Math 20580: 2nd group homework
More about diagonalization

(due Monday, November 22)

Please work with one or two of you classmates to solve the following problems concerning diagonalization. You are welcome to use matlab, calculators, etc to ease the computation here. Regardless, your solutions should not go into great detail about your computations but rather summarize them and explain how they solve the given problems. The solutions that you turn in shouldn't occupy more than three pages.

Problem 1. In order to buy a house, you borrow 200K from your local bank. The terms are as follows: you pay 5% interest compounded monthly on the remaining principal, you pay down this principal by making one payment per month of P dollars, and P is chosen so that the loan is repaid at the end of 30 years. What is P?

Hints: Let A_n be the amount of money you own at the end of month n. Note that for two specific times n, the value of A_n is known. Find a recursive relationship between A_{n+1} and A_n. Then let $\mathbf{v}_n = \begin{pmatrix} A_n \\ P \end{pmatrix}$ and write down a matrix relating $\mathbf{v}_{n+1}$ and $\mathbf{v}_n$.

Problem 2. Its cold season, and campus splits into two groups of people: those who have colds and those who don't. On any given day, a healthy person has a 10% chance of catching a new cold, and an afflicted person has a 20% chance of recovering from his or her cold. Given that cold season lasts a good 100 days, about what are the chances that a given person will have a cold at the end of the season? Explain why your answer has nothing to do with whether or not the person starts the season with a cold. (Hint: let p_n be the fraction of people with colds on day n and q_n the fraction of people without. What is the probable relationship between (p_{n+1}, q_{n+1}) and (p_n, q_n)?)

Problem 3. Consider the plane $H = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 + 2x_2 - 2x_3 = 0\}$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation $T(\mathbf{v}) = \mathbf{v}_H$ (i.e. the orthogonal projection of $\mathbf{v}$ onto H.

(a) Find the standard matrix A for T by computing the images $T(\mathbf{e}_j)$ of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

(b) Alternatively, find A by identifying a basis of eigenvectors for T, writing down a diagonalization of A, and then multiplying everything out. Which (if either) method for finding A seems easier to use?

(c) Note that if $\mathbf{v} = \mathbf{v}_H + \mathbf{v}_\perp$ is the usual decomposition of $\mathbf{v}$ into vectors contained in and orthogonal to H, then the reflection of $\mathbf{v}$ through H is given by $R(\mathbf{v}) = \mathbf{v}_H - \mathbf{v}_\perp$. Use this to find a nice relationship between $\mathbf{v}$, its projection $T(\mathbf{v})$ onto H, and its reflection $R(\mathbf{v})$ through H. Use this to find the standard matrix for R from the matrix for T.