Textbook problems:

p145: 1 (you may assume that \(P_1, \ldots, P_k\) are the points in \(\mathbb{C}\) where the denominator of \(R\) vanishes. Also, as far as I can tell, it does no good to assume that \(f\) has poles at these points.)

Solution. Let \(R = p/q\), where \(p, q\) are polynomials; let \(S \subset \mathbb{C}\) denote the finite set of points consisting of all zeroes of both \(p\) and \(q\). Then \(g := f/R\) is a holomorphic function on \(\mathbb{C} \setminus S\) satisfying \(|g| \leq 1\). Moreover, since \(S\) is finite, each point in \(S\) is an isolated singularity for \(g\). On account of our bound for \(g\), these singularities are all removable. I may therefore, by the removable singularity theorem, regard \(g\) as a holomorphic function defined on all of \(\mathbb{C}\). By continuity, I have \(|g| \leq 1\) even at points in \(S\). Therefore Liouville’s Theorem tells me that \(g\) is constant. That is, \(f = cR\) for some \(c \in \mathbb{C}\). □

p145: 5cde (justify your answers)

Solution.

c: 0 is a pole, because \(\lim_{z \to \infty} |1/z^3 + \cos z| \geq \lim_{z \to \infty} |z|^{-3} - 1 = \infty\).

d: 0 is essential for \(f(z) = z e^{1/z} e^{-1/z^2}\) because on the one hand, if \(z_n = -1/n\), then
\[
\lim_{n \to \infty} |f(z_n)| = \lim_{n \to \infty} -e^{-n^2 - n} = 0.
\]
Hence 0 is not a pole of \(f\). But on the other hand,
\[
\lim_{n \to \infty} |f(z_n)| = \lim_{n \to \infty} |ze^{n^2 + in}| = \lim_{n \to \infty} e^{n^2} = \infty,
\]
so that 0 is not removable either.

e: 0 is removable, because by L’Hôpital’s rule (as in last week’s homework), I have
\[
\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{\cos z}{1} = 1.
\]
In particular, there exists \(r > 0\) such that \(|z^{-1} \sin z| \leq 2\) for all \(z \in D^*(0, 1)\).

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Solution. Suppose the assertion is false: the alternative is that there exists \(N \in \mathbb{N}\) such that \(|f(z)(z - P)^N| \leq N\) for all \(z \in D^*(P, r)\). That is \(g(z) := f(z)(z - P)^N\) has a removable singularity at \(P\). By the removable singularity theorem, I may assume that \(g\) is well-defined and holomorphic on all of \(D(P, r)\). I may therefore write \(g(z) = (z - P)^k \tilde{g}(z)\) where \(k \in \mathbb{N}\) and \(\tilde{g} : D(P, r) \to \mathbb{C}\) is holomorphic and does not vanish at \(P\). This gives \(f(z) = \tilde{g}(z)(z - P)^{k-N}\) for all \(z \in D(P, r)\). If \(k \geq N\), then \(|f| \leq |\tilde{g}| \leq 2|\tilde{g}(P)|\) is bounded near \(P\), so that \(P\) is removable for \(f\). If \(k < N\), then \(\lim_{z \to P} |f(z)| = |\tilde{g}(P)| \lim_{z \to 0} |z - P|^{k-N} = \infty\), so that
$P$ is a pole for $f$. Regardless, $P$ is not essential, contrary to assumption. So the assertion is true after all. \hfill \square

**Problem 1.** The Laurent series for a holomorphic function depends on more than where you center your annulus. Consider, for instance, the function $f(z) = 1/\sin z$.

- Compute the principle part of the Laurent series for $f(z)$ on $D^*(0, \pi)$.

  *Solution.* Since $|\sin z| \to 0$ as $z \to 0$, it follows that $1/\sin z$ has a pole at 0. So I can write $f(z) = z^{-k} \tilde{f}(z)$ for some $k \in \mathbb{N}$ and $\tilde{f} : \mathbb{C} \to \mathbb{C}$ holomorphic with $\tilde{f}(0) \neq 0$. In particular $\lim_{z \to 0} z^k f(z) = \tilde{f}(0)$ exists and is non-zero. Using e.g. L'hôpital’s rule, one verifies that this happens only when $k = 1$ and that the limit in that case is 1. Using the fact that $\tilde{f}$ is defined by a power series centered at 0, I obtain

  \[
  z/\sin z = \tilde{f}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n
  \]

  so that $f(z) = 1/z + \sum_{n=0}^{\infty} c_{n+1} z^n$. The principal part of this is $1/z$.

- Find a rational function $R(z)$ such that $1/\sin z - R(z)$ has removable singularities on $|z| < 2\pi$.

  *Solution.* On $D(z, 2\pi)$, the function $\sin z$ has three zeroes: 0, $\pi$, and $-\pi$. Moreover,

  \[
  \lim_{z \to 0} \frac{\sin z}{z} = 1, \quad \lim_{z \to \pi} \frac{\sin z}{z-\pi} = \lim_{z \to -\pi} \frac{\sin z}{z+\pi}
  \]

  Hence I take $R(z) = \frac{1}{z} - \frac{1}{z-\pi} - \frac{1}{z+\pi}$, and compute that

  \[
  \lim_{z \to 0} \frac{1}{\sin z} - R(z) = \lim_{z \to 0} \frac{1}{\sin z} - \frac{1}{z} = \lim_{z \to 0} \frac{z - \sin z}{z \sin z} = 1,
  \]

  where the last limit can be established using e.g. power series or (my preference) a couple of applications of L’hôpital’s rule. At any rate, it follows that $\frac{1}{\sin z} - R(z)$ is bounded near $z = 0$ and that 0 is therefore a removable singularity. The same kind of argument shows that $\frac{1}{\sin z} - R(z)$ has removable singularities at $\pm \pi$.

- Compute the coefficients of the negative powers of $z$ in the Laurent series expansion for $1/\sin z$ on the annulus $\{\pi < |z| < 2\pi\}$.

  *Solution.* By the previous part and the removable singularities theorem, there is a holomorphic function $g : D(0, 2\pi) \to \mathbb{C}$ such that $g(z) = \frac{1}{\sin z} - R(z)$ for all $z$ other than 0 and $\pi$. Since $g$ is given on $D(0, 2\pi)$ by a power series centered at 0, it follows that the principal part of the Laurent series of $\frac{1}{\sin z}$ on $\{\pi < |z| < 2\pi\}$ is that of $R(z)$, which I compute
using geometric series tricks:

\[ R(z) = \frac{1}{z} - \frac{1}{z - \pi} - \frac{1}{z + \pi} = \frac{1}{z} \left( 1 - \frac{2}{1 - (\pi/z)^2} \right) \]

\[ = \frac{1}{z} \left( 1 - 2 \sum_{n=0}^{\infty} \left( \frac{\pi}{z} \right)^{2n} \right) \]

\[ = -\frac{1}{z} - \sum_{n=1}^{\infty} 2 \left( \frac{\pi}{z} \right)^{2n+1}. \]

So the negative coefficients in the Laurent series for \( \frac{1}{\sin z} \) are \(-1, 0, -2, 0, -2, 0, -2, \ldots \)

**Problem 2.** Suppose that \( f : D^*(0,1) \to \mathbb{C} \) is holomorphic and satisfies

\[ z^2 f'(z) = [f(z)]^2 + z. \]

Show that \( f \) has an essential singularity at 0.

**Solution.** Suppose rather that 0 is a pole or removable singularity of \( f \). Then, as usual, I can write \( f(z) = z^k g(z) \) where \( k \in \mathbb{Z} \) is the order of the singularity and \( g : D(0,1) \to \mathbb{C} \) is a holomorphic function that does not vanish at 0. Plugging into the given equation results in

\[ z^{k+2} g'(z) + k z^{k+1} g(z) = z^{2k} [g(z)]^2 + z. \]

When \( k \geq 1 \), this equation may be summarized as

\[ z^{k+1} h(z) = z \tilde{h}(z) \]

where neither \( h \) nor \( \tilde{h} \) vanish at 0. This is impossible since \( k \geq 1 \) implies \( k + 1 \geq 2 \). If on the other hand, \( k \leq 0 \), then the given equation becomes

\[ z^{k+1} h(z) = z^{2k} \tilde{h}(z) \]

where again neither \( h(0) \) nor \( \tilde{h}(0) \) is 0. This, too, is impossible since \( k \leq 0 \) implies that \( 2k \leq k + 1 \). In any case, I am led to conclude that 0 must be an essential singularity of \( f \).

**Problem 3.** Show that an isolated singularity of a holomorphic function \( f \) cannot be a pole of \( e^f \).

**Solution.** I let \( P \in \mathbb{C} \) denote the location of the isolated singularity, and I suppose (contrary to assertion) that \( P \) is a pole for \( e^f \). Then

\[ \infty = \lim_{z \to P} |e^{f(z)}| = \lim_{z \to P} e^{\text{Re} f(z)}. \]

Hence \( \lim_{z \to P} \text{Re} f(z) = \infty \). This implies that

\[ \lim_{z \to P} |f(z)| = \lim_{z \to P} \sqrt{(\text{Re} f(z))^2 + (\text{Im} f(z))^2} \geq \lim_{z \to \infty} |\text{Re} f(z)| = \infty. \]

Hence \( P \) must also be a pole for \( f \).
I now write $f(z) = (z - P)^{-k}g(z)$, where $k \in \mathbb{Z}_+$, and $g$ is holomorphic across $P$ with $a := g(P) \neq 0$. Taking $w$ to be some $k$th root of $-a^{-1}$, I set $z_n = P + w/n$. Then $z_n \to P$ and therefore

$$
\infty = \lim_{z \to P} e^{f(z)} = \lim_{n \to \infty} e^{f(z_n)} = \lim_{n \to \infty} e^{-n^k a^{-1} g(P+w/n)} = \lim_{n \to \infty} e^{-n^k} = 0
$$

From this contradiction, I conclude that $e^f$ cannot have a pole at $P$. \qed