**Homework 11**
(due Friday 12/6)

**PLEASE NOTE:** Problems 6 and 7 are new as of Monday 12/2.

**Problem 1.** Hubbard and Hubbard: 2.10.1, 2.10.5, 2.10.8

**Problem 2.** In coming to grips with a theorem about functions $f : \mathbf{R}^n \to \mathbf{R}^m$, it's almost always a good idea to consider the special case $m = n = 1$.

(a) What does the inverse theorem say for functions $f : \mathbf{R} \to \mathbf{R}$?
(b) Why is the proof of the inverse function theorem *much* less complicated in this case?
(c) Give an example of an invertible $C^1$ function $f : \mathbf{R} \to \mathbf{R}$ whose inverse is not $C^1$.

**Problem 3.** The function $f : \mathbf{C} \to \mathbf{C}$ given by $f(z) = e^z$ can be regarded as a function $f : \mathbf{R}^2 \to \mathbf{R}^2$—i.e. $f(x, y) = (f_1(x, y), f_2(x, y))$ where $x$ and $y$ are the real and imaginary parts of $f$ and $f_1$ and $f_2$ are the real and imaginary parts of $f$.

(a) Write down formulas for $f_1$ and $f_2$.
(b) Compute the Jacobian matrix $Jf(x, y)$, and show it is invertible at every $(x, y) \in \mathbf{R}^2$. Hence by the inverse function theorem, $f$ is 'locally invertible' with $C^1$ inverse near any $(x, y) \in \mathbf{R}^2$.
(c) Show that nevertheless, $f$ is not *globally* invertible: i.e. there is no function $g : \mathbf{C} \to \mathbf{C}$ satisfying $g \circ f(x, y) = (x, y)$ and $f \circ g(s, t) = (s, t)$ for all $(x, y), (s, t) \in \mathbf{R}^2$.
(d) Consider instead the function $f : \mathbf{C} \to \mathbf{C}$ given by $f(z) = z^2$. Again, rewrite $f$ in real terms as a function from $\mathbf{R}^2$ and use this to determine all points in $\mathbf{C}$ at which $f$ is locally invertible with $C^1$ inverse.

**Problem 4.** Consider the curve $(x^2 + y^2)^2 = x^2 - y^2$ from Problem 2-6 in Jones notes. Use the implicit function theorem to determine all points on the curve can where one (in principle) can express $y$ locally as a $C^1$ function of $x$? Near which points can one express $x$ locally as a $C^1$ function of $y$? Near which points can we do neither? How do your answers accord with what you see when you look at a plot of the curve?

**Problem 5.** Consider the function $f : \mathbf{R}^3 \to \mathbf{R}$ given by $f(x, y, z) = x^2 + y^2 + z^2 + xyz$.

(a) Use the implicit function theorem to determine the values of $c \in \mathbf{R}$ for which the level set $\{f(x, y, z) = c\}$ is a hypersurface. The remaining values of $c$ are called *critical values* of $f$.
(b) Use Mathematica (check out the command 'ContourPlot3D') or whatever to plot the level surfaces of $f$ for critical values of $c$ and also for values of $c$ that are a little above and below the critical values. In fact, I'd encourage you to plot some level surfaces

of $f$ *before* you complete the first part of this problem, and try to spot the critical values visually.

(c) Summarize what you see in a coherent narrative about how the level surfaces of $f$ change as $c$ goes from $-\infty$ to $\infty$. Subplots involving love, betrayal and/or absolutely adorable pets are welcome.

**Problem 6.** Jones notes: 5-7 (explain why one should expect to get the same answer for both functions), 5-17, 5-27 (there are a *lot* of critical points for this last one)

**Problem 7.** There are a lot of cool uses of the Lagrange multiplier method, but this has *got* to be one of the best. Let $a, b, c, d > 0$ be given numbers. Your goal is to find a plane quadrilateral $Q \subset \mathbf{R}^2$

- whose sides have (in order) exactly lengths $a, b, c, d$, and
- whose area is maximal.

Proceed as follows, drawing and labeling a picture of your quadrilateral $Q$ in order to help make sense of things.

(a) What further necessary conditions must $a, b, c, d$ satisfy so that this problem has a solution?

(b) Let $\alpha$ be the angle between the sides of length $a$ and $d$ and $\beta$ be the angle between the sides of length $b$ and $c$.

(c) Find a constraint equation $f(\alpha, \beta) = 0$ that must be satisfied by $\alpha$ and $\beta$.

(d) Explain why we must have $0 \leq \alpha, \beta \leq \pi$.

(e) Express the area $A(\alpha, \beta)$ of $Q$ in terms of $\alpha$ and $\beta$.

(f) Use the method of Lagrange multipliers to find a (very simple) condition satisfied by $\alpha$ and $\beta$ when the area $A$ is maximal. Be careful here to account for any critical points of $f$.

(g) Dust off a theorem from Euclidean geometry to interpret this condition geometrically.