functions u, v. Then*

$$\lim_{j\to\infty} dd^c u_j \wedge dd^c v_j = dd^c u \wedge dd^c v$$

weakly.

Though examples indicate that the integration by parts construction cannot be used to defined the wedge product of arbitrary positive closed currents, one need not restrict oneself to positive closed currents with locally bounded potentials. Indeed, Fornaess and Sibony [FS4] have shown that the integration by parts construction and Theorem 4.1 succeed when the unboundedness loci of u and v do not coincide too much. The precise condition they discovered is as follows. Let M_u denote the smallest closed set such that $p \notin M_u$ implies that u_j is bounded on a neighborhood of p. Let M_v be the corresponding set for v. Then the wedge product $dd^c u \wedge dd^c v$ is *admissible* provided that $M_u \cap M_v$ lies in the pseudoconvex envelope of its complement in W. In particular, things go well if at any point in W, at least one of the functions u or v is locally bounded.

Following [FS2] we refer to $\mathcal{I}^+ \stackrel{\mathrm{def}}{=} \overline{I^+_\infty}$ as the *extended indeterminacy set* of a minimally separating birational map f_+. We call the complement $\mathcal{D}^+ \stackrel{\mathrm{def}}{=} \mathbf{P}^2 \setminus \mathcal{I}^+$ of $\mathcal{I}^+$ the *dynamic domain* of f_+. We say that f_+ is *separating* if $\mathcal{I}^+ \cap \mathcal{I}^- = \emptyset$. In [Dil] we showed that the escape function $\tilde{G}^+$ for a separating birational map is continuous on $\pi^{-1}(\mathcal{D}^+)$. In particular, local potentials for μ^+ are bounded near any point in $\mathcal{D}^+$. Clearly, $\mathcal{D}^+ \cup \mathcal{D}^- = \mathbf{P}^2$ for a separating birational map, so it is clear from the discussion above that the wedge product $\mu = \mu^+ \wedge \mu^-$ is admissible for such a map. In order to show that μ is also invariant, we will need a couple of preliminary lemmas. We thank Eric Bedford for pointing these out to us and explaining their proofs.

Lemma 4.2 *Suppose that u and v are plurisubharmonic functions defined on the unit polydisk Δ^2, and that u is continuous. Then $dd^c u \wedge dd^c v$ has no atoms.*

Proof. It is enough to show that $dd^c u \wedge dd^c v$ attaches no mass to the origin. After subtracting off a constant, we can assume that $u(0,0) = 0$ and set $\omega(r) = \sup_{|x|,|y|<r} |u(x,y)|$. We choose a smooth compactly supported function $\psi : \Delta^2 \to [0,1]$ such that $\psi = 1$ on $\Delta^2/2$, and we set $\psi_r(x,y) = \psi(x/r, y/r)$ for $r > 0$. Let $\theta = dd^c||(x,y)||^2$. Then since $dd^c u \wedge dd^c v$ is positive, we have

$$\begin{aligned} dd^c u \wedge dd^c v(\mathbf{0}) &\leq \liminf_{r\to 0} \int_{\Delta^2} \psi_r \, dd^c u \wedge dd^c v \\ &= \liminf_{r\to 0} \int_{\Delta^2} u \, dd^c \psi_r \wedge dd^c v \\ &= \liminf_{r\to 0} ||u \, dd^c \psi_r||_\infty \, \mathrm{M}_{r\Delta^2}[dd^c v] \end{aligned}$$

$$\leq \liminf_{r\to 0} \frac{C\omega(r)}{r^2} \int_{r\Delta^2} \theta \wedge dd^c v$$

But,

$$\lim_{r\to 0} \frac{1}{r^2} \int_{r\Delta^2} \theta \wedge dd^c v = C'\nu(T, \mathbf{0}),$$

which is always finite (see e.g. [Dem], Consequence 4.4). Since $\omega(r)$ tends to 0 with r, we are done. $\square$

Lemma 4.3 *Suppose that u and v are plurisubharmonic functions on the unit polydisk $\Delta^2 = \{|x|, |y| < 1\}$. Assume that u is continuous and that its restriction to the x axis is harmonic. Assume that the restriction of v to the x axis is locally integrable (i.e. not identically $-\infty$). Then $dd^c u \wedge dd^c v$ concentrates no mass on the x axis.*

Proof. Since the conclusion is true if and only if it holds for every open subset of the x-axis, we can assume without loss of generality that the restriction of v to the x-axis is negative and (globally) integrable. By subtracting off $u(x, 0)$, we can assume that $u(x, y)$ vanishes on the x axis. To prove the lemma, it will suffice to show that $dd^c u \wedge dd^c v$ places no mass on the disk $D = \{(x, 0) : |x| < 1/4\}$.

Let $\psi : \Delta \to [0, 1]$ be a smooth, compactly supported function such that $\psi(z) = 1$ if $|z| \leq 1/2$. Let $\psi_r(z) = \psi(z/r)$, and let

$$\omega(r) = \sup\{u(x, y) : |x| < \frac{1}{2}, |y| < r\}.$$

Then

$$\begin{aligned}
dd^c u \wedge dd^c v(D) &= \lim_{r\to 0} \int_{\substack{|x|<1/4 \\ |y|<r}} dd^c u \wedge dd^c v \\
&\leq \lim_{r\to 0} \int_{\substack{|x|<1/2 \\ |y|<2r}} \psi_{2r}(y)\psi_{1/2}(x)\, dd^c u \wedge dd^c v \\
&= \lim_{r\to 0} \int_{\substack{|x|<1/2 \\ |y|<2r}} u\, dd^c[\psi_{2r}(y)\psi_{1/2}(x)] \wedge dd^c v. \qquad (8)
\end{aligned}$$

We shall have to deal separately with each of the integrals that arises from expanding

$$\begin{aligned}
dd^c[\psi_{2r}(y)\psi_{1/2}(x)] &= \psi_{1/2}(x)\, dd^c \psi_{2r}(y) + \psi_{2r}\, dd^c \psi_{1/2}(x) \\
&\quad + d\psi_{2r}(y) \wedge d^c \psi_{1/2}(x) + d\psi_{1/2}(x) \wedge d^c \psi_{2r}(y). \qquad (9)
\end{aligned}$$

Consider the part of the integral corresponding to the first term in equation (9). In the following computation, we take advantage repeatedly of the fact that ψ appears as a function of only one of the variables x and y.

$$
\begin{aligned}
\lim_{r\to 0} \int\limits_{\substack{|x|<1/2 \\ |y|<2r}} & u\psi_{1/2}(x)\, dd^c\psi_{2r}(y) \wedge dd^c v \\
&\le \lim_{r\to 0} \frac{\omega(r)}{r^2} \, ||dd^c\psi||_\infty \int\limits_{\substack{|x|<1/2 \\ |y|<2r}} dd^c v \wedge \frac{dy\wedge d\bar{y}}{2i} \\
&\le \lim_{r\to 0} \frac{C\omega(r)}{r^2} \int\limits_{\substack{|x|<1 \\ |y|<4r}} \psi_{4r}(y)\psi(x)\, dd^c v \wedge \frac{dy\wedge d\bar{y}}{2i} \\
&= \lim_{r\to 0} \frac{C\omega(r)}{r^2} \int\limits_{\substack{|x|<1 \\ |y|<4r}} v\psi_{4r}(y)\, dd^c\psi(x) \wedge \frac{dy\wedge d\bar{y}}{2i} \\
&\le \lim_{r\to 0} \frac{C\omega(r)}{r^2} \int\limits_{|x|<1} \int\limits_{|y|<4r} v(x,y)\, \frac{dy\wedge d\bar{y}}{2i} \wedge \frac{dx\wedge d\bar{x}}{2i}. \qquad (10)
\end{aligned}
$$

But for almost every $x \in \Delta$, we have that

$$
\lim_{r\to 0} \frac{1}{16\pi r^2} \int_{|y|<4r} v(x,y)\, \frac{dy\wedge d\bar{y}}{2i} \searrow v(x,0).
$$

Therefore, we can invoke the Lebesgue dominated convergence theorem and the fact that $\omega(r) \to 0$ with r to conclude that the limit in (10) is zero. This takes care of the contribution to (8) from the first term on the right side of (9). The contribution from the second term can be handled in a similar fashion.

We can apply Schwarz's inequality to the contribution from the third term on the right side of (9).

$$
\begin{aligned}
\lim_{r\to 0} & \left| \int\limits_{\substack{|x|<1/2 \\ |y|<2r}} u\, d\psi_{2r}(y) \wedge d^c\psi_{1/2}(x) \wedge dd^c v \right| \\
&\le \lim_{r\to 0} \left(\int\limits_{\substack{|x|<1/2 \\ |y|<2r}} |u|\, d\psi_{2r}(y) \wedge d^c\psi_{2r}(y) \wedge dd^c v \right)^{1/2}
\end{aligned}
$$

$$\times\left(\int_{\substack{|x|<1/2\\|y|<2r}} |u|\, d\psi_{1/2}(x)\wedge d^c\psi_{1/2}(x)\wedge dd^c v\right)^{1/2}.$$

By the same reasoning employed for the first term, we can show that each of the integrals in the last line behaves like $O(\omega(r))$ as r tends to zero. In particular, the contribution to (8) from the third term in (9) vanishes. An identical argument show that the contribution from the fourth term vanishes as well. □

Theorem 4.4 *The measure $\mu = \mu^- \wedge \mu^+$ for a separating birational map has no atoms and puts no mass on $\mathcal{C}^+$ and $\mathcal{C}^-$.*

Proof. As we noted above, we can find a neighborhood $U = U(p)$ of each point $p \in \mathbf{P}^2$ such that either $\tilde{G}^+$ or $\tilde{G}^-$ is continuous on $\pi^{-1}(U)$. Switching to local coordinates, we can assume that $U = \Delta^2$ is the unit polydisk, $p = (0,0)$ and $\pi^{-1} : \Delta^2 \to \mathbf{C}^3$ is a holomorphic section. Since $\mu^\pm = dd^c\tilde{G}^\pm \circ \pi^{-1}$, Lemma 4.2 shows that p is not an atom for μ.

In particular, μ places no mass on I^+ and no mass on any singular point of $\mathcal{C}^-$. To finish the proof, we need only show that μ places no mass on a neighborhood of each regular point of $\mathcal{C}^+\backslash I^+$ and $\mathcal{C}^-\backslash I^-$. Take a regular point $p \in \mathcal{C}^+\backslash I^+$, for instance. Let V be the irreducible component of $\mathcal{C}^+$ containing p. Since $\tilde{G}^-$ is continuous near $\pi^{-1}(I^+)$, we have from Proposition 2.1 that $\tilde{G}^-$ is not identically equal to $-\infty$ on V. That is, local potentials for μ^- are locally integrable on V. On the other hand, $f_+(V)$ is a point $p^- \in I^-$, and $\tilde{G}^+$ is continuous in a neighborhood of p^-. We apply the formula $\tilde{G}^+ \circ f_+ = (\deg f_+)\tilde{G}^+$ to conclude that $\tilde{G}^+$ is continuous on a neighborhood of $\pi^{-1}(V \setminus I^+)$. Therefore, local potentials for μ^+ are continuous on a neighborhood of p. Moreover, let $\pi^{-1} : U \to \mathbf{C}^3$ be a section defined on a neighborhood of p. Then the local potential $\tilde{G}^+ \circ \pi^{-1}$ for μ^+ satisfies

$$\tilde{G}^+ \circ \pi^{-1}(q) = \frac{1}{\deg f_+}\tilde{G}^+ \circ \tilde{f}_+ \circ \pi^{-1}(q) = \frac{1}{\deg f_+}(\tilde{G}^+(\tilde{p}^-)) + \log|\lambda(q)|$$

for all $q \in V \cap U$, some holomorphic function $\lambda : V \cap U \to \mathbf{C}^*$, and some $\tilde{p}^- \in \mathbf{C}^3$ (independent of q) such that $\pi(\tilde{p}^-) = p^-$. It follows that local potentials for μ^+ are harmonic on $U \cap V$. We can take U to be a small polydisk about p such that $V \cap U$ is identified with the x-axis. Lemma 4.3 now applies to finish the proof. $\square$

Corollary 4.5 *The measure μ associated with a separating birational map is invariant. That is, given any measurable subset $E \subset \mathbf{P}^2$*

$$\mu(f_+^{-1}(E)) = \mu(E).$$

Proof. Recall from Proposition 2.1 that $f_+ : \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ is a biholomorphism. Therefore if $E \subset \mathbf{P}^2 \setminus \mathcal{C}^-$, equation (7) holds rigorously. We need only consider further the case where $E \subset \mathcal{C}^-$. By the previous Theorem, we have that $\mu(E) = 0$. Furthermore, under *any* reasonable definition, $f_+^{-1}(E)$ will be a subset of $\mathcal{C}^+$. Hence, $\mu(f_+^{-1}(E)) = 0$, too. $\square$

5 Pushforwards of Non-closed Positive Currents

In [Dil] we considered images of currents under a birational map largely in order to prove Theorem 3.6. That theorem says approximately that iterated pullbacks of positive closed (1,1) currents tend to converge to μ^+ when properly normalized. One can view the theorem as a multi-variable analogue of the fact, from one variable complex dynamics, that pre-images of a non-exceptional point are dense in the Julia set. However, for many purposes, Theorem 3.6 is not strong enough. One must allow for positive (1,1) currents that are not closed. Here we will consider closed currents that have been "truncated" by contraction with cutoff functions.

Our goal in this section is to prove the analogue of Theorem 1.6 in [BS3] for birational maps. Actually, we will prove two such analogues. One imposes a weak hypothesis on the map but a somewhat restrictive hypothesis on the current. The other places less restriction on the current but only in exchange for a stronger hypothesis concerning the map. Substantial technical details aside, the proof that we give—especially Lemmas 5.2 and 5.3—largely follows the one given in [BS3]. However, at the conclusion of the proof, our approach diverges from [BS3] somewhat and follows Section VII.3 of [FS3] more closely instead. Throughout this section, let $U \subset \mathbf{P}^2$ be a given open set, T a positive closed (1,1) current on U, $\psi : U \to \mathbf{C}$ a smooth function with compact support, and $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ a birational map. First, we borrow an idea from [RS] to provide a workable definition of $f_{+*}(\psi T)$.

Let $\Gamma \subset \mathbf{P}^2 \times \mathbf{P}^2$ be the irreducible analytic subvariety obtained as the closure of the graph of $f_+|_{\mathbf{P}^2 \setminus I^+}$. Let $\alpha, \beta : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ be projection onto the first and second coordinates. Since Γ might be singular, we consider a desingularization $\tilde{\Gamma} \to \Gamma$ of Γ. Abusing notation slightly, we continue to use α and β to denote the pullback to $\tilde{\Gamma}$ of the projection functions. It is evident that the exceptional set of $\alpha : \tilde{\Gamma} \to \mathbf{P}^2$ is the one-dimensional "vertical" curve $\alpha^{-1}(I^+)$. It is also clear that $\alpha : \tilde{\Gamma} \setminus \alpha^{-1}(I^+) \to \mathbf{P}^2 \setminus I^+$ is a biholomorphism. Therefore, we can lift T to a positive closed (1,1) current $\alpha^* T$ on $\alpha^{-1}(U) \subset \tilde{\Gamma}$ by pushing forward with α^{-1} on $U \setminus I^+$ and then extending trivially across $\alpha^{-1}(I^+)$. The extension theorem of Harvey and Polking [HP] guarantees that $\alpha^* T$ is positive and closed on $\alpha^{-1}(U)$. We define $f_{+*}(\psi T)$ by its action on test forms:

$$\langle f_{+*}(\psi T), \varphi \rangle = \langle \alpha^* T, (\psi \circ \alpha)\, \beta^* \varphi \rangle$$

We can assume with no loss of generality that ψ is real and non-negative in what follows. Clearly, this assumption implies that both ψT and $f_{+*}(\psi T)$ are positive currents.

If $U = \mathbf{P}^2$ and $\psi \equiv 1$, then the rather abstract definition of pushforward we have just given coincides with the one given in section 2. More is true, in fact.

Proposition 5.1 *Suppose that $\chi_j : \mathbf{P}^2 \to [0,1]$ are smooth functions such that χ_j vanishes on a neighborhood of $\mathcal{C}^-$ and $\operatorname{supp}(1 - \chi_j)$ decreases to $\mathcal{C}^-$ as $j \to \infty$. Then for any test form φ, we have*

$$\langle f_{+*}(\psi T), \varphi \rangle = \lim_{j \to \infty} \langle f_{+*} T, \chi_j (\psi \circ f_-) \varphi \rangle = \lim_{j \to \infty} \langle \psi T, f_+^*(\chi_j \varphi) \rangle.$$

The pushforward in the middle expression and the pullback in the right-hand expression can be understood to take place with respect to a biholomorphic map.

Proof. What is needed is to show that $f_{+*}(\psi T)$ concentrates no mass on $\mathcal{C}^-$. Note that $\beta^{-1}(\mathcal{C}^-) = \alpha^{-1}(\mathcal{C}^+)$ can be divided into two components: $\overline{\beta^{-1}(\mathcal{C}^- \setminus I^-)} = \alpha^{-1}(I^+)$ and $\beta^{-1}(I^-) = \overline{\alpha^{-1}(\mathcal{C}^+ \setminus I^+)}$. We have that $\alpha^* T$ concentrates no support on the first component by definition, and $\beta^* \varphi$ is identically zero on the second component. Therefore, the restriction of $\alpha^*(\psi T)$ to $\beta^{-1}(\mathcal{C}^-)$ contributes nothing to the pairing $\langle \alpha^* \psi T, \beta^* \varphi \rangle$. $\square$

This proposition shows that $f_{+*}(\psi T)$ concentrates no mass on $\mathcal{C}^-$ and that $f_{+*}(\psi T) = 0$ if $\operatorname{supp} T \subset \mathcal{C}^+$. Before stating our main theorem, we offer a word of caution concerning pushforwards of non-closed currents. If the support of ψ intersects $\mathcal{C}^-$, then it is likely that $\psi \circ f_-$ will be discontinuous and $f_+(U)$ will fail to be open. That is, it is not clear that $f_{+*}(\psi T)$ can be expressed as a locally closed current times a smooth function. Therefore, while the expression $f^2_{+*}(\psi T)$ is generally well-defined, the expression $(f_{+*})^2(\psi T) = f_{+*}(f_{+*}(\psi T))$ is not.

We now state and prove two lemmas which are crucial for the main theorems in this section. Note that the hypotheses of the lemmas are somewhat weaker than those of the theorems.

Lemma 5.2 *Suppose that f_+ is minimally separating and that T admits a wedge product with μ^+. Then there exists a constant $C > 0$, independent of n, such that*

$$\mathrm{M}[f^n_{+*}(\psi T)] \leq C \deg(f^n_+)$$

If in addition local potentials for T are continuous in a neighborhood of each point of $I^+_\infty \cap U$, then

$$\lim_{n\to\infty} \frac{1}{\deg f^n_+} \int_{\mathbf{P}^2} f^n_{+*}(\psi T) \wedge \Theta = \int_{\mathbf{P}^2} \psi\, T \wedge \mu^+.$$

Proof. Let $\mathcal{C}^+_n = \cup_0^{n-1} f_+^{-1}(\mathcal{C}^+)$ and $\mathcal{C}^-_n$ denote the critical sets of f^n_+ and f^n_-, respectively. Let $\chi_j : \mathbf{P}^2 \to [0,1]$ be a sequence of smooth functions such that $\chi_j \equiv 0$ in a neighborhood of $\mathcal{C}^-_n$ and $\operatorname{supp}(1-\chi_j)$ decreases to $\mathcal{C}^+_n$. Then we have

$$\begin{aligned} \frac{1}{\deg f^n_+} \mathrm{M}[f^n_{+*}(\psi T)] &\leq \frac{C}{\deg f^n_+} \langle f^n_{+*}(\psi T), \Theta \rangle \\ &= \lim_{j\to\infty} \frac{C}{\deg f^n_+} \langle \psi T, (\chi_j \circ f^n_+)\, f^{n*}_+ \Theta \rangle, \end{aligned}$$

by Proposition 5.1. Note that T admits a wedge product with $f^{n*}_+\Theta$ (viewed as a positive closed (1,1) current). This follows from the fact that local potentials for $f^{n*}_+\Theta$ are unbounded only at points in I^+_n. Furthermore, recall that these local potentials may be taken to decrease to local potentials for μ^+. Therefore, we can apply Theorem 4.1 and continue to compute

$$\lim_{j\to\infty} \frac{1}{\deg f^n_+} \langle \psi T, (\chi_j \circ f^n_+)\, f^{n*}_+ \Theta \rangle \quad = \quad \lim_{j\to\infty} \frac{1}{\deg f^n_+} \int_{\mathbf{P}^2} (\chi_j \circ f^n_+) \psi\, T \wedge f^{n*}_+ \Theta$$

$$\begin{aligned} &\leq \frac{1}{\deg f_+^n}\int_{\mathbf{P}^2}\psi\, T\wedge f_+^{n*}\Theta \\ &\to \int_{\mathbf{P}^2}\psi\, T\wedge\mu^+. \end{aligned}$$

Therefore, the first conclusion of the lemma holds. Whether the second conclusion holds depends on whether the last inequality is actually an equality. To obtain equality, it is enough to know that $T\wedge f_+^{n*}\Theta$ concentrates no mass on $\mathcal{C}_n^+ = (f_+^n)^{-1}(\mathcal{C}_n^-)$.

We claim that given the extra hypotheses, this is so. Note first that $f_+^{n*}\Theta$ is smooth everywhere except at I_n^+. Furthermore, $f_+^{n*}\Theta$ restricted to $\mathcal{C}_n^+$ is actually zero away from I_n^+. Indeed, since f_+^n maps any irreducible component V of $\mathcal{C}_n^+$ onto a point p in I_n^-, a potential for $f_+^{n*}\Theta$ on $V\setminus I_n^+$ will be given by $u\circ f_+^n$ where u is local potential for Θ on a neighborhood of p. Clearly, $u\circ f_+^n$ is constant on $V\setminus I_n^+$. Therefore $T\wedge f_+^{n*}\Theta$ concentrates no mass on $\mathcal{C}_n^+\setminus I_n^+$. Finally, Lemma 4.2 and the extra hypothesis of this lemma imply that $T\wedge f_+^{n*}\Theta$ concentrates no mass on I_n^+, either. $\square$

Lemma 5.3 *Suppose that the first conclusion of Lemma 5.2 holds and* $\deg f_+^n$ *tends to* ∞ *with* n. *Then the sequences* $\frac{\partial f_{+*}^n(\psi T)}{\deg f_+^n}$, $\frac{dd^c f_{+*}^n(\psi T)}{\deg f_+^n}$ *tend to zero in the mass norm as* $n\to\infty$.

Proof. Let λ be a test one-form on $\mathbf{P}^2$ such that $||\lambda||_\infty\leq 1$. Let $\tilde{\Gamma}$ be the desingularization of the graph of f_+^n, with coordinate projections α and β. Choose a compactly supported smooth function $\psi_1 : U\to[0,1]$ such that $\psi_1\equiv 1$ on a neighborhood of $\operatorname{supp}\psi$. Then

$$\begin{aligned} |\langle f_{+*}^n\psi T, d\lambda\rangle| &= |\langle\alpha^*T, d(\psi\circ\alpha)\wedge\beta^*\lambda\rangle| \\ &\leq \langle\alpha^*T, \alpha^*(d\psi)\wedge\alpha^*(d^c\psi)\rangle^{\frac12}\langle\alpha^*T, -i(\psi_1\circ\alpha)\,\beta^*\lambda\wedge\beta^*\bar{\lambda}\rangle^{\frac12} \\ &= \langle T, d\psi\wedge d^c\psi\rangle^{1/2}\langle f_{+*}^n(\psi_1 T), -i\lambda\wedge\bar{\lambda}\rangle^{1/2} \\ &\leq C(\deg f_+)^{n/2} \end{aligned}$$

The first inequality is essentially Schwarz's inequality. The second equality follows from the facts that T concentrates no mass on I^+, that by definition α^*T concentrates no mass on $\alpha^{-1}(I^+)$, and that $\alpha:\tilde{\Gamma}\setminus\alpha^{-1}(I^+)\to\mathbf{P}^2\setminus I^+$ is a biholomorphism. Thus $\langle\alpha^*T, \alpha^*(d\psi\wedge d^c\psi)\rangle = \langle T, d\psi\wedge d^c\psi\rangle$, as asserted. The last inequality follows from the previous lemma. Since $\deg f_+^n$ tends to

∞ with n, we see that $\frac{1}{\deg f_+^n}\partial f_{+*}^n(\psi T)$ tends to 0 at a rate independent of λ—i.e. in the mass norm.

If $\rho : \mathbf{P}^2 \to \mathbf{C}$ is a test function with $||\rho||_\infty \leq 1$, then we have

$$\begin{aligned}
|\langle f_{+*}^n(\psi T), dd^c\rho\rangle| &= \langle \alpha^* T, (\rho\circ\beta)\, dd^c(\psi\circ\alpha)\rangle \\
&= \langle (\alpha^* T)\wedge \alpha^*(dd^c\psi), \rho\circ\beta\rangle \\
&\leq \mathrm{M}[\alpha^* T\wedge \alpha^* dd^c\psi] \\
&= \mathrm{M}[T\wedge dd^c\psi].
\end{aligned}$$

The last equality holds because $\alpha^*T\wedge\alpha^* dd^c\psi$ puts no mass on $\alpha^{-1}(I^+)$. Dividing through by $\deg f_+^n$ finishes the proof. $\square$

Now suppose that all hypotheses of Lemma 5.2 are satisfied. Let $\mathcal{S}$ denote the set of all limit points of the sequence of currents $\{f_{+*}^n(\psi T)/\deg f_+^n\}$. The conclusion of Lemma 5.2 implies that $\mathcal{S}$ is non-empty and that any $S\in\mathcal{S}$ satisfies

$$||S|| = \int_{\mathbf{P}^2} \psi\, T\wedge\mu^+.$$

Lemma 5.3 further implies that all elements of $\mathcal{S}$ are closed. The first of our two convergence theorems addresses the case where T is defined on all of $\mathbf{P}^2$.

Theorem 5.4 *Suppose that f_+ is minimally separating and $U = \mathbf{P}^2$. If T admits a wedge product with μ^+ and local potentials for T are continuous near each point of I_∞^+, then*

$$\lim_{n\to\infty} \frac{f_{+*}^n(\psi T)}{\deg f_+^n} = c\mu^-,$$

in the weak topology on currents, where

$$c = \int_{\mathbf{P}^2} \psi\, T\wedge\mu^+.$$

Proof. Clearly ψT is dominated by $||\psi||_\infty T$. Remark 4.16 from [Dil] (similar to Theorem 3.6) implies that

$$\lim_{n\to\infty}\frac{f_{+*}^n T}{\deg f_+^n} = \lim_{n\to\infty}\frac{f_-^{n*} T}{\deg f_+^n} = ||T||\cdot\mu^-.$$

Therefore, any element of $\mathcal{S}$ is dominated by $||T|| \cdot ||\psi||_\infty \, \mu^-$. Corollary 3.7 then implies that any element of $\mathcal{S}$ is a multiple of μ^-—i.e. $S = ||S|| \cdot \mu^-$. The remarks preceding the statement of this theorem determine $||S||$ uniquely. $\square$

The second of our convergence theorems addresses the case where T is not globally defined on $\mathbf{P}^2$, but our proof requires that f_+ be *completely separating*—i.e. the iterates of f_- form a normal family on a neighborhood of $\mathcal{I}^+$. We recall from [Dil] that this condition automatically implies that f_+ is separating. Unfortunately, however, there are examples of completely separating maps f_+ for which f_- is *not* completely separating. For our present purposes, the relevant properties of completely separating maps are summarized by

Theorem 5.5 *Suppose that $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is a completely separating birational map. Then the set of points on which iterates of f_+ form a normal family is exactly equal to $\mathbf{P}^2 \setminus \operatorname{supp} \mu^+$. Moreover, given any compact set $K \subset \mathcal{D}^+$, there exists a constant $C > 0$ such that $\operatorname{dist}(f_+^n(K), \mathcal{I}^+) > C$ for all $n \geq 0$.*

Proof. See [Dil]

Theorem 5.6 *Suppose that f_+ is completely separating and $U \subset \mathcal{D}^+$. Then*

$$\lim_{n\to\infty} \frac{f_{+*}^n(\psi T)}{\deg f_+^n} = c\mu^-,$$

in the weak topology on currents, where

$$c = \int_{\mathbf{P}^2} \psi \, T \wedge \mu^+.$$

Proof. First we claim that $f_-^* S = f_{+*} S$ for every $S \in \mathcal{S}$. Indeed, since $U \subset \mathcal{D}^+$, we have that $\operatorname{supp}(\psi T) \subset\subset \mathcal{D}^+$. We know from Theorem 5.5 that there exists a constant $C > 0$ such that $\operatorname{dist}(f_+^n(\operatorname{supp}(\psi T)), \mathcal{I}^+) > C$ for all $n \geq 0$. Therefore, S is compactly supported in $\mathcal{D}^+$. In particular, $\nu(S, p)$ vanishes at every point of I^+, and Theorem 2.3 suffices to establish the claim.

Next we claim that the set of limits $\mathcal{S}$ satisfies the relation

$$(\deg f_+)\mathcal{S} \subset f_-^* \mathcal{S}.$$

To see this let $S = \lim_{k\to\infty} \frac{f_{+*}^{n_k}(\psi T)}{\deg f_+^{n_k}}$ be an element of $\mathcal{S}$. Let S' be a limit point of the subsequence obtained by replacing n_k with $n_k - 1$. After refining if necessary, we can assume that S' is unique. Let χ_j be as in Proposition 5.1. For any test form φ, we have

$$\begin{aligned}
\langle f_-^* S', \varphi\rangle &= \langle f_{+*} S', \varphi\rangle \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \langle \frac{f_{+*}^{n_k-1}(\psi T)}{\deg f_+^{n_k-1}}, f_+^*(\chi_j \varphi)\rangle \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \langle \frac{f_{+*}^{n_k}(\psi T)}{\deg f_+^{n_k-1}}, \chi_j \varphi\rangle \\
&= (\deg f_+) \lim_{j\to\infty} \langle S, \chi_j \varphi\rangle \\
&= (\deg f_+)\langle S|_{\mathbf{P}^2\setminus\mathcal{C}^-}, \varphi\rangle.
\end{aligned}$$

That is, $f_-^* S' = (\deg f_+) S|_{\mathbf{P}^2\setminus\mathcal{C}^-}$. However, if S concentrates any mass on $\mathcal{C}^-$, then we have

$$(\deg f_+)||S|_{\mathbf{P}^2\setminus\mathcal{C}^-}|| < (\deg f_+)||S|| = (\deg f_+)||S'|| = ||f_-^* S'||,$$

which is a contradiction. Therefore $f_-^* S' = S$, and the second claim is verified.

Because elements of $\mathcal{S}$ avoid a uniform neighborhood of $\mathcal{I}^+$, we can apply Theorem 3.6 to see that

$$\mathcal{S} \subset \lim_{n\to\infty} \frac{f_-^{n*}\mathcal{S}}{\deg f_-^n} = \{c\mu^-\},$$

where c is uniquely determined by the discussion preceding Theorem 5.4. □

6 Support of μ^+

Following Proposition 2.1, we described an action of a birational map on the collection of closed subsets of $\mathbf{P}^2$. There is, however, no reasonable sense in which a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ sends open sets to open sets. In fact, if $U \subset \mathbf{P}^2$ is open, it is not hard to show that $f_+(U)$ is open if and only if U either avoids or contains $\mathcal{C}^+$. On the other hand, f_+ does preserve *closures of* open sets. It is always true that $f_+(\overline{U}) = \overline{\operatorname{int} f_+(\overline{U})}$. Moreover, this action is bijective since $f_+(f_-(\overline{U})) = \overline{U}$. Therefore, it makes sense to talk about open sets whose closures are invariant under a birational map. Theorem 5.4 implies that such sets can intersect the supports of μ^+ and μ^- in only a rather limited number of ways.

Theorem 6.1 *Suppose that $U \subset \mathbf{P}^2$ is an open set whose closure is invariant under a minimally separating birational map. If* $\operatorname{supp}\mu^+ \cap U \neq \emptyset$, *then* $\operatorname{supp}\mu^- \subset \overline{U}$.

Proof. Assume there is a point $p \in \operatorname{supp}\mu^+ \cap U$. Pick a smooth function $\psi : \mathbf{P}^2 \to [0,1]$ such that $\operatorname{supp}\psi \subset U$ and and $\psi(p) = 1$. Then

$$\int_{\mathbf{P}^2} \psi\Theta \wedge \mu^+ \stackrel{\text{def}}{=} c > 0.$$

Hence by Theorem 5.4,

$$\frac{f^n_{+*}(\psi\Theta)}{\deg f^n_+} \to c\mu^-,$$

and since $\operatorname{supp} f^n_{+*}(\psi\Theta) \subset \overline{U}$ for every n, we have $\operatorname{supp}\mu^- \subset \overline{U}$ as well. $\square$

At this point we do not know whether sets like $\mathcal{I}^+$, $\operatorname{supp}\mu^+$, etc. can have interior. However, the theorem just proved shows that if such sets do have interior, then the map f_+ must be rather special.

Corollary 6.2 *The following are true for a minimally separating birational map.*

1. *If* $\operatorname{supp}\mu^+$ *omits one point in* $\operatorname{supp}\mu^-$, *then* $\operatorname{supp}\mu^+$ *is nowhere dense.*
2. *If $\mathcal{I}^+$ omits one point in* $\operatorname{supp}\mu^-$ *(or, more particularly, in $\mathcal{I}^-$), then $\mathcal{I}^+$ is nowhere dense.*
3. *If both* $\operatorname{supp}\mu^+$ *and* $\operatorname{supp}\mu^-$ *have non-empty interior, then* $\operatorname{supp}\mu^+ = \overline{\operatorname{int}\operatorname{supp}\mu^+} = \overline{\operatorname{int}\operatorname{supp}\mu^-} = \operatorname{supp}\mu^-$.
4. *If both $\mathcal{I}^+$ and $\mathcal{I}^-$ have non-empty interior, then* $\operatorname{supp}\mu^+ = \operatorname{supp}\mu^- = \mathcal{I}^+ = \mathcal{I}^-$.

Proof. 1: The complement of $\operatorname{supp}\mu^+$ is an open set with invariant closure. If $\operatorname{supp}\mu^-$ intersects this set, then $\operatorname{supp}\mu^+$ is contained in its closure. 2: The complement of $\mathcal{I}^+$ is another open set with invariant closure. If $\operatorname{supp}\mu^-$ intersects this set, then $\mathcal{I}^+ \subset \operatorname{supp}\mu^+$ lies in its boundary. Statements 3 and 4 follow immediately from 1 and 2. $\square$

Stronger conclusions are possible for separating birational maps. Though we proved it already in our previous paper, we note that Corollary 6.2 immediately implies that $\mathcal{I}^+$ and $\mathcal{I}^-$ are nowhere dense for separating birational maps. The presence of an attracting periodic point implies that $\operatorname{supp}\mu^+$ is also nowhere dense.

Corollary 6.3 *If f_+ is separating and p is an attracting periodic point, then* $\operatorname{supp}\mu^+$ *lies in the boundary of any connected component of the basin of p. In particular,* $\operatorname{supp}\mu^+$ *is nowhere dense.*

Proof. Iterates of f_+ form a normal family on the interior of the basin of p, so $\operatorname{supp}\mu^+$ does not intersect the interior. On the other hand, we showed in [Dil] that attracting periodic points of separating birational maps belong to $\operatorname{supp}\mu^-$. Since the closure of basin of p is invariant under f_+, Theorem 6.1 implies that $\operatorname{supp}\mu^+$ lies in the closure of the basin. Applying this reasoning to f_+^k, where k is the period of p shows that $\operatorname{supp}\mu^+$ lies in the boundary of any connected component of the basin of p. $\square$

Corollary 6.4 *If f_+ is completely separating, then* $\operatorname{supp}\mu^-$ *is nowhere dense. Moreover,* $\operatorname{supp}\mu^+$ *is equal to the boundary of any connected component of any attracting basin.*

Proof. Since iterates of f_- form a normal family in a neighborhood of $\mathcal{I}^+$, we have that $\mathcal{I}^+ \cap \operatorname{supp}\mu^-$ is empty. On the other hand $\mathcal{I}^+ \subset \operatorname{supp}\mu^+$. Thus, the first statement follows from Corollary 6.2. The second statement follows from Corollary 6.3, Theorem 5.5, and the fact that iterates of f_+ do not form a normal family on any open set intersecting the boundary of an attracting basin. $\square$

For completely separating birational maps we can apply Theorem 5.6 instead of Theorem 5.4. This gives a description of $\operatorname{supp}\mu^+$ in terms of stable manifolds.

Theorem 6.5 *Suppose that f_+ is completely separating. Then* $\operatorname{supp}\mu^-$ *is equal to the closure of the unstable manifold of any saddle periodic point.*

Proof. Let p be a saddle periodic point, and W^u_{loc} be a local unstable manifold through p. Let $\chi : \mathbf{P}^2 \to [0,1]$ be a smooth function supported in a small neighborhood of p. We can assume that $\operatorname{supp}\chi \cap W^u_{loc}$ is relatively compact in W^u_{loc}.

Since f_+ is completely separating, we know that $p \in \mathcal{D}^+$ (iterates of f_- cannot form a normal family near p). From [Dil] we also know that $\tilde{G}^+$ cannot be pluriharmonic on $\pi^{-1}(W^u_{loc})$. Therefore,

$$c = \int_{\mathbf{P}^2} \chi\,[W^u_{loc}] \wedge \mu^+ > 0.$$

We can apply Theorem 5.6 to conclude that

$$\lim_{n\to\infty} \frac{1}{\deg f_+^{kn}} f_{+*}^{kn}(\chi[W^u_{loc}]) = c\mu^-,$$

where k is the period of p. Since $f_{+*}^{kn}(\chi[W^u_{loc}])$ is supported on the global unstable manifold of p for all n, we have that μ^- is supported on the closure of the unstable manifold. The opposite inclusion was proven for separating birational maps in [Dil] □

7 Mixing properties of μ.

In this section, we prove that the measure $\mu = \mu^+ \wedge \mu^-$ defined in Section 4 is mixing for f_+. First, however, we prove a couple of related results that are of interest in their own right. First we recall the notion of a non-wandering point. Let $f : X \to X$ be a continuous map of a metric space X. We define a pre-order $\prec$ for points in X by saying that $q \prec p$ if for any neighborhoods $U \ni p, V \ni q$, there exist arbitrarily large n such that $f^n(U)$ intersects V. We call $p \in X$ *non-wandering* if $p \prec p$. We extend these definitions to birational maps $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ by disregarding any points of indeterminacy for f_+^n in the intersection $f_+^n(U) \cap V$.

Corollary 7.1 *Given a minimally separating birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$, we have $q \prec p$ for every $p \in \operatorname{supp}\mu^+$ and $q \in \operatorname{supp}\mu^-$. In particular, every point in $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^-$ is non-wandering.*

Proof. Let $U \ni p$, $V \ni q$ be neighborhoods and ψ be a positive test function supported on U and non-vanishing at p. Then $\langle \mu^+, \psi\Theta\rangle$ is positive. Hence, by Theorem 5.4 $\lim_{n\to\infty} \frac{1}{d^n} f^n_{+*}(\psi\Theta) = c\mu^-$ for some $c > 0$. Therefore, we must have $\emptyset \neq f^n_+(\operatorname{supp}\psi) \cap V \subset f^n_+(U) \cap V$ for n sufficiently large. This proves the first statement in the corollary. The second statement follows immediately by taking $p = q$ in the first. $\square$

It is perhaps important to note what we have *not* shown here—namely that the restriction of f_+ to $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^-$ is topologically transitive. The non-empty intersection $f^n_+(U) \cap V$ obtained in the proof might fail to contain points of either $\operatorname{supp}\mu^-$ or $\operatorname{supp}\mu^+$, let alone points in $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^-$. However, for separating birational maps stronger conclusions can be drawn by passing to the measure theoretic level.

Corollary 7.2 *Suppose that f_+ is separating, and $\psi : \mathbf{P}^2 \to \mathbf{C}$ is smooth, then*

$$\lim_{n\to\infty} \frac{f^n_{+*}(\psi\mu^-)}{\deg f^n_+} = \left(\int_{\mathbf{P}^2} \psi\,\mu\right) \cdot \mu^-$$

in the weak topology on currents.

Proof. Since f_+ is separating, local potentials for μ^- are continuous on a neighborhood of every point in $\mathcal{I}^+$. Therefore, all hypotheses of Theorem 5.4 are fulfilled by setting $T = \mu^-$. $\square$

Example 7.3 *It is important that we require some regularity from ψ in the above corollary. Suppose that f_+ is a polynomial diffeomorphism of $\mathbf{C}^2$ with at least two attracting periodic points. Then $\operatorname{supp}\mu^-$ intersects the basins of both points. If χ is the characteristic function for one of the basins, we have that $\chi\mu^- = f_{+*}\chi\mu^-/\deg f_+$ is a non-trivial forward invariant current.*

Example 7.4 *It is also important that f_+ be birational. We illustrate this point with an example in which μ^+ plays the role that μ^- played in Corollary 7.2. Consider the holomorphic map $f : \mathbf{P}^2 \to \mathbf{P}^2$ whose restriction to $\mathbf{C}^2$ is $f(x, y) = (x^2, y^2)$. The restriction of the current μ^+ (usually denoted by T in this context) to $\mathbf{C}^2$ is $dd^c \log\max\{1, |x|, |y|\}$. In particular, the restriction of μ^+ to $U = \{|x| < 1\}$ is $dd^c \log^+ |y|$, which simply doubles under pullback by f_+. Let $\rho : [0, 1] \to [0, 1]$ be any smooth function such that $\rho(0) = 1$ and*

$\rho(1) = 0$. If $\psi = \rho(|x|)$, then we have that $\psi \circ f^n$ tends uniformly to one on compact subsets of U. Since μ^+ has finite mass, we have $f^{n*}(\psi\mu^+)/2^n \to \mu^+|_U$ as n tends to infinity.

$f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is said to be *mixing* with respect to an invariant measure μ (assume for the sake of simplicity that μ does not charge I^+, so that $f_{+*}\mu$ is unambiguous) if for any measurable subsets $A, B \subset \mathbf{P}^2$, we have

$$\lim_{n\to\infty} \mu(f_+^n(A) \cap B) = \mu(A) \cap \mu(B).$$

Bedford and Smillie [BS3] showed that polynomial diffeomorphisms of $\mathbf{P}^2$ are mixing with respect to the measure $\mu = \mu^+ \wedge \mu^-$. We now generalize their result.

Theorem 7.5 *A separating birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is mixing with respect to the measure $\mu = \mu^+ \wedge \mu^-$.*

The main idea behind the proof of this theorem appears in [BS3]. It is, however, somewhat more delicate to make their idea succeed at this level of generality. As in Lemma 5.2 we let $\mathcal{C}_n^+$ denote the critical set of f_+^n, where we take $\mathcal{C}_\infty^+$ to be the (increasing) union of $\mathcal{C}_n^+$ over all finite $n > 0$.
Proof. Since μ is a Borel measure, it is enough (see [KH]) to show that for any two smooth functions $\psi, \varphi : \mathbf{P}^2 \to \mathbf{C}$ we have

$$\lim_{n\to\infty} \int_{\mathbf{P}^2} \varphi \cdot (\psi \circ f_+^n)\, d\mu = \left(\int_{\mathbf{P}^2} \varphi\, d\mu\right)\left(\int_{\mathbf{P}^2} \psi\, d\mu\right).$$

Even though $\psi \circ f_+^n$ might be discontinuous at points in I_n^+, the first integral makes sense because μ does not charge $\mathcal{C}_n^+ \supset I_n^+$.

Clearly, we lose no generality by assuming that ψ and φ take values only in the interval $[0,1]$ and that φ is supported in a coordinate polydisk D. We can also assume that $D \cap \mathcal{I}^- = \emptyset$. To see this, note that because f_+ is separating we can write $\varphi = \varphi^+ + \varphi^-$ where $\operatorname{supp}\varphi^+ \cap \mathcal{I}^-$ and $\operatorname{supp}\varphi^- \cap \mathcal{I}^+$ are empty. Then by invariance of μ, we can write

$$\int_{\mathbf{P}^2} \varphi \cdot (\psi \circ f_+^n)\, d\mu = \int_{\mathbf{P}^2} \varphi^+ \cdot (\psi \circ f_+^n)\, d\mu + \int_{\mathbf{P}^2} \psi \cdot (\varphi^- \circ f_-^n)\, d\mu.$$

and deal with the first and second integrals separately. The arguments that follow address only the first integral, but those needed for the second integral are completely analogous.

We choose a local potential g^- for μ^- on a neighborhood of $\overline{D}$ in such a way that g^- vanishes at every $z \in I_n^+ \cap D$ (this can be arranged, since I_n^+ is finite, by adding on an appropriate pluriharmonic function). We let

$$\omega^-(r) = \max_{B_{I_n^+}(r)} |g^-(z)|,$$

and note that $\lim_{r\to 0} \omega^-(r) = 0$ since local potentials for μ^- are continuous near $\mathcal{I}_n^+$.

We choose smooth functions $\chi_j : \mathbf{P}^2 \to [0,1]$ such that χ_j vanishes in a neighborhood of $\mathcal{C}_n^+$ and that $\operatorname{supp}(1-\chi_j)$ decreases to $\mathcal{C}_n^+$ as j increases. For sufficiently small r we choose smooth, compactly supported functions $\rho_r : D \to [0,1]$ as follows. Let $\rho : B_0(1) \to [0,1]$ be a smooth, compactly supported and radially symmetric function satisfying $\rho \equiv 1$ on $B_0(1/2)$. Using local coordinates on D, we then set

$$\rho_r(z) = \sum_{w \in D\cap I_n^+} \rho\left(\frac{z-w}{r}\right).$$

In what follows we will repeatedly use the fact that if T is a positive closed $(1,1)$ current on $\mathbf{P}^2$, and η is a continuous function with absolute value less than one everywhere, then

$$|\langle T, \eta\, dd^c\rho_r\rangle|, |\langle T, \eta\, d\rho_r \wedge d^c\rho_r\rangle| \le \frac{C}{r^2} |\langle T, \rho_{2r}\,\theta\rangle|,$$

where $\theta = dd^c||z||^2$ is the (local) Euclidean Kähler form on D, and C depends on ρ but not on η or r.

To prove Theorem 7, we need to know that

Lemma 7.6

$$\int_{\mathbf{P}^2} \varphi \cdot (\psi \circ f_+^n)\, d\mu = \lim_{r\to 0} \langle \mu^+, g^-(1-\rho_r)\, dd^c[\varphi \cdot (\psi \circ f_+^n)]\rangle.$$

In particular, the limit on the right side exists and is independent of ρ.

Assuming this lemma, the proof of the theorem proceeds largely as in [BS3]. That is, we expand

$$
\begin{aligned}
\int_{\mathbf{P}^2} \varphi \cdot (\psi \circ f_+^n)\, d\mu \; &= \; \lim_{r\to 0} \langle \mu^+, g^-(1-\rho_r)\varphi\, dd^c(\psi \circ f_+^n)\rangle \\
&\quad +\langle \mu^+, g^-(1-\rho_r)\, d\varphi \wedge d^c(\psi \circ f_+^n)\rangle \\
&\quad +\langle \mu^+, g^-(1-\rho_r)\, d(\psi \circ f_+^n) \wedge d^c\varphi\rangle \\
&\quad +\langle \mu^+, g^-(1-\rho_r)(\psi \circ f_+^n)\, dd^c\varphi\rangle.
\end{aligned}
\tag{11}
$$

and deal with each term on the right side separately. Taking advantage of invariance and the fact that μ^+ does not charge algebraic curves, we rewrite and bound the first term on the right side as follows.

$$
\begin{aligned}
&|\langle \mu^+, g^-(1-\rho_r)\varphi\, dd^c(\psi \circ f_+^n)\rangle| \\
&= \lim_{j\to\infty} \frac{1}{d^n} |\langle f_{-*}^n \mu^+, \chi_j g^-(1-\rho_r)\varphi\, dd^c(\psi \circ f_+^n)\rangle| \\
&= \lim_{j\to\infty} \frac{1}{d^n} |\langle dd^c f_{-*}^n(\psi\mu^+), \chi_j(1-\rho_r) g^-\varphi\rangle| \le \frac{C\, ||g^-||_\infty}{d^n}.
\end{aligned}
$$

The last inequality follows from the proof of Lemma 5.3. The constant C is independent of r and n, and the L^∞ norm of g^- is evaluated on $\overline{D}$. The second (and equivalently, the third) term on the right side of equation (11) is controlled in the same way as the first. The resulting upper bound for the second term is $C/d^{n/2}$. These considerations show that the contribution from the first three terms to the right side of (11) tends to zero as d increases at a rate which is independent of r. Therefore, the only relevant term is the fourth one, which can be rewritten as

$$
\begin{aligned}
&\lim_{r\to 0} \langle \mu^+, g^-(1-\rho_r)(\psi \circ f_+^n)\, dd^c\varphi\rangle \\
&= \lim_{r\to 0} \lim_{j\to\infty} \frac{1}{d^n} \langle f_{+*}^n \mu^+, \chi_j g^-(1-\rho_r)(\psi \circ f_+^n)\, dd^c\varphi\rangle \\
&= \lim_{r\to 0} \lim_{j\to\infty} \frac{1}{d^n} \langle f_{+*}^n(\psi\mu^+), \chi_j g^-(1-\rho_r)\, dd^c\varphi\rangle \\
&= \langle f_{+*}^n(\psi\mu^+), g^-\, dd^c\varphi\rangle,
\end{aligned}
$$

since $f_{+*}^n(\psi\mu^+)$ does not charge $\mathcal{C}_n^+$ or I_n^+. We conclude that

$$
\lim_{n\to\infty} \int_{\mathbf{P}^2} \varphi \cdot (\psi \circ f_+^n)\, d\mu \; = \; \lim_{n\to\infty} \langle f_{+*}^n(\psi\mu^+), g^-\, dd^c\varphi\rangle
$$

$$
\begin{aligned}
&= c\,\langle \mu^+, g^-\, dd^c\varphi\rangle = c\,\langle \mu^+ \wedge dd^c g^-, \varphi\rangle \\
&= c\int \varphi\, d\mu,
\end{aligned}
$$

where the second equality follows from Corollary 7.2, and

$$
c = \int \psi\, d\mu.
$$

□

Proof of Lemma 7.6. By definition of μ and Theorem 4.4, we have

$$
\begin{aligned}
\int_{\mathbf{P}^2} \varphi\cdot(\psi\circ f_+^n)\, d\mu &= \lim_{r\to 0} \int_{\mathbf{P}^2} (1-\rho_r)\cdot\varphi\cdot(\psi\circ f_+^n)\, d\mu \\
&= \lim_{r\to 0} \langle \mu^+, g^-\, dd^c[(1-\rho_r)\cdot\varphi\cdot(\psi\circ f_+^n)]\rangle \\
&= \lim_{r\to 0} \langle \mu^+, g^-(1-\rho_r)\, dd^c[\varphi\cdot(\psi\circ f_+^n)]\rangle \\
&\quad -\langle \mu^+, g^-\varphi\cdot(\psi\circ f_+^n)\, dd^c\rho_r)\rangle \\
&\quad +\langle \mu^+, g^-\, d^c[\varphi\cdot(\psi\circ f_+^n)]\wedge d\rho_r\rangle \\
&\quad +\langle \mu^+, g^-\, d^c\rho_r \wedge d[\varphi\cdot(\psi\circ f_+^n)]\rangle.
\end{aligned}
\tag{12}
$$

Our task is to show that the last three terms in the last expression vanish with r. The second term is most easily eliminated.

$$
\begin{aligned}
\lim_{r\to 0} |\langle \mu^+, g^-\varphi\cdot(\psi\circ f_+^n)\, dd^c\rho_r)\rangle| &\le \lim_{r\to 0} \frac{C\omega^-(r)}{r^2}\langle \rho_{2r}\theta, \mu^+\rangle \\
&\le \lim_{r\to 0} C\omega^-(r) \max_{p\in I_n^+} \nu(\mu^+, p) = 0.
\end{aligned}
$$

The third and fourth terms in (12) are equal, so we deal only with the third. We break this term up further.

$$
\begin{aligned}
|\langle \mu^+, g^-\, d^c(\varphi\cdot\psi\circ f_+^n)\wedge d\rho_r\rangle| &\le |\langle \mu^+, g^-(\psi\circ f_+^n)\, d^c\varphi\wedge d\rho_r\rangle| \\
&\quad + \ |\langle \mu^+, g^-\varphi\, d^c(\psi\circ f_+^n)\wedge d\rho_r\rangle|.
\end{aligned}
\tag{13}
$$

To deal with the first term in this new decomposition, we apply Schwarz's inequality.

$$
\begin{aligned}
&|\langle \mu^+, g^-(\psi\circ f_+^n)\, d^c\varphi\wedge d\rho_r\rangle| \\
&\qquad \le \ |\langle \mu^+, (g^-(\psi\circ f_+^n))^2\, d\rho_r\wedge d^c\rho_r\rangle|^{1/2} |\langle \mu^+, d\varphi\wedge d^c\varphi\rangle|^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C\omega^-(r)}{r}|\langle \mu^+, \rho_{2r}\theta\rangle|^{1/2} \\
&\le C\omega^-(r)\max_{p\in I_n^+}[\nu(\mu^+,p)]^{1/2}
\end{aligned}
$$

which tends to zero with r. It remains only to address the second term on the right side of (13). We apply Schwarz's inequality again and take advantage of the fact that μ^+ does not charge curves to compute

$$
\begin{aligned}
&|\langle \mu^+, g^-\varphi\, d^c(\psi\circ f_+^n)\wedge d\rho_r\rangle| \\
&\quad\le |\langle \mu^+, (1-\rho_{r/2})\, d(\psi\circ f_+^n)\wedge d^c(\psi\circ f_+^n)\rangle|^{1/2}|\langle \mu^+, (g^-\varphi)^2\, d\rho_r\wedge d^c\rho_r\rangle|^{1/2} \\
&\quad\le C\omega^-(r)\lim_{j\to\infty}|\langle \mu^+, \chi_j(1-\rho_{r/2})\, d(\psi\circ f_+^n)\wedge d^c(\psi\circ f_+^n)\rangle|^{1/2} \\
&\quad= \frac{C\omega^-(r)}{d^{n/2}}\lim_{j\to\infty}|\langle \mu^+, (\chi_j\circ f_-^n)(1-\rho_{r/2}\circ f_-^n)\, d\psi\wedge d^c\psi\rangle| \\
&\quad\le \frac{C\omega^-(r)}{d^{n/2}}.
\end{aligned}
$$

Since the last quantity vanishes with r, we are done. □

Corollary 7.7 *If $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is a separating birational map, then either* $\operatorname{supp}\mu \subset \mathcal{I}^+$ *or $\mu(\mathcal{I}^+) = 0$. In particular, either $\mu(\mathcal{I}^+) = 0$ or $\mu(\mathcal{I}^-) = 0$.*

Proof. By the previous theorem f_+ is ergodic with respect to μ. By definition of $\mathcal{I}^+$, we have $f_+(\mathcal{I}^+) = \mathcal{I}^+$ (modulo I^+, which has measure zero). Therefore, $\mu(\mathcal{I}^+)$ is either zero or one. In the latter case, we conclude from the fact that $\mathcal{I}^+$ is closed that $\operatorname{supp}\mu \subset \mathcal{I}^+$. Finally, $\mathcal{I}^+\cap\mathcal{I}^- = \emptyset$ by hypothesis, so at least one of the two sets must have measure zero. □

References

[BS1] Eric Bedford and John Smillie. *Polynomial diffeomorphisms of* $\mathbf{C}^2$*: currents equilibrium measure and hyperbolicity.* Inventiones Math. **103**(1991), 69–99.

[BS2] Eric Bedford and John Smillie. *Polynomial diffeomorphisms of* $\mathbf{C}^2$*, II: stable manifolds and recurrence.* J. Amer. Math. Soc. **4**(1991), 657–679.

[BS3] Eric Bedford and John Smillie. *Polynomial diffeomorphisms of* $\mathbf{C}^2$*, III: ergodicity, exponents and entropy of the equilibrium measure.* Math. Ann. **294**(1992), 395–420.

[Bro] Hans Brolin. *Invariant sets under iteration of rational functions.* Ark. Mat. **6**(1965), 103–144.

[Dem] Jean-Pierre Demailly. Monge-Ampère operators, Lelong numbers, and intersection theory. In Complex Analysis and Geometry (Vincenzo Ancona and Alessandro Silva, editors), pages 115–193. Plenum Press, 1993.

[Dil] Jeffrey Diller. *Dynamics of birational maps of* $\mathbf{P}^2$. Indiana Univ. Math. J. **45**(1996), 721–772.

[Fav] Charles Favre. Points périodic d'applications birationnelles de $\mathbf{P}^2$. Preprint.

[FS1] John Erik Fornæss and Nessim Sibony. *Complex Hénon mappings in* $\mathbf{C}^2$ *and Fatou-Bieberbach domains.* Duke Math. J. **65**(1992), 687–708.

[FS2] John Erik Fornæss and Nessim Sibony. Complex dynamics in higher dimension, II. In *Modern Methods in Complex Dynamics*, volume 137 of *Ann. of Math. Stud.*, pages 135–182. Princeton Univ. Press, 1995.

[FS3] John Erik Fornæss and Nessim Sibony. Complex dynamics in higher dimensions. In *Complex Potential Theory*, volume 439 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* Kluwer Aca. Publ., 1996. Notes partially written by Estela A. Gavosto.

[FS4] John Erik Fornæss and Nessim Sibony. *Oka's inequality for currents and applications.* Math. Ann. **301**(1996), 399–419.

[Fri1] Shmuel Friedland. *Entropy of polynomial and rational maps.* Ann-Math **133**(1991), 359–368.

[Fri2] Shmuel Friedland. *Entropy of rational self-maps of projective varieties.* In *Dystems and related topics*, volume 9 of *Adv. Ser. Dyn. Syst.*, pages 128–140. World Sci. Publishing, 1994.

[FM] Shmuel Friedland and John Milnor. *Dynamical properties of plane polynomial automorphisms.* Ergodic Theory and Dynamical Systems **9**(1989), 67–99.

[HP] R. Harvey and J. Polking. *Extending analytic objects.* cpam **28**(1975), 701–727.

[HO1] John Hubbard and Ralph Oberste-Vorth. *Hénon mappings in the complex domain I. The global topology of dynamical space.* Ins. Hautes Etudes Sci. Publ. Math. **79**(1994), 5–46.

[HO2] John Hubbard and Ralph Oberste-Vorth. *Hénon mappings in the complex domain II. Projective and inductive limits of polynomials.* In *Real and complex dynamical systems*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 89–132. Kluwer Acad. Publ., 1995.

[HP] John Hubbard and Peter Papadopol. *Superattractive fixed points in* $\mathbf{C}^n$. IUMJ **43**(1994), 321–366.

[HPV] John Hubbard, Peter Papadopol, and Vladimir Veselov. A compactification of Hénon mappings in $\mathbf{C}^2$ as a dynamical system. Preprint.

[KH] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995.

[Kli] Maciej Klimek. *Pluripotential Theory.* Oxford University Press, 1991.

[RS] Alexander Russakovski and Bernard Shiffman. Value distribution for sequences of rational mappings and complex dynamics. Preprint.

[Sib] Nessim Sibony. Dynamique des applications rationelles de $\mathbf{P}^2$. Preprint.

[Siu] Yum-Tong. Siu. *Analyticity of sets associated to Lelong numbers and the extension of positive closed currents.* InvMath **27**(1974), 53–156.

[Sko] Henri Skoda. *Extension problems and positive currents in complex analysis.* In Contributions to Several Complex Variables (Alan Howard and Pit Mann Wong, editors), Aspects of Mathematics, pages 299–328, 1986.