Review Sheet for Exam 2

Standard disclaimer: The following represents a sincere effort to help you prepare for our exam. It is not guaranteed to be perfect. There might well be minor errors or (especially) omissions. These will not, however, absolve you of the responsibility to be fully prepared for the exam. If you suspect a problem with this review sheet, please bring it to my attention (bounty points are possible).

Review session: Sunday Nov 23, 5-6 PM in DBRT 125.

Time and place: the exam will take place Monday, Nov 24 in class. It will cover the material from chapters 3 and 4 of Shifrin.

Ground Rules: the exam is closed book and no calculators are allowed. All you'll need are sharp pencils and a good eraser. No pens please!

Format: Similar to exam 1.

Specific things to know

- **Most important statements and definitons:** Directional derivative; differentiable/derivative; chain rule, gradient, (unit) tangent vector/speed/length (of a differentiable curve); row/column/null space, rank (of a matrix); singular matrix; linear (in)dependent/spanning/basis (sequence of vectors); dimension (of a subspace); rank theorem.
- **Further terminology (beyond stuff introduced in chapters 1 and 2):** partial derivative; gradient; (unit) tangent vector/speed/length/curvature (of a differentiable curve); $m \times n$ (consistent/homogeneous) linear system; augmented matrix; elementary row operation; elementary matrix; (reduced) echelon form.
- **Some true/false skills:** Know the relationships among the derivative and partial/directional derivatives of a function; conditions guaranteeing differentiability, equality of mixed partial derivatives; ; understanding the relationship between gradient and level set of a scalar valued function; equivalent reformulations of singular/nonsingular for a square matrix; different ways of looking at and relationships among row/column/null spaces of a matrix; recognizing linearly (in)dependent and spanning sets of vectors.
- **Computational skills:** Computing directional/partial derivatives and (the standard matrix of) the derivative of a differentiable function; using the definition of differentiable to verify a formula for the derivative of a given function; using the chain rule; finding the tangent (hyper)plane of a level set or tangent line for a curve; finding the (unit) tangent vector/speed/length of a curve.

  Using elementary row operations to put a matrix in reduced/echelon form; finding the elementary matrix associated to an elementary row operation; giving the general solution of a linear system; finding the inverse of a matrix; finding bases for row/column/null spaces of a matrix; determining whether a given sequence of vectors is linearly independent or spans a given subspace; finding a basis for a subspace or the orthogonal complement of a subspace (given, say, a spanning set for the subspace). Determining the dimension of a subpace or its orthogonal complement.
- **Proof skills:** Go over old proof-oriented homework problems.

Unsolicited advice. The best things you can do to prepare are

- Make sure you thoroughly understand the solutions to all the homework problems. Even if you got full credit on a given problem, it's worth comparing your solution with mine and/or trying to write out the solution from scratch. If you haven't already done the warmup problems, you can use them to practice.
- Make yourself a glossary of terms and theorems and memorize them.
- Get a good night's sleep.

In linear algebra courses that I've taught, I have often included the blurb below concerning the importance of being able to look at linear systems from different points of view. It's suitable enough for this class that I'm including it in this review.

Editorial comment (many ways of saying the same thing): One thing that is crucial to understanding what's going on this class is to remember that we now have several equivalent ways to express a linear system of equations. To illustrate this, I let A be an $m \times n$ matrix with jth column $\mathbf{a}_j \in \mathbf{R}^m$ and ij entry $a_{ij} \in \mathbf{R}$, and we let $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$ be vectors with ith entries x_i, b_i respectively. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Then the following equations all say exactly the same thing.

m equations in n unknowns.:

$$\begin{array}{rcccl} a_{11}x_1 + a_{12}x_2 + & \dots & a_{1n}x_n & = & b_1. \\ a_{21}x_1 + a_{22}x_2 + & \dots & a_{2n}x_n & = & b_2. \\ \vdots & & & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & a_{mn}x_n & = & b_m. \end{array}$$

vector equation:

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \dots + x_n\mathbf{a_n} = \mathbf{b}.$$

matrix/vector equation:

$$A\mathbf{x} = \mathbf{b}.$$

linear transformation equation:

$$T(\mathbf{x}) = \mathbf{b}.$$

A question about any of these equations will likely reduce to some question about what happens when we row reduce the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & & & & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

to put it in echelon or reduced echelon form. For instance, the questions "is $\mathbf{b}$ in the range of T?" and "does $\mathbf{b}$ belong to span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?" are the same as determining whether the augmented matrix represents a consistent system: i.e. "if we reduce $[A|\mathbf{b}]$ to echelon form, do all pivots occur before the last column?" Many statements and computations in linear algebra that seem mysterious at first are really just straightforward consequences of thinking about a linear system from as many of the above points of view as possible.