1. Continuous functions and open sets

Definition 1.1. *Let $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a function and $S \subset \mathbf{R}^m$ a set. The* preimage *of S by f is the set*

$$f^{-1}(S) = \{x \in U : f(x) \in S\}.$$

So for instance, if $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = x^2$, then $f^{-1}([0,1]) = [-1,1]$, $f^{-1}([1,4]) = [-2,-1] \cup [2,1]$, and $f^{-1}([-4,-1]) = \emptyset$. The following result characterizes continuity in terms of preimages of open sets.

Theorem 1.2. *Let $U \subset \mathbf{R}^n$ be open. A function $f : U \to \mathbf{R}^m$ is continuous (at all points in U) if and only if for each open $V \subset \mathbf{R}^m$, the preimage $f^{-1}(V)$ is also open.*

Proof. Suppose that f is continuous on U and that $V \subset \mathbf{R}^m$ is open. Given a point $a \in f^{-1}(V)$, we have (by definition of $f^{-1}(V)$) that $f(a) \in V$. Since V is open, there exists $\epsilon > 0$ such that $B(f(a), \epsilon) \subset V$. And since f is continuous at a, there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies that $\|f(x) - f(a)\| < \epsilon$. That is, $x \in B(a, \delta)$ implies $f(x) \in B(f(a), \epsilon)$. In particular $f(x) \in V$. Thus $B(a, \delta) \subset f^{-1}(V)$, which means that $f^{-1}(V)$ is open. This proves the 'only if' part of the theorem.

For the 'if' part of the theorem, let us suppose that $f^{-1}(V)$ is open whenever $V \subset \mathbf{R}^m$ is open and aim to prove that f is continuous at each point in U: given $a \in U$ and $\epsilon > 0$, we know that the ball $B(f(a), \epsilon)$ is open in $\mathbf{R}^m$. By assumption then, so is $f^{-1}(B(f(a), \epsilon)$. As $a \in f^{-1}(B(f(a)), \epsilon)$, it follows that there exists $\delta > 0$ such that $B(a, \delta) \subset f^{-1}(B(f(a), \epsilon))$. In other words $\|x - a\| < \delta$ implies that $\|f(x) - f(a)\| < \epsilon$. This proves that f is continous at a. □

Corollary 1.3. *Let $U \subset \mathbf{R}^n$ be open and $f : U \to \mathbf{R}$ be a continuous, scalar-valued function. Then the sets $\{x \in U : f(x) < 0\}$, $\{x \in U : f(x) > 0\}$ and $\{x \in U : f(x) \neq 0\}$ are all open subsets of $\mathbf{R}^n$. If the domain U of f is all of $\mathbf{R}^m$, then $\{x \in \mathbf{R}^n : f(x) = 0\}$ is closed.*

Proof. By definition, we have $\{x \in U : f(x) < 0\} = f^{-1}((-\infty, 0))$ and $\{x \in U : f(x) > 0\} = f^{-1}((0, \infty))$, so the fact that these sets are open follows immediately from the assumption that f is continous and the previous theorem. Since finite unions of open sets are open, it follows that

$$\{x \in U : f(x) \neq 0\} = \{x \in U : f(x) < 0\} \cup \{x \in U : f(x) > 0\}.$$

is open. Finally, when $U = \mathbf{R}^n$, we have that

$$\{x \in \mathbf{R}^n : f(x) \neq 0\} = \mathbf{R}^n \setminus \{x \in \mathbf{R}^n : f(x) \neq 0\}$$

is the complement of an open set and therefore closed. □