Homework 10
(due Friday, Nov 22)

Problem 1. Solve the following ODEs and initial value problems left over from last week's homework.

(a) $y'' + 2y' + 5y = 12e^{-t}$
(b) $y'' + y' + 4y = \cos 3t$ (note that this one has changed from last week).
(c) $y'' - 4y' - 5y = 4e^{-2t}$, $y(0) = 0$, $y'(0) = -1$
(d) $y'' - 2y' - 3y = 3e^{2t}$.
(e) $y'' + 2y' = 3 + 4\sin 2t$.

Problem 2. (Another remnant of last week) Find the correct *form* for a particular solution y_p of $y'' + y = t(1 + \sin t)$. That is, you don't need to actually determine the constants that appear in your solution.

Problem 3. Use Newton's method with initial guess $p_0 = (1, -1)$ to approximate a solution of

$$\begin{aligned} y^2 e^x &= 3 \\ 2ye^x + 10y^4 &= 0. \end{aligned}$$

to 6 decimal places.

Problem 4. Use Newton's method to find the unique point of intersection (to 6 decimal places) between the ellipses

$$\begin{aligned} 3x^2 + y^2 &= 7 \\ x^2 + 4y^2 &= 8 \end{aligned}$$

that lies in the first quadrant. Choose your initial guess by graphing the two ellipses by hand and then staring at the picture.

Problem 5. Note that the point $(1, 2)$ satisfies the equation

$$x^4 - 3xy + y^3 - 3 = 0.$$

Use the idea behind Newton's method (replacing a function by its linearization) to find x (to six decimal places) so that $(x, 2.15)$ that satisfies the same equation.

Problem 6. Let $f : \mathbf{R} \to \mathbf{R}$ be a function and $a \in \mathbf{R}$ be a fixed point of f; i.e. $f(a) = a$. Suppose that f is differentiable at a and in fact $|f'(a)| < 1$. Show that there exists $\epsilon > 0$ such that if $|x - a| < \epsilon$ then the sequence $x, f(x), f^2(x), \ldots$ converges to a.

Problem 7. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function and a is a root of f; i.e. $f(a) = 0$. Let $N(x) = x - f(x)/f'(x)$ be the 'Newton improvement function'. Show that if f is twice differentiable at a, then there exists $\epsilon > 0$ such that if $|x - a| < \epsilon$, then the sequence $x, N(x), N^2(x), \ldots$ converges to a. (Hint: what is $N'(a)$?)

Problem 8. Consider the equation $\cos x = x$ (where the 'angle' x is as usual in calculus, given in radians).

(a) Show that the equation has a unique solution $x \in \mathbf{R}$.

(b) Show that cos is a contraction mapping on $[0, 1]$. Hence the sequence $0, \cos 0, \cos(\cos 0), \ldots$ converges to the solution of $\cos x = x$.

(c) How does the equation change if one uses degrees as units? Explain why the same conclusions hold in parts (a) and (b), only better.

(d) (Extra credit) Can you find a system of units for the angle x where the first conclusion holds but the second fails?