CONTRACTION PROPERTIES OF THE POINCARE SERIES OPERATOR

David E. Barrett and Jeffrey Diller

University of Michigan
Cornell University

1. Introduction

In a striking pair of papers, McMullen gave a new proof of the contraction properties of Thurston's "skinning map"—an iteration on the Teichmüller space of a Riemann surface. His approach was to reduce the problem to the study of a pushforward operator (called the Poincaré series operator) for quadratic differentials [Mc2], and then show that this pushforward operator is itself contracting [Mc1]. Our aim in this paper is to give new proofs of McMullen's estimates on the norm of the Poincaré series operator. Our methods differ significantly from McMullen's, especially in that we avoid the notion of "amenability," and some of the related combinatorial arguments, in favor of more complex analytic and geometric tactics. Our methods have the advantage of yielding estimates that are completely explicit in terms of the injectivity radii of the Riemann surfaces involved. On the other hand, our methods address only the case of covering surfaces with finitely generated fundamental group. This is not too serious a shortcoming, since McMullen uses only the finite topology case in his applications to the skinning map

In the rest of this introduction, we will provide some basic definitions, state our main results, and explain the organization of this paper. The introductions to McMullen's papers do a wonderful job of summarizing the connections between quadratic differentials and Teichmüller theory, and between Teichmüller theory and Thurston's program. A good reference on quadratic differentials is [Ga]. Buser's book [Bu] offers a point of view on Riemann surfaces that is particularly well–suited to the methods we use here.

Let X be a Riemann surface. A quadratic differential on X is an expression of the form $\phi = \phi(z)\,dz^2$ in local coordinates. Put more abstractly, a quadratic differential is a section of the square of the holomorphic cotangent bundle of X. ϕ is

---

1991 *Mathematics Subject Classification.* 30F30, 30F60.

First author supported in part by a grant from the National Science Foundation. Second author supported in part by an NSF Postdoctoral Fellowship.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

called holomorphic if its local trivializations $\phi(z)$ are holomorphic. "Taking absolute values"—$|\phi| = |\phi(z)|\,|dz|^2$—identifies any quadratic differential with a measure on X in a coordinate independent fashion. Thus it is natural to consider the L^1 norm $||\phi||$ of ϕ. We denote the space of all L^1 holomorphic quadratic differentials on X by $Q(X)$. If X is of finite type (i.e. X is obtained from a compact surface by removing finitely many points) then the dimension of $Q(X)$ is finite and determined by the genus and number of punctures of X.

Now suppose that $\pi : Y \to X$ is a holomorphic covering of one Riemann surface by another. Then there is a natural corresponding pushforward operator $\Theta : Q(Y) \to Q(X)$, similar to pushforward of measures. Given $\phi \in Q(Y)$, one defines $\Theta\phi$ by

$$(\Theta\phi)(z) = \sum_{w\in\pi^{-1}(z)} (\pi_w^{-1})^*\phi.$$

Taking absolute values shows that this sum converges in L^1. In fact, we have $||\Theta\phi|| \le \int_X \pi_*|\phi| = \int_Y |\phi| = ||\phi||$, so that Θ has an operator norm no greater than one. But L^1 convergence of holomorphic functions implies uniform convergence on compact sets, so the sum defining $\Theta\phi$ converges pointwise to a holomorphic quadratic differential. For historical reasons Θ is known as the *Poincaré series operator.*

With this notation, we now describe the main results and organization of this paper. When we say that a constant depends only on the topology of a surface, we mean that it can be taken as a function of the number of generators of the fundamental group of the surface.

Theorem 1.1. *Suppose that X is a Riemann surface of finite type and that Y is a Riemann surface of infinite type with finitely generated fundamental group. Let $\pi : Y \to X$ be a holomorphic covering map. Then the norm of the corresponding Poincaré series operator satisfies*

$$||\Theta|| < 1 - k < 1. \tag{1.1}$$

Furthermore, $k > 0$ may be taken to depend only on the topology of X and Y, and on the length ℓ of the shortest closed geodesic on X. As a function of ℓ, k may be taken to be continuous and increasing.

The metric implied in the statement of the theorem is the Poincaré (hyperbolic) metric—that is, the constant curvature -1 metric that X (and Y) inherits from the Poincaré metric on the unit disk Δ. A hyperbolic Riemann surface is of finite or infinite type according to whether it has finite or infinite area, respectively, in the Poincaré metric. Theorem 1.1 is essentially the same as Theorem 1.4 of [Mc1]. It includes, among other things, an affirmative answer to "Kra's theta conjecture," which asserts Theorem 1.1 in the case $Y = \Delta$.

After fixing some notation and stating a few preliminary facts in Section 2, we prove two estimates in Section 3 that constitute the main part of the proof of

Theorem 1.1. Although either estimate would suffice for the proof, we choose to elaborate on the first. Both estimates depend on the existence of small solutions to a particular differential equation on the covering surface. Namely,

Theorem 1.2. *Let Y be a Riemann surface of infinite type and with finitely generated fundamental group. Let ω_A be the Poincaré area form on Y and ℓ_Y be the length of the shortest closed geodesic on Y. Then there is a (1,0) form η on Y such that*

$$\overline{\partial}\eta = \omega_A \qquad \text{and} \qquad \langle\eta\rangle \leq t,$$

where $\langle\eta\rangle$ is the pointwise length of η in the Poincaré metric, and t can be taken to satisfy

$$t \leq \frac{C}{\ell_Y^k}$$

for constants C and k that depends only on the topology of Y.

A proof of this theorem in the case of infinite type surfaces without cusps can be found in [Di]. In the appendix to this paper, we describe the fairly straightforward modifications one needs to make to [Di] to obtain the same theorem for infinite type surfaces with cusps

The proof of Theorem 1.1 given in Section 3 has the virtue of being very short. However, the constant k that it provides is not very explicit. In sections 4 and 5 we revisit Theorem 1.1 with an eye to estimating k in more detail. Section 4 presents some detailed results about the geometry of a hyperbolic Riemann surface. Most of these results are well–known, but to our knowledge, Theorem 4.4 has not been employed elsewhere.

In Section 5 we use the results from Section 4 to provide a value of k that is completely explicit in its dependence on t and ℓ. The next two theorems follow as corollaries.

Theorem 1.3. *The constant k in Theorem 1.1 can be taken to be*

$$k = \lambda^{1/\ell^C}$$

where $\lambda < 1$ and $C > 0$ depend only on the topologies of X and Y.

This theorem gives a rather weak value for k, but if one is willing to fix the covering surface (as, for example, in Kra's Theta Conjecture) a much stronger result is possible.

Theorem 1.4. *The constant k in Theorem 1.1 can be taken to be*

$$k = C\ell,$$

where C is a constant depending only on the constant t in Theorem 1.2 and on the topologies of X and Y.

Note that the statement of this theorem would be absurd if ℓ could be arbitrarily large. We will rely repeatedly below on the fact that, excepting annuli and the disk, ℓ is always bounded above among hyperbolic Riemann surfaces of a given topological type. For instance, we will assert without comment that $\tanh \ell \sim C\ell$. We conclude Section 6 and this paper by presenting an example from [Mc1] which shows that the inequality given by Theorem 1.4 is sharp.

2. Preliminaries.

In this section we make definitions, introduce notation, and state several results that we will need below.

The Poincaré metric. Any hyperbolic Riemann surface carries a complete, constant curvature -1 metric, which we call the *Poincaré metric* (or *hyperbolic metric*). Holomorphic covering maps preserve this metric locally, so we use the notation dA and ds to refer to the associated length and area densities, regardless of which surface we are working on. We will also confuse the measure dA with the corresponding area two form ω_A.

For the convenience of the reader who wishes to reproduce any local coordinate computation that we omit, we recall that on the unit disk, $\Delta = \{|z| < 1\}$, the Poincaré metric has the form

$$ds = \frac{2|dz|}{1 - |z|^2}; \tag{2.1}$$

on the upper half plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$ it has the form

$$\frac{|dz|}{\operatorname{Im} z}; \tag{2.2}$$

on the annulus $A = A_R = \{e^{-R} < |z| < e^R\}$, it has the form

$$\frac{\pi|dz|}{2R|z|\cos(\frac{\pi \log|z|}{2R})}; \tag{2.3}$$

and on the punctured unit disk $\Delta^* = \{0 < |z| < 1\}$, it has the form

$$-\frac{|dz|}{|z|\log|z|} \tag{2.4}$$

Given a hyperbolic Riemann surface $X \neq \Delta$ and a point $p \in X$, there will be some largest R such that the set of points lying within Poincaré distance R of p is a topological disk. We call this R the injectivity radius $I(p)$ of X at p.

More on Pushforward Operators. The discussion in the introduction and the references listed in the bibliography provide sufficient background on pushforwards of quadratic differentials, but since we will want to consider similar operators applied to functions, forms, and densities, we offer more discussion here.

Suppose we have a holomorphic covering $\pi : Y \to X$ of one hyperbolic Riemann surface by another. The following discussion will apply specificly to a 1-form η on Y, but with minor modifications it will apply equally well to forms of any degree (including functions) and quadratic differentials. Given a point $p \in X$, we let $\langle \eta(p) \rangle$ denote the length of the covector $\eta(p)$ as measured by the Poincaré metric. We define the L^1 norm of η by

$$||\eta||_1 = \int_Y \langle \eta \rangle \, dA.$$

We also define the 1-density associated with η by $|\eta| = \langle \eta \rangle \, ds$ (for two forms ω, we set $|\omega| = \langle \omega \rangle \, dA$; for functions $F : Y \to \mathbb{C}$, we set $|F| = \langle F \rangle$; etc.). Finally, we define the pushforward of η by π as

$$\Theta\eta(p) = \sum_{q \in \pi^{-1}(p)} [(\pi_q^{-1})^* \eta](p),$$

and the pushforward of the corresponding density as

$$\pi_*|\eta| = \left(\sum_{q \in \pi^{-1}(p)} \langle \eta(q) \rangle \right) ds.$$

If $||\eta||_1$ is finite and η is holomorphic then Cauchy estimates ([Ho] Theorem 1.2.4) imply that the sums defining $\Theta\eta$ and $\pi_*|\eta|$ converge locally uniformly. Clearly,

$$\pi_*|\eta| \geq |\Theta\eta|$$

The same remarks will hold for a (not necessarily holomorphic) 1–form $F\,\eta$ where F is holomorphic and L^1, and $\langle \eta \rangle \leq C$ on Y, since $\langle \Theta(f\eta) \rangle \leq C\pi_* \langle F \rangle$. In particular, if η is the (1,0) form guaranteed by Theorem 1.2, then

$$\overline{\partial}(\Theta(F\,\eta)) = \Theta(\overline{\partial}(F\,\eta)) = \Theta(F\,\omega_A) = (\Theta F)\, dA, \tag{2.5}$$

since the sums defining both $\Theta(F\,\eta)$ and $(\Theta F)\,dA$ converge locally uniformly.

3. Two Estimates on $||\Theta||$

This section contains the core of the proof of Theorem 1.1. We prove two lemmas that bound the norm of the Poincaré series operator below 1. We elaborate on the first of the two lemmas in order to relate the amount of contraction to the constant t given by Theorem 1.2, the topology of X, and the length of the shortest closed geodesic on X.

Throughout the rest of this paper X will denote a hyperbolic Riemann surface of finite type (i.e. finite volume), g will denote the genus of X and ℓ will denote the length of the shortest closed geodesic on X. By $\overline{X}$ we mean the compact Riemann surface of genus g that one obtains by adding a single point to each end of X. We let $P = \overline{X} \setminus X$ denote the (finite) set of *punctures* of X and we let $|P|$ denote the cardinality of P. By a *topological constant*, or a constant *depending only topology*, we will mean a constant that can be prescribed purely in terms of g and $|P|$.

As in Theorem 1.1, $\pi : Y \to X$ will be a holomorphic covering of X by another Riemann surface Y. We assume that Y satisfies the conclusion of Theorem 1.2 and take $\Theta : Q(Y) \to Q(X)$ to be the Poincaré series operator corresponding to π. The surface Y will enter into estimates on $||\Theta||$ only through the constant t given by Theorem 1.2.

Suppose that $\phi \in Q(Y)$ satisfies $||\phi|| = 1$. We assume without loss of generality that $\Theta\phi \not\equiv 0$. Since Θ is linear, we can prove Theorem 1.1 by appropriately estimating

$$1 - ||\Theta\phi||.$$

Because $\Theta\phi \in Q(X)$, we see that $\Theta\phi$ extends to a meromorphic quadratic differential on $\overline{X}$ with at worst a simple pole at each point in P. Consequently, if Z denotes the set of zeroes of $\Theta\phi$, the Riemann-Roch Theorem implies both that $Z \cup P$ is non–empty and that $|Z|$ is bounded above by a number depending only on g and $|P|$.

We define a meromorphic function $F : Y \to \overline{\mathbb{C}}$ by

$$\phi = F\pi^*(\Theta\phi) \tag{3.1}$$

We will want to apply Θ to F so we prove

Lemma 3.1. *Let U be any relatively compact subset of $X \setminus Z$. Then the L^1 norm of F is finite on $\pi^{-1}(U)$.*

Proof. There exists a constant ϵ such that $\langle\Theta\phi\rangle \geq \epsilon$ on U. Thus,

$$\begin{aligned}\int_{\pi^{-1}(U)} \langle F\rangle \, dA &= \int_{\pi^{-1}(U)} \frac{\langle F\rangle}{\langle\phi\rangle}\langle\phi\rangle \, dA \\ &= \int_{\pi^{-1}(U)} \langle\pi^*(\Theta\phi)\rangle^{-1}|\phi| \\ &\leq \frac{1}{\epsilon}. \quad \square\end{aligned}$$

Note that

$$\Theta\phi = \Theta(F\pi^*(\Theta\phi)) = (\Theta F)(\Theta\phi), \tag{3.2}$$

so that $\Theta F \equiv 1$ off Z.

Now we come to the first of our two main lemmas. Let $K \subset \overline{X}$ be any closed set such that $Z \cup P \subset K$ and bK is smooth (allowing point components) and compact in X. For all $r \geq 0$, let K_r be the set of points whose distance from K is no greater than r. We define

$$m(r) = \min_{p \in bK_r} \langle\Theta\phi\rangle.$$

Lemma 3.2. *For any* $r_0 < \infty$

$$1 - ||\Theta\phi||_1 \geq \int_0^{r_0} m(r)[t^{-1}\operatorname{Area}(X \setminus K_r) - \operatorname{Length}(bK_r)]\,dr. \tag{3.3}$$

Proof. Since a measure and its pushforward have the same mass, we have

$$\begin{aligned}
1 - ||\Theta\phi||_1 &= \int_X \pi_*|\phi| - |\Theta\phi| \\
&= \int_X (\pi_*\langle F\rangle - 1)\langle\Theta\phi\rangle\,dA \geq \int_{X\setminus K} (\pi_*\langle F\rangle - 1)\langle\Theta\phi\rangle\,dA \\
&\qquad (\text{since } \pi_*\langle F\rangle \geq \langle\Theta F\rangle = 1) \\
&= \int_0^\infty \int_{bK_r} (\pi_*\langle F\rangle - 1)\langle\Theta\phi\rangle\,ds\,dr \\
&\geq \int_0^{r_0} \int_{bK_r} (\pi_*\langle F\rangle - 1)\langle\Theta\phi\rangle\,ds\,dr \\
&\geq \int_0^{r_0} m(r)\left[\int_{bK_r} \pi_*(\langle F\rangle\,ds) - \operatorname{Length}(bK_r)\right]dr.
\end{aligned}$$

Now let η be the $(1,0)$ form guaranteed by Theorem 1.2. Using this form, we obtain

$$\begin{aligned}
\int_{bK_r} \pi_*(\langle F\rangle\,ds) &\geq t^{-1}\left|\int_{bK_r} \Theta(F\eta)\right| \\
&= t^{-1}\left|\int_{X\setminus K_r} \overline{\partial}\Theta(F\eta)\right| \\
&= t^{-1}\left|\int_{X\setminus K_r} (\Theta F)\,dA\right| \quad (\text{by } (2.5)) \\
&= t^{-1}\operatorname{Area}(X \setminus K_r),
\end{aligned}$$

which is what we need to complete the proof. $\square$

Proof of Theorem 1.1. In order to use Lemma 3.2, we need to define the set K. Were P empty, we would simply take $K = Z$. Then K_r could be no worse than a disjoint union of $|Z|$ disks of Poincaré radius r—i.e. direct computation in local coordinates reveals that $\operatorname{Area}(K_r), \operatorname{Length}(bK_r) \leq 2\pi|Z|\sinh r$. For each $p \in P$, we add a set K_p to K as follows:

By dividing the universal cover of X by a deck transformation corresponding to a simple closed curve about p, one obtains a natural holomorphic covering map $\pi_p : \Delta^* \to X$ which extends to a holomorphic map of Δ into $\overline{X}$ such that $\pi_p(0) = p$. Fix a number $0 < x < 1$, and set $K_p = \pi_p(\{0 < |z| < x)$. Then another direct computation shows that $\operatorname{Area}(K_{p,r}), \operatorname{Length}(bK_{p,r}) \leq -2\pi e^r/\log x$.

Thus K is the union of Z and all the sets K_p, and we have that

$$\mathrm{Area}(K_r), \mathrm{Length}(bK_r) \leq 2\pi \left(|Z| \sinh r - \frac{|P| e^r}{\log x} \right).$$

By Lemma 3.2 we have

$$(3.4). \quad 1 - ||\Theta\phi|| \geq t^{-1} \int_0^{r_0} m(r) \left(\mathrm{Area}(X) - (1+t) \left(|Z| \sinh r - \frac{|P| e^r}{\log x} \right) \right) dr$$

Since $A(X)$ depends only on g and $|P|$, and $|Z|$ is bounded above in terms of g and $|P|$, we can choose x and r_0 depending only on g and $|P|$ so that the integrand remains positive for all $0 < r < r_0$.

The proof will be complete once we address $m(r)$. In sections 5 and 6, we derive explicit bounds for $m(r)$. For now, we argue abstractly for a bound. We can assume that $||\Theta\phi|| > \frac{1}{2}$; if this were not true, then we would already have that $1 - ||\Theta\phi|| > \frac{1}{2}$. $m(r)$ will vary continuously with $\Theta\phi$. In particular, scaling $\Theta\phi$ will scale $m(r)$ by the same amount. Since $Q(X)$ is finite dimensional, and we have a lower bound on $||\Theta\phi||$, we see that $m(r)$ admits a positive lower bound independent of ϕ. It is well–known in Teichmüller theory that $Q(X)$ varies continuously with the location of X in the Teichmüller space for surfaces quasiconformally equivalent to X. Hence we may also assume that our lower bound on $m(r)$ varies continuously. Now ℓ^{-1} is a continuous exhaustion function on the Teichmüller space of X. So a positive lower bound on ℓ forces X to lie in a compact subset of Teichmüller space. On this set, we can choose our lower bound on $m(r)$ to be independent of X. Such a bound suffices for our purposes, and it finishes the proof. □

We conclude this section with our second, alternative estimate on $1 - ||\Theta\phi||$. Although we do not pursue it further here, one could also use this lemma to prove Theorem 1.1.

Lemma 3.3. *Let $\chi : X \to \mathbb{R}$ be any smooth, compactly supported function vanishing in a neighborhood of Z, and let t be the constant given by Theorem 1.2. Then*

$$(3.5) \qquad 1 - ||\Theta\phi|| \geq \left(\inf_X \frac{\langle \Theta\phi \rangle}{\langle \overline{\partial}\chi \rangle} \right) \int_X (t^{-1}\chi - \langle \overline{\partial}\chi \rangle) \, dA.$$

So if one picks any function χ that vanishes near Z, one obtains an estimate on the extent to which Θ shrinks ϕ. It requires more work, though, to eliminate the dependence on ϕ from the righthand side.

Proof. As in the last lemma, we write $1 - ||\Theta\phi||$ as

$$\begin{aligned} \int_X \pi_*|\phi| - |\Theta\phi| &\geq \inf_X \frac{\langle \Theta\phi \rangle}{\langle \overline{\partial}\chi \rangle} \left(\int_X \pi_*|\phi| \frac{\langle \overline{\partial}\chi \rangle}{\langle \Theta\phi \rangle} - \int_X |\Theta\phi| \frac{\langle \overline{\partial}\chi \rangle}{\langle \Theta\phi \rangle} \right) \\ &= \inf_X \frac{\langle \Theta\phi \rangle}{\langle \overline{\partial}\chi \rangle} \left(\int_X \pi_*|\phi| \frac{\langle \overline{\partial}\chi \rangle}{\langle \Theta\phi \rangle} - \int_X \langle \overline{\partial}\chi \rangle \, dA \right). \end{aligned}$$

Now let η be the form guaranteed by Theorem 1.2 and F be the meromorphic function defined by 3.1.

$$\begin{aligned}
\int_X \pi_*|\phi|\frac{\langle\overline{\partial}\chi\rangle}{\langle\Theta\phi\rangle} &= \int_X \pi_*\left(|\phi|\frac{\langle\pi^*\overline{\partial}\chi\rangle}{\langle\pi^*\Theta\phi\rangle}\right)\\
&\geq \left|\int_X \Theta(F\langle\overline{\partial}\chi\circ\pi\rangle\, dA)\right|\\
&\geq \left|\int_X \Theta(t^{-1}F(\overline{\partial}\chi\circ\pi)\wedge\eta)\right|\\
&= t^{-1}\left|\int_X \Theta(F(\chi\circ\pi)\, dA)\right|\\
&= t^{-1}\left|\int_X (\Theta F)\chi\, dA\right|\\
&= t^{-1}\int_X \chi\, dA,
\end{aligned}$$

which is what we need to finish the proof. $\square$

4. More Preliminaries

In order to derive more explicit bounds for $||\Theta||$, we need some detailed results concerning the geometry of a hyperbolic Riemann surface X.

Local Coordinates. Recall that the injectivity radius of X at p is the largest number $I(p)$ such that $\{q\in X : d(p,q)<I(p)\}$ is a topological disk. By *standard coordinates* about p, we will mean a uniformization map $\pi_p:\Delta\to X$ that maps 0 to p. π_p will be a local isometry in the Poincaré metric, and it will map the sub-disk $\{|z|<\tanh(I(p)/2)\}$ injectively onto the disk of radius $I(p)$ about p. Consequently, π_p defines local coordinates in the usual sense on this disk.

Suppose that γ is a shortest path between two points $p_1,p_2\in X$. The notion of injectivity radius can also be used to give local coordinates about γ. If $I_\gamma = \min\{I(p):p\in\gamma\}$, then we have

Lemma 4.1. *Let $\pi_X:\mathbb{H}\to X$ be a uniformization map and $\tilde\gamma$ be any lift of γ. Then π_X maps the set*

$$U_{\tilde\gamma} = \{z\in\mathbb{H} : \mathrm{dist}(z,\tilde\gamma) < I_\gamma/3\}.$$

injectively onto the corresponding neighborhood U_γ of γ.

Proof. Suppose that π_X is not injective on $U_{\tilde\gamma}$—i.e. that there are points $z_1,z_2\in U_{\tilde\gamma}$ such that $\pi_X(z_1)=\pi_X(z_2)$. Let $w_j\in\tilde\gamma$ be chosen as close as possible to z_j, $j=1,2$. Note that

$$d(w_1,w_2) = d(\pi_X(w_1),\pi_X(w_2)) \leq d(\pi_X(w_1),\pi_X(z_1)) + d(\pi_X(w_2),\pi_X(z_2)) < 2I_\gamma/3$$

since z_1 and z_2 have the same image. On the other hand

$$d(w_1, z_2) \leq d(w_1, w_2) + d(w_2, z_2) < 2I_\gamma/3 + I_\gamma/3 = I_\gamma.$$

But now we have a contradiction, because both z_1 and z_2 lie within I_γ of w_1, whereas we know that π_X is injective on the disk of radius I_γ about w_1 □

We will also be concerned with coordinate neighborhoods of punctures and of simple, closed geodesics. Given either a puncture p or a simple closed geodesic γ, let T be the corresponding deck transformation on the universal cover Δ of X. Then there is a natural covering of X by the $\Delta/\{T^n\}$. In the case of a puncture, we obtain a holomorphic cover

$$\pi_p : \Delta^* \to X$$

which one can extend holomorphically past the origin by setting $\pi_p(0) = p$. We define the *cusp* $\mathcal{C} = \mathcal{C}_p$ about p to be the image under π_p of the set

$$\{0 < |z| < e^{-\pi}\}.$$

In the case of a simple closed geodesic, we obtain a holomorphic cover

$$\pi_\gamma : A_R \to X,$$

where $R = \pi^2/\operatorname{Length}(\gamma)$, and π_γ maps $\{|z| = 1\}$ onto γ bijectively. We define the *collar* $\mathcal{C} = \mathcal{C}_\gamma$ about γ to be the image under π_γ of the set $A_{R'}$ where

$$\tan \frac{\pi R'}{2R} = \frac{1}{\sinh(\frac{1}{2}\operatorname{Length}(\gamma))} \tag{4.1.}$$

This is equivalent to setting

$$\mathcal{C}_\gamma = \left\{ p \in X : \operatorname{dist}(p, \gamma) = \sinh^{-1} \frac{1}{\sinh(\frac{1}{2}\operatorname{Length}(\gamma))} \right\}.$$

We refer to the covering map associated with a closed geodesic (or a puncture) as *standard coordinates* about the geodesic (or the puncture, respectively) We will generally use standard coordinates for computations performed on cusps and collars—a practice justified by the next theorem. By a *short* geodesic, we mean a closed geodesic of length less than $2\sinh^{-1} 1$. Let Γ denote the set of all short geodesics on X.

Collar Theorem. *(see [Bu] sections 4.1 and 4.4 or [Ke]) The following statements hold for X.*

(1) *If γ is a simple closed geodesic on X, then π_γ is injective on $A_{R'}$. Similarly, if p is a puncture, then π_p is injective on $\{|z| < e^{-\pi}\}$.*
(2) *$|\Gamma| \leq 3g - 3 + |P|$, and all geodesics in Γ are simple.*

(3) *Cusps and collars about short geodesics are mutually disjoint from one another.*

(4) *Let $\mathcal{C}$ be a collar of a short geodesic or a cusp. Then*

$$I(p) \geq \sinh^{-1} e^{-\operatorname{dist}(p, b\mathcal{C})}$$

for every $p \in \mathcal{C}$.

(5) *Any point p that does not lie in a cusp or in the collar of a short geodesic satisfies $I(p) \geq \sinh^{-1} 1$.*

For reference purposes, we designate several important subsets of X.

$$\begin{aligned} X_{core} &= X \setminus \bigcup_{p \in P} \mathcal{C}_p \\ X_{core}(s) &= \{p \in X : \operatorname{dist}(p, X_{core}) \leq s\} \\ X_{thick} &= X_{core} \setminus \bigcup_{\gamma \in \Gamma} \mathcal{C}_\gamma \\ X_{thick}(s) &= \{p \in X : \operatorname{dist}(p, X_{thick}) \leq s\}. \end{aligned}$$

We refer to X_{core} as the *core* of X and X_{thick} as the *thick part* of X. One can derive useful diameter estimates for the core and thick part of X in terms of ℓ and topology.

Lemma 4.2. *There exists a topological constant C_1 such that any two points in the same connected component of X_{thick} are joined by a path in X_{thick} of length less than C_1. There exist topological constants C_2, C_3 such that any two points in X_{core} are joined by a path γ in X_{core} satisfying*

$$\operatorname{Length}(\gamma) \leq C_2 + C_3 \log \frac{1}{\ell}.$$

The first statement in this Lemma follows from Lemma 4.1, the absolute lower bound on injectivity radius among points in X_{thick}, and the fact that the area of a component of X_{thick} depends only on topology (see [Di] for a full proof of a similar result). The second statement follows quickly from the first statement and the Collar Theorem.

Harnack's inequality and change in the size of a quadratic differential. By restating the classical Harnack inequality for harmonic functions in an invariant form, we are able to obtain useful pointwise estimates for the size of a quadratic differential in the Poincaré metric.

Lemma 4.3. *Let X be a hyperbolic Riemann surface, $W \subset X$ a domain, and $h : W \to \mathbb{R}^+$ a positive harmonic function. Then*

$$\langle d \log h(p) \rangle \leq \frac{1}{\tanh(\frac{1}{2} \operatorname{dist}(p, bW))}.$$

Proof. In standard coordinates about p, we have that h is positive and harmonic on the disk $\{|z| < \tanh(\frac{1}{2}\operatorname{dist}(p, bW))\}$. After rotating coordinates, we may suppose that $\left.\frac{d\tilde{h}(re^{i\theta})}{dr}\right|_{r=0}$ is maximal when $\theta = 0$. Then

$$\begin{aligned}\langle d\log h(p)\rangle &= \frac{1}{2}\left.\frac{d\log h(r)}{dr}\right|_{r=0} \\ &= \lim_{r\to 0}\frac{1}{2r}\log\frac{h(r)}{h(0)} \\ &\leq \lim_{r\to 0}\frac{1}{2r}\log\frac{\tanh(\frac{1}{2}\operatorname{dist}(p,bW))+r}{\tanh(\frac{1}{2}\operatorname{dist}(p,bW))-r} \\ &= \frac{1}{\tanh(\frac{1}{2}\operatorname{dist}(p,bW))}.\end{aligned}$$

We have used the classical Harnack inequality in the transition between the second and third lines. ☐

Theorem 4.4. *Suppose that ψ is a holomorphic quadratic differential on X with zero set Z. Suppose that $W \subset X \setminus Z$ is a domain such that*

$$\langle\psi(p)\rangle \leq M$$

for all $p \in W$. Set $\rho(p) = \min\{d(p, bW), 1\}$. Then if $\gamma \subset W$ is a parametrized path connecting p_1 and p_2, we have

$$\frac{\langle\psi(p_1)\rangle}{\langle\psi(p_2)\rangle} \geq \left(\frac{\langle\psi(p_2)\rangle}{CM}\right)^{-1+e^{\int_\gamma \frac{ds}{\tanh(\rho/2)}}}.$$

C is a universal constant.

This theorem is especially useful when p_2 can be chosen so that $\langle\psi(p_2)\rangle$ is close to M.

Proof. Fix $p \in W$. In standard coordinates about p, we have $\psi = f(z)\,dz^2$, and $\langle\psi\rangle = |f(z)|(1-|z|^2)^2/4$, where f is defined on $\{|z| < \tanh(\rho(p)/2)\}$. If we take $C = (1-\tanh^2(1/2))^2$, then $\log(4M/C|f|)$ is a positive harmonic function on $\{|z| < \tanh(\rho(p)/2)\}$. We apply Harnack's inequality and obtain

$$\langle d\log\log(M/C\langle\psi\rangle)\rangle = \langle d\log\log(4M/C|f|)\rangle|_{z=0} \leq \frac{1}{\tanh(\rho/2)}.$$

After integrating this becomes

$$|\log\log(M/C\langle\psi(p_1)\rangle) - \log\log(M/C\langle\psi(p_2)\rangle)| \leq \int_\gamma \frac{ds}{\tanh(\rho/2)}.$$

The theorem follows after we exponentiate and rearrange. ☐

Maxima of quadratic differentials. We now prove two results about the maximum value $\langle\psi\rangle$ of a non-zero element $\psi \in Q(X)$ and about the location in X where the maximum occurs. As before, let Z be the zeroes of ψ.

Lemma 4.5. *$\langle\psi\rangle$ realizes its supremum at a point $p_{max} \in X_{core}$. There exist absolute constants C_1, C_2 such that*

$$\frac{||\psi||}{\mathrm{Area}(X)} \leq \langle\psi(p_{max})\rangle \leq \frac{C_1||\psi||}{\ell^2}. \tag{4.2}$$

and

$$\mathrm{dist}(p_{max}, Z) \geq C_2. \tag{4.3}$$

Proof. Let $\mathcal{C}$ be a cusp in X, and in standard coordinates on $\mathcal{C}$ write $\psi = f(z)\,dz^2/z$. Since ψ is integrable, it has at worst a simple pole at any puncture. So f extends holomorphically across $z = 0$. Let $M = \max_{|z|\leq e^{-\pi}} |f(z)|$. By the maximum principle, this maximum is realized at some point in the boundary of the cusp. We have then that

$$\langle\psi\rangle = |zf(z)(\log|z|)^2| \leq M|z|(\log|z|)^2$$

for all $|z| \leq e^{-\pi}$ with equality achieved at some point where $|z| = e^{-\pi}$. One can check by hand that $r(\log r)^2$ is increasing on the interval $0 < r \leq e^{-\pi}$. Hence the maximum of $\langle\psi\rangle$ on $\overline{\mathcal{C}}$ occurs on $b\mathcal{C}$. It follows that the maximum of $\langle\psi\rangle$ on X occurs in X_{core}.

The left estimate in 4.2 is immediate. To get the right estimate, we work in standard coordinates about p_{max}, writing $\psi = f(z)\,dz^2$ for some holomorphic function f, and $\langle\psi\rangle = |f|\frac{(1-|z|^2)^2}{4}$. In particular $|f(0)| = 4\langle\psi(p_{max})\rangle$. The disk $\{|z| < \tanh(I(p_{max})/2)\}$ injects into X, so we have

$$\begin{aligned}
||\psi|| = \int_X \langle\psi\rangle dA &\geq \int_0^{2\pi}\int_0^{\tanh(I(p_{max})/2)} |f(re^{i\theta})| r\,dr\,d\theta \\
&\geq 2\pi \int_0^{\tanh(I(p_{max})/2)} r\left|\int_{|z|=r} \frac{f(z)}{z}\,dz\right| dr \\
&= 2\pi|f(0)|\int_0^{\tanh(I(p_{max})/2)} r\,dr = 4\pi\langle\psi(p_{max})\rangle \tanh^2(I(p_{max})/2).
\end{aligned}$$

Since $p_{max} \in X_{core}$, we have $I(p_{max}) \geq \ell/2$. Thus we have

$$\langle\psi(p_{max})\rangle \leq \frac{||\psi||}{4\pi\tanh^2(\ell/4)} \leq \frac{C_1||\psi||}{\ell^2}$$

for $\ell < 1$ and some absolute constant C_1.

To get 4.3, note that

$$\langle\psi(p_{max})\rangle < \operatorname{dist}(p_{max}, Z) \cdot \max_X \langle d\langle\psi\rangle\rangle. \tag{4.4}$$

Moreover, working in standard coordinates about any point in X, we estimate

$$\begin{aligned}\langle d\langle\psi\rangle\rangle &\leq \frac{|f'(0)|}{8} = \frac{1}{16\pi}\left|\int_{|z|=1/2} \frac{f(z)\,dz}{z^2}\right| \\ &\leq C\langle\psi(p_{max})\rangle.\end{aligned}$$

4.3 follows after we use this estimate in 4.4. $\square$

Lemma 4.6. *Let $M(s) = \max\{\langle\psi(p)\rangle : p \in X_{thick}(s)\}$. Then there exists a topological constant C_1 such that*

$$M(0) \geq C_1||\psi||\ell$$

Secondly, if $0 \leq s \leq t$, we have

$$M(s) \geq e^{s-t}M(t).$$

Finally, if $p_{max}(s) \in X_{thick}(s)$ is a point where $M(s)$ is achieved, then

$$\operatorname{dist}(p_{max}(s), Z) \geq C_2$$

for some absolute constant C_2.

Proof. Suppose first that at least half of the mass of ψ is concentrated outside collars of short geodesics. It follows that $\langle\psi\rangle \geq ||\psi||/2\,\text{Area}(X)$ at some point in X_{thick} or in a cusp. Arguing as in the previous lemma, we see that in fact this occurs in X_{thick}. Since $\text{Area}(X)$ depends only on the topology of X, the estimate on $M(0)$ holds.

Now suppose that at least half the mass of ψ is concentrated inside collars of short geodesics. By (2) of the Collar Theorem, there exists a collar $\mathcal{C} = \mathcal{C}_\gamma$ containing a definite fraction of the mass of ψ. In standard coordinates on $\mathcal{C}$, we write $\psi = f(z)\,dz^2/z^2$ and let $M = \max_{\overline{\mathcal{C}}}|f| = \max_{b\mathcal{C}}|f|$. Then for $R' \leq R = \pi^2/\text{Length}(\gamma)$ satisfying 4.1, we have

$$C||\psi|| \leq \int_{\mathcal{C}}|\psi| = \int_0^{2\pi}\int_{e^{-R'}}^{e^{R'}} \frac{|f(z)|}{|z|^2} r\,dr\,d\theta \leq \int_0^{2\pi}\int_{e^{-R'}}^{e^{R'}} \frac{M}{r}\,dr\,d\theta = 4\pi MR'$$

Using 2.3 to express $\langle\psi\rangle$ in standard coordinates on $b\mathcal{C}$ gives

$$\begin{aligned}M(0) \geq \max_{b\mathcal{C}}\langle\psi\rangle &= \frac{4MR^2}{\pi^2}\cos^2\frac{\pi R'}{2R} \\ &\geq \frac{C||\psi||R^2}{R'}\tanh^2\frac{\text{Length}(\gamma)}{2} \geq C||\psi||\,\text{Length}(\gamma) \\ &\geq C_1||\psi||\ell\end{aligned}$$

To prove the inequality relating $M(s)$ and $M(t)$, we again use standard coordinates on $\mathcal{C}$ and write $\psi = f(z)\,dz^2/z^2$. The inequality then follows from the maximum principle, equation 2.3, and a straightforward computation. One consequence of the inequality between $M(s)$ and $M(t)$ is that for $\operatorname{dist}(p, p_{max}(s)) < 1/2$, we have $\langle\psi(p)\rangle \leq C\langle\psi(p_{max}(s))\rangle$ since $p \in X_{thick}(s+1/2)$. The lower bound on $\operatorname{dist}(p_{max}(s), Z)$ is then established just as it was in the previous lemma. □

5. Explicit bounds for $||\Theta||$

In this section we return to the proof of Theorem 1.1. Our goal is to obtain an expression for the constant k in the theorem which is explicit in terms of the length ℓ of the shortest closed geodesic on X and the constant t that arises in Theorem 1.2. In order to accomplish our goal, we will apply the results from the previous section to the bound 3.3 appearing in the conclusion of Lemma 3.2. As in Section 2, we let $\phi \in Q(Y)$ satisfy $||\phi|| = 1$, and we consider $\psi = \Theta\phi$. Without loss of generality we can assume that $||\psi|| \geq 1/2$. Unless, otherwise stated, we will assume implicitly that constants in this section depend only the topology of X. We will also assume for ease of stating results that $\ell < 1$ and $t > 1$.

In order to apply Lemma 3.2, we must first choose a compact set $K \subset X$. Our choice here will differ from the one we made in the proof given in Section 2 for Theorem 1.1.

Lemma 5.1. *There are constants $C_1, \ldots, C_5$ such that if $s = \log C_1 t$ and U is any connected component of $X_{thick}(s)$, we have*

(1) $\operatorname{Area}(U) - t\operatorname{Length}(bU) \geq C_2$;
(2) $I(p) \geq C_3/t$ *for all* $p \in U$;
(3) *Given any $p_1, p_2 \in U$, there exists a parametrized path $\gamma \subset U$ connecting p_1 and p_2 such that* $\operatorname{Length}(\gamma) < C_4 + C_5 \log t$.

Proof. We will find a constant s that guarantees (1) and then show that (2) and (3) follow. Call the genus of U, g_U. Let $A_1, \ldots, A_n \subset X$ denote the embedded annuli bounded by short closed geodesics on one side and components of bU on the other side. We allow for the possibility that for some values of j, A_j is actually part of a cusp—in this case, we have a puncture rather than a short closed geodesic bounding one side of A_j. In what follows, it makes sense to treat the puncture as a geodesic of length 0.

By the Gauss–Bonnet Theorem,

$$\operatorname{Area}(U) = 2\pi(2g_U + n - 2) - \sum \operatorname{Area}(A_j).$$

If $n = 0$, U is all of X, which gives (1) automatically. Otherwise, either $g_U \geq 1$ and $n \geq 1$, or $g_U = 0$ and $n \geq 3$. Thus

$$\operatorname{Area}(U) \geq 2\pi(n/3) - \sum \operatorname{Area}(A_j).$$

Let ℓ_j denote the length of the geodesic component of bA_j, and let L_j denote the length of the other component. Then

$$\operatorname{Area}(U) - t \operatorname{Length}(bU) \geq 2\pi(n/3) + \sum((t-1)\operatorname{Area}(A_j) - t(\operatorname{Area}(A_j) + L_j)).$$

By differentiating, one shows that

$$\operatorname{Area}(A_j) + L_j = \frac{e^{-s}\ell_j}{\tanh \ell_j/4}$$

is an increasing function of ℓ_j. It will be greatest when the two boundary components of A_j coincide, $\ell_j = L_j$, and $\operatorname{Area}(A_j) = 0$. In this case, s measures the distance from the geodesic to the edge of the collar containing A_j. Therefore

$$\sinh(\ell_j/2) = 1/\sinh s.$$

and

$$\begin{aligned} \operatorname{Area}(U) - t \operatorname{Length}(bU) &\geq 2n\left(\frac{\pi}{3} - t\sinh^{-1}\frac{1}{\sinh s}\right) \\ &\geq \frac{\pi}{3} \end{aligned}$$

if

$$s \geq \sinh^{-1}\frac{1}{\sinh(\pi/6t)}.$$

Since $t > 1$, the quantity on the right side is bounded above by $\log(C_1 t)$.

(2) is a direct consequence of item (4) of the Collar Theorem above. (3) follows from the diameter estimate for X_{core} given by Lemma 4.2, and from the definitions of s and $X_{thick}(s)$. $\square$

We now fix U to be the component of $X_{thick}(s)$ containing the point $p_{max}(s)$, where s is given by the previous lemma, and $p_{max}(s)$ is given in Lemma 4.6. We set $K' = \overline{X \setminus U}$, we let Z be the set of zeroes of ψ, and we define $K = K' \cup Z$. As usual, $K'(r), Z(r), K(r)$ denote the closed sets of points within distance r of each of K', Z, K, respectively. Recall that the cardinality $|Z|$ is controlled by the genus and number of punctures of X. Since bU is contained entirely in cusps and collars of short geodesics on X, one can show by direct computation that there are constants C_1, C_2 such that

$$\begin{aligned} \operatorname{Length}(bK(r)) &\leq \operatorname{Length}(bU) + C_1 r \\ \operatorname{Area}(X \setminus K(r)) &\geq \operatorname{Area}(U) - C_2 r \end{aligned}$$

for all $r < 1$. Hence,

$$\operatorname{Area}(X \setminus K(r)) - t\operatorname{Length}(bK(r)) \geq \operatorname{Area}(U) - t\operatorname{Length}(bU) - Ctr$$

is positive on an interval of length comparable to t^{-1}.

Lemma 5.2. *There are positive constants C_1, C_2 such that if $r < C_1/t$, then any two points in $X \setminus K(r)$ can be joined by a path in $X \setminus K(C_2 r)$.*

Proof. We first claim that for C_2 small enough, any connected component of $K(C_2 r)$ which intersects $K'(C_2 r)$ must actually lie in $K'(r)$. Hence, such components do not separate components of $X \setminus K(r)$. Indeed if V is a component of $K(C_2 r)$ which intersects $K'(C_2 r)$, and $p \in V \setminus K'(C_2 r)$, then p is joined to $K'(C_2 r)$ by a chain of disks of radius $C_2 r$ about points in Z. Hence, $\mathrm{dist}(p, K'(C_2 r)) \leq 2C_2|Z|r$. Choosing C_2 smaller than $(2|Z|+1)^{-1}$ proves the claim.

So now we point out that if the Lemma is false for small C_2, we can assume that some curve γ in $bZ(C_2 r)$ separates components of $X \setminus K(r)$. If we assume that $r < C_1/t \leq C_1 I$, where I is the minimum injectivity radius among points in U, and C_1 is small enough, we have that γ, and hence some component W of $X \setminus K(r)$ lies entirely inside a hyperbolic disk of radius $C_3 C_2 r$. But this forces W to lie within distance $C_2 r + C_3 C_2 r$ of some point in Z. Since $\mathrm{dist}(W, Z) \geq r$, we have a contradiction for C_2 small enough. □

Lemma 5.3. *Let $r \leq C_1/t$ be as in the previous Lemma. Any two points in in $\overline{X \setminus K(r)}$ are joined by a path $\gamma \subset \overline{X \setminus K(C_2 r)}$ with the following properties.*

1. *γ consists of length minimizing geodesic segments and at most one connected segment of each connected component of $bK(C_2 r)$.*
2. $\mathrm{Length}(\gamma) \leq C_3 + C_4 \log t$.
3. $\mathrm{Length}(\gamma \cap Z(s)) \leq C_5 s$ *for all $s > 0$.*

Proof. Given any two points p_1, p_2 in $\overline{X \setminus K(r)}$, first choose γ to be a length minimizing geodesic connecting p_1 to p_2. By (3) of Lemma 5.1, we know that $\mathrm{Length}(\gamma) \leq C_1 + C_2 \log t$. If γ intersects a component V of $K(C_2 r)$, let $q_1, q_2 \in bK(C_2 r)$ be the first and last points of that intersection. By the previous lemma,we know that there is a segment in bV connecting q_1 to q_2. We replace $\gamma \cap V$ with this segment of bV, possibly increasing the length of γ, but by no more than a constant times r. After carrying out this modification of γ on each component of $K(C_2 r)$, we obtain a new path connecting p_1 and p_2 satisfying item (1). Item (2) of the lemma follows since after modification the length of γ has increased by at most a constant times t^{-1}. It is not difficult to verify item (3) by considering geodesic segments of γ and segments of γ in $bK(C_2 r)$ separately. □

Recall that in Section 3, we defined

$$m(r) = \min_{bK(r)} \langle \psi \rangle.$$

We are now in a position to estimate $m(r)$. Fix a point $p_1 \in bK(r)$ and let γ be a path from p_1 to $p_{max}(s) = p_2$ satisfying the conclusions of Lemma 5.3. We can apply Lemma 4.4, taking the set W in the statement of that lemma to be $X_{thick}(s+1) \setminus Z$. By Theorem 4.6 we can assume that the number M in Theorem

4.4 satisfies $M \le C\langle\psi(p_{max}(s))\rangle$. We have for all $p \in \gamma$ that

$$\mathrm{dist}(p, bW) \ge \min\{1, \mathrm{dist}(p, Z)\}.$$

Hence,

$$\langle\psi(p_1)\rangle \ge \langle\psi(p_{max}(s))\rangle \exp\left(C_1 \int_\gamma \frac{ds}{\min\{1, \mathrm{dist}(p, Z)\}}\right). \tag{5.1}$$

To estimate the integral, we split the domain of integration into those points where $\mathrm{dist}(p, Z) > 1$ and those points where $\mathrm{dist}(p, Z) < 1$. We have

$$\int_{\gamma\setminus Z(1)} ds \le \mathrm{Length}(\gamma) \le C_1 + C_2 \log t$$

in the first case. In the second case, we have by Lemma 5.3

$$\begin{aligned}\int_{\gamma\cap Z(1)} \frac{ds}{\mathrm{dist}(p, Z)} &= \int_1^{1/C_2 r} \mathrm{Length}(\gamma \cap Z(1/u))\, du \\ &= \int_1^{1/C_2 r} \frac{C\, du}{u} = C_1 + C_2 \log \frac{1}{r}\end{aligned}$$

.

Altogether, the integral in equation 5.1 is dominated by

$$C_1 + C_2 \log t + C_3 \log \frac{1}{r}.$$

We apply Lemma 4.6 and the assumption that $||\psi|| \ge 1/2$ to estimate $\langle\psi(p_{max}(s))\rangle$, concluding by Theorem 4.4 that

$$m(r) \ge \ell \exp(-At^B r^{-C})$$

for positive constants A, B, C. Inserting this estimate (valid for $r < D/t$) into equation 3.3 gives

$$\begin{aligned}1 - ||\psi|| &\ge \frac{\ell}{t} \int_0^{D/t} (E - Ftr) \exp(-At^B r^{-C})\, dr \\ &\ge \frac{\ell}{t^2} \exp(-At^B)\end{aligned}$$

for positive constants A and B. Since the right side does not depend on ψ, we have proved

Theorem 5.4. *Let X be a Riemann surface of finite type. Let Y be a Riemann surface of infinite type with finitely generated fundamental group. Suppose that $\pi : Y \to X$ is a holomorphic covering map and $\Theta : Q(Y) \to Q(X)$ is the corresponding pushforward operator. Then*

$$||\Theta|| \leq 1 - k,$$

for a positive constant k. We can take

$$k = \frac{\ell}{t^2}\lambda^{-t^C},$$

where λ and C are positive constants depending only on the topology of X, t is the number associated with Y by Theorem 1.2, and ℓ is the length of the shortest closed geodesic on X.

Theorem 1.4 is an immediate consequence of Theorem 5.4. In particular, we have

Corollary 5.5. *Suppose that $\pi : Y \to X$ is a holomorphic cover of a surface of finite type by a disk or an annulus. Then the norm of the corresponding Poincaré series operator satisfies*

$$||\Theta|| \leq 1 - C\ell$$

for some constant C depending only on the topology of X.

This corollary holds because we can take $t = 1$ if Y is a disk or an annulus (see [Di]). On the other hand, by Theorem 1.2 one can take $t = C/\ell_Y^D$, where C, D are constants depending only the topology of Y and ℓ_Y is the length of the shortest closed geodesic on Y. Since holomorphic covers are local isometries we have $\ell_Y \geq \ell$. Theorem 1.3 follows.

We close this paper with an example that demonstrates that Corollary 5.6 is sharp. The example appears in [Mc1]

Let $\gamma \subset X$ be the shortest closed geodesic on a surface X of finite type. Let $\pi_\gamma : A_R \to X$ be the covering map giving standard coordinates on the collar about γ. We consider $\Theta\phi$ where $\phi = dz^2/z^2 \in Q(A_R)$. Since π_γ is injective on $A_{R'}$ (R' as in equation 4.1, we have

$$||\Theta\phi|| \geq \int_{A_{R'}} |\phi| - \int_{A_R \setminus A_{R'}} |\phi|.$$

An easy computation now shows that

$$||\Theta|| \geq \frac{||\Theta\phi||}{||\phi||} \geq 1 - C\ell,$$

for some absolute constant C.

6. Concluding Remarks

We stress again that the covering surface Y enters into the proof of Theorem 1.1 only through Theorem 1.2. If one is unconcerned with tying the bound on $1 - ||\Theta||$ to the geometry of X, then one can replace the assumption that Y has finitely generated fundamental group with the assumption that there exists a bounded solution of $\overline{\partial}\eta = \omega_A$ on Y. The proof of the existence of η given in [Di] relies chiefly on the existence of a Green's function on Y that admits adequate uniform upper bounds.

On the other hand, McMullen [Mc1] showed that $||\Theta|| < 1$ whenever the cover $\pi : Y \to X$ is non-amenable. It would be interesting to know whether one can demonstrate the existence and appropriate boundedness of a Green's function on Y from the assumption that there exists a non-amenable cover of some finite type surface X by Y.

Appendix: Extending Theorem 1.2 to infinite type surfaces with punctures.

In [Di] we proved Theorem 1.2 for infinite type Riemann surfaces Y without punctures. It is not difficult to extend the proof in that paper to handle infinite type surfaces with punctures. Let Y be a Riemann surface of infinite type and finitely generated fundamental group, and let $\overline{Y}$ be the compact bordered Riemann surface obtained by adjoining the ideal boundary of Y to Y. Note that $b\overline{Y}$ consists of a non-empty, disjoint union of simple closed curves. The proof in [Di] consists of the following main steps:

(1) Given any connected component $\overline{\gamma}$ of bY, there is a simple closed geodesic $\gamma \subset Y$ homotopic to $\overline{\gamma}$. One can write down a (1,0) form η_γ such that $\overline{\partial}\eta_\gamma = \omega_A$ and $\langle\eta\rangle \leq 1$ on the annulus between $\overline{\gamma}$ and γ.

(2) After choosing appropriate cutoff functions on each such annulus, finding a global bounded solution to $\overline{\partial}\eta = \omega_A$ on Y reduces to solving the equation with ω_A replaced by a form ω_0 with compact support. Because Y is of infinite type, there exists a Green's function $G(p, q)$ with pole at q on $\overline{Y}$. We set

$$h(q) = \int_{\overline{Y}} G(z, w)\, \omega_0.$$

Then if $\eta_0 = \partial h/4$, we have $\overline{\partial}\eta_0 = \Delta h = \omega_0$.

(3) It is not hard to show that $\langle\eta_0\rangle$ is controlled by estimates involving ω_0 and $G(z, w)$. We then relate the size of G to the geometry of Y via the estimate

$$G(z, w) \leq \log^+ \operatorname{dist}(z, w) + \frac{C}{\ell^k}$$

where ℓ is the length of the shortest closed geodesic on Y.

(4) The estimate on G depends in turn on a sort of inradius estimate for the core $Y_{core} \subset Y$ that one obtains by removing all boundary annuli. In particular, we show
$$d(p, bY_{core}) \leq C_1 + C_2 \log \frac{1}{\ell}$$
for all $p \in Y_{core}$.

To handle the case where Y has punctures, one needs to make the following modifications to the proof of Theorem 1.2

(1) As on boundary annuli, one can write down an explict bounded solution of $\overline{\partial}\eta = \omega_A$ on the cusp about a puncture in $\overline{Y} \setminus Y$. After choosing further appropriate cutoff functions on the cusps, one again reduces the differential equation to one with compactly supported data.
(2) We use the same method to obtain the global form η_0. The geometric estimates on G and the inradius of Y_{core} are the same as before. However, one must also remove cusps from Y to obtain Y_{core}.
(3) The proof of the inradius estimate for Y_{core} becomes slighly more elaborate in the presence of cusps. It proceeds roughly as follows: Any $p \in Y_{core}$, is joined to bY_{core} by a path $\gamma \subset Y_{core}$ that decomposes into (i) length minimizing geodesic segments lying in components of $Y_{thick} \cap Y_{core}$, (ii) length minimizing geodesic segments lying in collars of short geodesics, and (iii) connected arcs in the boundaries of cusps or collars of short geodesics. The number of pieces in the decomposition is bounded above in terms of topology. The Gauss-Bonnet Theorem gives a topological upper bound on the area of a component of $Y_{thick} \cap Y_{core}$. Combined with a slight variant of Lemma 4.1, this implies a topological upper bound for the length of a type (i) piece of γ. Direct computation shows that pieces of type (ii) have length no greater than $C \log \frac{1}{L}$ where L is the length of the core geodesic of the collar. Likewise, another computation shows that the length of pieces of type (iii) admits an absolute upper bound.

References

[BaDi] D. Barrett and J. Diller, *Poincaré series and holomorphic averaging*, Invent. Math. **110** (1992), 23–27.

[Bu] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston, 1992.

[Di] J. Diller, *A canonical $\overline{\partial}$ problem for Riemann surfaces*, To appear in Indiana Univ. Math J.

[Ga] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, John Wiley & Sons, New York, 1987.

[Ho] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North–Holland, New York, 1966.

[Ke] L. Keen, *Collars on Riemann surfaces*, Discontinuous Groups and Riemann Surfaces (L. Greenberg, ed.), Princeton University Press, Princeton NJ, 1974, pp. 263–268.

[Mc1] C. McMullen, *Amenability, poincaré series, and holomorphic averaging*, Invent. Math. **97** (1989), 95–127.

[Mc2] C. McMullen, *Iteration on teichmüller space*, Invent. Math. **99** (1990), 425–454.

[Mc3] C. McMullen, *Amenable Coverings of Complex Manifolds and Holomorphic Probability Measures*, Invent. Math. **110** (1992), 29–37.

Ann Arbor, MI 48109-1109 USA
E-mail address: barrett@math.lsa.umich.edu

Ithaca, NY 14853 USA
E-mail address: diller@math.cornell.edu