Homework 8
(due Friday, Nov 8)

Problem 1. Do problems 9, 10, 20, 21 from the book *Differential Equations*, 2nd ed, by Polking, Boggess and Arnold). In each problem describe the asymptotic behavior as t increases of the solution $y(t)$ when $y(0) = 0$ and when $y(0) - 2$.

Problem 2. Consider the initial value problem

$$y' = y^{2/3}, \quad y(0) = 0.$$

Show that for any $C > 0$, the function

$$y(t) = \begin{cases} 0 & \text{if} \quad t \le C \\ \left(\frac{t-C}{3}\right)^3 & \text{if} \quad t > C \end{cases}$$

is a solution. Be especially careful with the point $t = C$. Why does this not contradict the existence/uniqueness theorem for solutions of first order ODEs?

Problem 3. Verify that applying Picard iteration to the initial value problem $y' = y$, $y(0) = 1$ with initial guess $y_0(t) \equiv 1$ gives rise to $y_n(t)$ equal to the nth order Taylor polynomial centered at 0 for e^t.

Problem 4. Suppose that $f, g : U \to \mathbf{R}$ are C^1 functions such that $f(t, y) < g(t, y)$ for all $(t, y) \in \mathbf{R}^2$. Suppose further that on some open interval $I \subset \mathbf{R}$ we have solutions $y_1, y_2 : I \to \mathbf{R}$ satisfying the same initial condition $y_1(t_0) = y_2(t_0)$ but different ODEs: $y_1' = f(t, y_1)$, $y_2' = f(t, y_2)$. Show that $y_1(t) < y_2(t)$ for all $t > t_0$ and $y_1(t) > y_2(t)$ for all $t < t_0$. This is a little trickier than it might first appear. It might help to follow this outline.

- Show that there is a small open interval $(t_0-\delta, t_0+\delta)$ about t_0 on which the conclusions hold.
- Explain that one can draw similar conclusions at *any* point t_1 where we have equality $y_1(t_1) = y_2(t_1)$.
- Explain that to conclude the argument it suffices to show that t_0 is the *only* point where y_1 and y_2 agree.
- Finish with a proof by contradiction.

Problem 5. Let $y(x)$ denote the height, at horizontal position x, of a cable freely suspended between two points. Assume for simplicity's sake that the downward force of gravity is equal to the mass of an object (i.e. $g = 1$) and that the density of the cable is 1. Note that the forces acting on any small segment of the cable are

- the downward force of gravity due to the mass of the segment.
- a force exerted by the portion of the cable to the right of the segment.

- a force exerted by the portion of the cable to the left of the segment.

Each of these forces has a horizontal and a vertical component, and since the cable is at rest, the forces must add to zero at every point of the cable. Use these observations to write down a differential equation satisfied by $y(x)$. Then solve this equation to derive a formula for the shape of the hanging cable.

Problem 6. Redo the previous problem for the situation of an ideal suspension bridge. That is, assume the cable is massless but that it supports a bridge whose weight is spread out evenly (assume the density is 1 again) in the horizontal direction. Believe it or not, this makes the problem much simpler.