Final Exam (Due Tuesday, December 15 by noon)
Math 60370, Fall '09

For a perfect score, it is sufficient to correctly solve four of the following eight problems. If you think you've solved four, then turn in just those problems. If you're unable to solve four problems completely, you're welcome to submit solutions to five problems, but you should point out any gaps (that you're aware of) in your arguments.

For assistance, You can use your notes, the textbook, old homework solutions, and your professor as resources, but nothing else. Do not, in particular, talk to other people about the exam or consult with other books.

1. Let $f : U \to \mathbf{C}$ be a non-constant holomorphic function on a domain $U \subset \mathbf{C}$. Let $N(z) = z - f(z)/f'(z)$. Show that if $f(z_0) = 0$, then there is a disk $D(z_0, r)$ to which $N(z)$ extends as a well-defined holomorphic function satisfying $N(z_0) = z_0$. Then show for $r > 0$ small enough, that

$$\lim_{n \to \infty} N^n(z) \to z_0$$

uniformly for $z \in D(z_0, r)$. Note that $f'(z_0)$ *can* vanish and be careful to explain why the converge is uniform rather than merely pointwise.

2. Explain why there exists a holomorphic function $f : \mathbf{C} \setminus [-1, 1] \to \mathbf{C}$ such that $e^{f(z)} = \frac{z-1}{z+1}$.

3. Show that if $f : \mathbf{C} \to \mathbf{C}$ is entire and

$$\int_{\mathbf{C}} |f(x + iy)| \, dx \, dy < \infty,$$

then $f \equiv 0$.

4. Find all automorphisms of $\mathbf{C} \setminus \{0, 1, \infty\}$ and prove there are no others.

5. Show that a bounded harmonic function $h : D^*(0, 1) \to \mathbf{R}$ extends to a harmonic function $h : D(0, 1) \to \mathbf{R}$.

6. (Partial fractions decomposition) Let $R(z) = P(z)/Q(z)$ be a rational function such that P and Q have no common factors and $\deg P < \deg Q$. Suppose $Q(z) = \Pi_{j=1}^{k}(z - z_j)^{m_j}$. Show that one can find constants $c_{j\ell} \in \mathbf{C}$ such that

$$R(z) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \frac{c_{j\ell}}{(z - z_j)^{\ell}}.$$

(Note: There's a proof of this statement by pure linear algebra, but I'm looking for a complex analysis proof here.)

7. Give an example of a holomorphic function $f : D(0,1) \to D(0,1)$ such that $f(0) = 1/2$ and $f'(0) = 3/4$, but show that there is no holomorphic function $f : D(0,1) \to D(0,1)$ such that $f(0) = 1/2$ and $f'(0) = 4/5$.

8. Suppose that $f : \mathbf{H} \to \mathbf{C}$ is holomorphic and that $I \subset \mathbf{R} = b\mathbf{H}$ is an open interval such that $\lim_{z \to x} f(x) = 0$ for every $x \in I$. Show that $f \equiv 0$.