1. The Extreme Value Theorem

Let us first review some pertinent definitions and facts about subsets of $\mathbf{R}$.

Definition 1.1. *A set $X \subset \mathbf{R}$ of real numbers is* bounded above *if there exists $M \in \mathbf{R}$ such that $x \leq M$ for any $x \in X$. We call $M \in \mathbf{R}$ an* upper bound *for X. Moreover,*

- *if $M \in X$, then we call M the* maximum *for X;*
- *if $M \leq M'$ for any other upper bound M' for X, then we call M the* least upper bound *(or* supremum*) of X.*

Upper bounds are never unique (if they exist at all), but least upper bounds and maxima are always unique. The maximum for X is also the least upper bound for X, but the reverse is not always true. Indeed, a bounded set (e.g. $(0,1)$) need not admit a maximum, but the 'completeness property' of $\mathbf{R}$ says that such a set $X \subset \mathbf{R}$ *always* has a least upper bound. We denote this quantity by $\sup X$. We then extend the completeness axiom to empty and unbounded subsets of $\mathbf{R}$ with the conventions that $\sup \emptyset = -\infty$ and that $\sup X = \infty$ if X is not bounded above.

We leave the reader to puzzle out the analogue of this discussion for lower bounds, greatest lower bounds (infima), and minima of subsets of $\mathbf{R}$.

Proposition 1.2. *Let $X_1, \dots, X_k \subset \mathbf{R}$ be a finite collection of sets and $X = X_1 \cup \dots \cup X_k$. Then*

$$\sup X = \max_{1 \leq j \leq k} \sup X_j.$$

Theorem 1.3 (Extreme Value Theorem)**.** *Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is continuous and that $K \subset \mathbf{R}^n$ is a compact subset of the domain of f. Then there exist $\mathbf{p}, \mathbf{q} \in K$ such that*

$$f(\mathbf{q}) \leq f(\mathbf{x}) \leq f(\mathbf{p})$$

for all $\mathbf{x} \in K$.

Another way to state the conclusion is to say that the image

$$f(K) := \{f(\mathbf{x}) : \mathbf{x} \in K\}$$

of K by f has a maximum and minimum.

The proof requires a bit of notation. A *cube* in $\mathbf{R}^n$ is a product $C = [a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbf{R}^n$ of closed intervals, all of the same length $b_j - a_j$. We call this length the *side* of C, and we call the point $(a_1, \dots, a_n)$ (a bit misleadingly) the *bottom vertex* of C. Similarly, $(b_1, \dots, b_n)$ is the *top vertex.* Note that cubes are compact. Note further that by halving each of the intervals $[a_j, b_j]$, one can partition any cube $C \subset \mathbf{R}^n$ into a union of 2^n smaller cubes, each with side-lengths equal to half the side-length of C.

Proof. I will prove only the existence of $\mathbf{p}$, since the argument for $\mathbf{q}$ is similar.

First I use the boundedness of K: there exists $R > 0$ such that $K \subset B(\mathbf{0}, R)$. So the cube $C_0 := [-R, R]^n$ contains K. Subdividing C into 2^n cubes with side length half that of C, we can apply Proposition 1.2 to choose one of these, call it C_1, such that $\sup f(C_1 \cap K) = \sup f(K)$. Repeating this proceedure, we obtain an infinite nested sequence

$$C_0 \supset C_1 \supset C_2 \supset \dots$$

of cubes C_j such that

- $\sup f(C_j \cap K) = \sup f(K)$; and
- the side length of C_j is $2^{1-j} R$.

Now I use the completeness property of $\mathbf{R}$.

Lemma 1.4. *The intersection $\bigcap_{j=0}^{\infty} C_j$ of all the cubes C_j contains exactly one point $\mathbf{p} \in \mathbf{R}^n$.*

Proof. Since $C_j \subset C_0$, we have that all coordinates of the bottom vertex of C_j are bounded above by R. Let $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{R}^n$ be the point whose ith coordinate is the least upper bound of all the ith coordinates of all the bottom corners of cubes C_j.

Let a_i, b_i denote the ith coordinates of the bottom/top vertices of some particular cube C_J in our sequence. Then certainly $a_i < b_i$. In fact, however, since the cubes are nested, b_i is larger than the ith coordinate of the bottom vertex of any other cube C_j, too. That is, b_i is an upper bound for the ith coordinates of all the bottom vertices. Since p_i is the *least* upper bound, I see that $a_i \leq p_i \leq b_i$. And since this is true for each coordinate i, I see that $\mathbf{p} \in C_J$. Finally, since $J \geq 0$ was arbitrary, I conclude that $\mathbf{p} \in \bigcap C_j$. That is, there exists some point $\mathbf{p}$ in the intersection.

To see that there is a unique such point, suppose that $\mathbf{q} \in \bigcap C_j$ is another. Then $\mathbf{q} \in C_j$ implies that $|q_i - p_i| \leq 2^{1-j}R$ for all $1 \leq i \leq n$. Letting $j \to \infty$, we see that $\mathbf{p} = \mathbf{q}$. That is, $\mathbf{p}$ is the *only point* that lies in all the cubes. □

Next I use the closed-ness of K.

Lemma 1.5. $\mathbf{p} \in K$.

Proof. If $\mathbf{p}$ is an interior point of K, then certainly $\mathbf{p} \in K$. If $\mathbf{p}$ is a boundary point of K, then $\mathbf{p} \in K$ because K is closed. So it suffices to show that $\mathbf{p}$ is not an exterior point of K.

Assume in order to get a contradiction that it is. Then there exists $\delta > 0$ such that $B(\mathbf{p}, \delta) \cap K = \emptyset$. On the other hand, if j is large enough (specifically, $2^{1-j}R < \delta/\sqrt{n}$), the fact that $\mathbf{p} \in C_j$ implies that $C_j \subset B(\mathbf{p}, \delta)$. Since $\sup f(C_j \cap K) = \sup f(K)$, we have in particular that $K \cap C_j \neq \emptyset$. So $K \cap B(\mathbf{p}, \delta) \neq \emptyset$. That is, we have our contradiction. □

Finally, I use continuity of f. For any $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{p}\| < \delta$ implies $|f(\mathbf{x}) - f(\mathbf{p})| < \epsilon$. In particular, $f(\mathbf{x}) < f(\mathbf{p}) + \epsilon$ for all $\mathbf{x} \in B(\mathbf{p}, \delta)$. Again taking j large enough, I have that $C_j \subset B(\mathbf{p}, \delta)$. Hence

$$f(\mathbf{p}) \leq \sup f(K) = \sup f(K \cap C_j) \leq f(\mathbf{p}) + \epsilon.$$

Letting $\epsilon \to 0$ shows that $\sup f(K) = f(\mathbf{p})$. □