Homework 9

(due Friday, Nov 15)

Problem 1. Solve the following ODEs and initial value problems (from Polking, Boggess and Arnold)

(a) $y'' - y' - 2y = 0$
(b) $y'' + 4y' + 5y = 0$
(c) $4y'' + 4y' + y = 0$
(d) $y'' + 2y' + 3y = 0$, $y(0) = 1$, $y'(0) = 0$
(e) $y'' + 2y' + 5y = 12e^{-t}$
(f) $y'' + 4y = \cos 3t$
(g) $y'' - 4y' - 5y = 4e^{-2t}$, $y(0) = 0$, $y'(0) = -1$
(h) $y'' + 2y' + 4y = 0$.
(i) $\frac{d^4y}{dt^4} - 4\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} = 0$.
(j) $y''' + y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$.
(k) $y'' - 2y' - 3y = 3e^{2t}$.
(l) $y'' + 2y' = 3 + 4\sin 2t$.

Problem 2. Recall that if $z : I \to \mathbf{C}$ is a complex-valued function defined on an open interval $I \subset \mathbf{R}$, then one can write $z = u + iv$ where $u, v : I \to \mathbf{R}$ are real-valued functions. It follows that $z'(t) = u'(t) + iv'(t)$. Use this fact to verify that

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}.$$

for any *complex* constant λ.

Problem 3. Find the correct *form* for a particular solution y_p of $y'' + y = t(1 + \sin t)$. That is, you don't need to actually determine the constants that appear in your solution.

Problem 4. Verify that $y(t) = t^{-1}$ solves

$$2t^2y'' + 3ty' - y = 0.$$

Use this information to find the general solution of the ODE on the interval $(0, \infty)$.

Problem 5. Show that no solution $y(t)$ of the homogeneous linear equation $y'' - t^2y = 0$ can be found by solving the first order equations $y' - ty = 0$, $y' + ty = 0$. Hence the technique of factoring differential operators (at least when it is applied naively) does not extend to help solve linear homogeneous equations with non-constant coefficients. Why not?

Problem 6. Civilization collapses and in particular, everything about trig functions is forgotten. Some time after the collapse, we rediscover the existence and uniqueness theorem for solutions to second order linear ODEs. We then define functions $y = C, S : \mathbf{R} \to \mathbf{R}$ to be the unique solutions of $y'' + y = 0$ satisfying the initial conditions $C(0) = 1$, $C'(0) = 0$ and $S(0) = 0$ and $S'(0) = 1$. From this, prove each of the following.

(a) $C' = -S$ and $S' = C$.
(b) $C(t)^2 + S(t)^2 = 1$ for all $t \in \mathbf{R}$.
(c) $C(t+s) = C(t)C(s) - S(t)S(s)$ and $S(t+s) = S(t)C(s) + C(t)S(s)$ for all $t, s \in \mathbf{R}$.
(d) (Extra credit) There is a number $t_0 > 0$ such that $C(t) > 0$ when $t \in [0, t_0)$ but $C(t_0) = 0$. The hard part here is to show that there exists *any* point s where $C(s) = 0$, let alone a first such point. Proving the existence of s tests your understanding of differential calculus from math 165 (or wherever you learned that stuff).
(e) $C(t)$ decreases from 1 to 0 on $[0, t_0]$ whereas $S(t)$ increases from 0 to 1.
(f) Work out similar statements for C and S on the intervals $[t_0, 2t_0]$, $[2t_0, 3t_0]$, and $[3t_0, 4t_0]$.
(g) C and S are both periodic with period $4t_0$.

Problem 7. Extra credit: (A bit of analysis) Let $y : \mathbf{R} \to \mathbf{R}$ be a function that is differentiable at every point.

(a) Show by example that y' need not be continuous.
(b) Give an example where $y'(0) > 0$ but (nevertheless) y fails to be increasing on $(-\epsilon, \epsilon)$ no matter how small you choose $\epsilon > 0$.
(c) Prove that y' satisfies the conclusion of the intermediate value theorem; say specifically, if $y'(0) < 0$ and $y'(1) > 0$, then there exists $t \in (0, 1)$ such that $y'(t) = 0$.

I'll give credit for solutions to each of these separately.