Inverting linear transformations between finite dimensional vector spaces

This is just a note to clean up a proof I got bogged down on in class. You'll see that I'm using a slightly different argument here.

Theorem 0.1. *Suppose $T : V \to W$, $S : W \to V$ are linear transformations between vector space of the same finite dimension. Then the following are equivalent.*

- $T \circ S = \mathrm{id}$.
- $S \circ T = \mathrm{id}$.
- $S = T^{-1}$.

Proof. The third assertion implies, by definition, both of the others. Hence it suffices to show that either of the first two assertions implies the third.

So suppose the first assertion holds: $T \circ S = \mathrm{id}$. Then on the one hand $T(V) \subset W$ automatically. But on the other hand, since $S(W) \subset V$, I see that $W = T(S(W)) \subset T(V)$, too. Thus $W = T(V)$. That is, T is surjective. But since $\dim V = \dim W < \infty$, this means that T is bijective. Hence the inverse function T^{-1} exists. Moreover,

$$T^{-1} = T^{-1} \circ \mathrm{id} = T^{-1} \circ (T \circ S) = (T^{-1} \circ T) \circ S = \mathrm{id} \circ S = S.$$

Hence $S = T^{-1}$, as desired.

The proof that the second assertion implies the third is identical, with the letters S and T switched throughout. □