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Math 20810: Honors Algebra I
Fall Semester 2008
Final Exam
Thursday, December 18

This examination contains 6 problems, not counting the extra credit at the end. All vector spaces on this exam are finite dimensional and real (i.e. vector spaces over $\mathbf{R}$). On computational problems please show enough work (or give enough explanation) to justify your answer.

Scores

Question	Possible	Actual
1	20	
2	30	
3-6	50	
Total	100	

GOOD LUCK

1. Respond to all of the following. (5 points each)

(a) Define linearly independent.

(b) State the *Cayley-Hamilton Theorem*.

(c) Define *eigenvector*.

(d) Define the *adjoint* of a linear transformation.

2. Following are twelve assertions. Many are false. Find five false ones and, on the next page, give specific counterexamples. Note that you need *not* justify your examples. It might encourage you to know that for every false statement there's a fairly simple counterexample involving vector spaces with small dimensions. (6 points each)

(a) If V is a finite dimensional inner product space and $H \subset V$ is a subspace, then there is a unique subspace $H' \subset V$ complementary to H.

(b) If two square matrices are similar, then they have the same determinant.

(c) A union of two subspaces is a subspace.

(d) Suppose V is an inner product space and H is a subspace. Then for every $\mathbf{v}$ in V, the norm (i.e. length) of $\mathbf{v}$ is at least as large as the length of the orthogonal projection of $\mathbf{v}$ onto H.

(e) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator, then there is a vector $\mathbf{v} \in \mathbf{R}^2$ such that $\{\mathbf{v}, T(\mathbf{v})\}$ is a basis for $\mathbf{R}^2$.

(f) If $A \in M_{n\times n}(\mathbf{R})$ and t is a scalar, then $\det(tA) = t\det(A)$.

(g) If A is an invertible matrix, then A and A^{-1} have the same characteristic polynomial.

(h) If $\dim V > \dim W$ and $T : V \to W$ is linear, then T is not surjective.

(i) Let V be a finite dimensional inner product space and $H_1, H_2 \subset V$ be orthogonal subspaces. Then $\dim H_1 + \dim H_2 \leq \dim V$.

(j) If $T : V \to V$ is an operator, then $\ker T$ is a T-invariant subspace.

(k) All linear operators are diagonalizable.

(l) If the characteristic polynomial of a linear operator $T : \mathbf{R}^2 \to \mathbf{R}^2$ is $x^2 - 2x$, then T is not invertible.

Your answers to Problem 2:

3. (15 points) Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Based on your computation of A^{-1}, what is $\det A$?

4. (15 points) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -5 & -4 \\ -2 & 2 & 1 \end{bmatrix}.$$

(a) Show that $(0, 2, -1)$ is an eigenvector of T and find the corresponding eigenvalue. Do this *before* finding the characteristic polynomial of T!

(b) Find the characteristic polynomial of T.

(c) Find a basis $\mathcal{B} \subset \mathbf{R}^3$ and a diagonal matrix D such that T has matrix D relative to $\mathcal{B}$. Or explain why this can't be done.

5. (10 points) Consider the linear system $\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$.

- Find the least squares solution.
- Using your answer, determine how far the right side of the equation is from the column space of the matrix on the left side.

6. (10 points) Let $V = P_2(\mathbf{R})$ and consider the inner product $\langle p(x), q(x) \rangle = p(1)q(1) + p(0)q(0) + p(-1)q(-1)$.

(a) Explain briefly why this is non-degenerate. That is, why is $\|p(x)\| = 0$ only if $p(x)$ is the zero polynomial?

(b) Find an orthogonal basis for $P_2(\mathbf{R})$ relative to this inner product.

7. (Extra credit—5 points) Suppose that V is a finite dimensional vector space and $T : V \to V$ is a linear operator such that $T^2 = \text{id}$. Show that there is a basis for V relative to which the matrix for T is diagonal with block form

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Note that the two blocks can have different sizes and either one might be empty.