Review Sheet for Exam 1

Standard disclaimer: The following represents a sincere effort to help you prepare for our exam. It is not guaranteed to be perfect. There might well be minor errors or (especially) omissions. These will not, however, absolve you of the responsibility to be fully prepared for the exam. If you suspect a problem with this review sheet, please bring it to my attention (bounty points are possible).

Time and place: the exam will take place Thursday, Oct 8 during our usual tutorial time. I will allow you to work until 4:55 (5 minutes before the start of the math4all talk that day). The exam will cover the material from sections 1.1-1.4 and 4.1-4.4 from Shifrin.

Ground Rules: the exam is closed book and no calculators are allowed. All you'll need are sharp pencils and a good eraser. No pens please!

Format: I typically ask three kinds of questions:

(1) A page where I ask you to correctly state several definitions and/or named theorems.
   When stating a definition or theorem, be careful to introduce your notation. For instance don't start out "*V is a subspace if...*" Instead say, "*A subset $V \subset \mathbf{R}^n$ is a subspace if...*"

(2) 'True/false' type questions where you're asked to identify the false statements and provide specific counterexamples to each.
   Concerning the nature of your counterexamples, note the word *specific* above. If I include the assertion *All real numbers have real square roots*, don't say '*negative numbers don't have real square roots*'. Say simply e.g. '-7'. And you don't have to justify that your counterexamples serve their purposes. So you don't have to say '*-7 doesn't have a real square root because it's negative and the square of a real number is always non-negative.*' Just '-7' will suffice.

(3) Some computational or short proof problems.
   For these, I will feel free (but not obliged) to recycle problems from the homework, particularly if I ask you to prove something.

Unsolicited advice. The best things you can do to prepare are

- Make sure you thoroughly understand the solutions to all the homework problems. Even if you got full credit on a given problem, it's worth comparing your solution with mine and/or trying to write out the solution from scratch.
- Memorize key statements and definition. The glossary I've linked from the schedule page includes nearly (in case I missed something) all of these, and you can use the versions I provided as the canonical ones.
- Get a good night's sleep.

Editorial comment (many ways of saying the same thing): One thing that is crucial to understanding what's going on this class is to remember that we now have several equivalent ways to express a linear system of equations. To illustrate this, I let A be an $m \times n$ matrix with jth column $\mathbf{a}_j \in \mathbf{R}^m$ and ij entry $a_{ij} \in \mathbf{R}$, and we let $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$ be vectors with ith entries x_i, b_i respectively. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Then the following equations all say exactly the same thing.

m **equations in** n **unknowns.:**

$$\begin{array}{rcccl} a_{11}x_1 + a_{12}x_2 + & \dots & a_{1n}x_n & = & b_1. \\ a_{21}x_1 + a_{22}x_2 + & \dots & a_{2n}x_n & = & b_2. \\ \vdots & & & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & a_{mn}x_n & = & b_m. \end{array}$$

vector equation:

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \dots + x_n\mathbf{a_n} = \mathbf{b}.$$

matrix/vector equation:

$$A\mathbf{x} = \mathbf{b}.$$

linear transformation equation:

$$T(\mathbf{x}) = \mathbf{b}.$$

A question about any of these equations will likely reduce to some question about what happens when we row reduce the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & & & & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

to put it in echelon or reduced echelon form. For instance, given a vector $\mathbf{b} \in \mathbf{R}^m$, the following statements are all different ways of saying the same thing.

- $\mathbf{b} \in \operatorname{col} A$.
- $\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$.
- $A\mathbf{x} = \mathbf{b}$ is consistent.
- there exists a vector $\mathbf{x} \in \mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{b}$.

For specific A and $\mathbf{b}$ we can determine whether the statements are true by applying row operations to put the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ in echelon form and looking at where the pivots appear. The statements are true if and only if no pivots show up in the last column (the one corresponding to $\mathbf{b}$).

Here is another instance of the same phenomenon: the following statements are all equivalent.

- The nullspace of A is trivial.
- The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
- For any $\mathbf{b} \in \mathbf{R}^m$, the linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution.
- The vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent.

Equivalence of these statements is readily apparent if you know the definitions of the terms and have become adept at going back and forth between various ways of presenting a linear system. And again, give a specific matrix A you can use Gaussian elimination (how?) to determine whether these statements are true or false.