Textbook problems:

GK, p249: 65.

Solution. False. For a counterexample choose any negative subharmonic function that is not harmonic. For instance, \( u(z) = \log |z| \) on \( D(0,1) \). Or (if you prefer something bounded) \( u(z) = |z|^2 - 1 \) on \( D(0,1) \).

Problem 1. (This problem expands on exercise 5 in Krantz; it also depends on your knowing a bit about differential forms—wedge product and Green's/Stokes' Theorem mostly). Let \( \eta = A \, dx + B \, dy \) be a 1-form on an open set \( \Omega \subset \mathbb{C} \). We define \( \ast \eta \) to be the 1-form \(-B \, dx + A \, dy\).

(a) Let \( u : \Omega \to \mathbb{R} \) be a \( C^2 \) function. One calls \( \ast du \) (sometimes written \( 'd' \)) the conjugate differential of \( u \). Show that \( d \ast du = \Delta u \, dx \wedge dy \). Hence \( \ast du \) is closed if and only if \( f \) is harmonic.

Solution. I compute

\[
d \ast du = d(-u_y \, dx + u_x \, dy) = (-u_{xy} \, dx + u_{yx} \, dy) \wedge dx + (u_{xx} \, dx + u_{yy} \, dy) \wedge dy = -u_{yy} \, dy \wedge dx + u_{xx} \, dx \wedge dy = \Delta u \, dx \wedge dy.
\]

The third equality holds because \( dx \wedge dx = dy \wedge dy = 0 \) and the fourth because \( dy \wedge dx = -dx \wedge dy \).

(b) Show that if \( v \) is a second \( C^2 \) function, then \( du \wedge \ast dv = dv \wedge \ast du \).

Solution.

\[
du \wedge \ast dv = (u_x \, dx + u_y \, dy) \wedge (-v_y \, dx + v_x \, dy) = u_x v_x \, dx \wedge dy - u_y v_y \, dy \wedge dx = (u_x v_x + u_y v_y) \, dx \wedge dy.
\]

Since the last expression is symmetric in \( u \) and \( v \), it follows that it is also equal to \( dv \wedge \ast du \).

(c) Show that if \( u \) is harmonic and \( \Omega \) is simply connected that \( \ast du = dv \) where \( v : \Omega \to \mathbb{R} \) is any harmonic conjugate for \( u \). Deduce from this a simple expression for \( \ast d \log |z| \).

Solution. Since \( \Omega \) is simply connected, we know there exists a harmonic conjugate \( v \) for \( u \). Since \( u + iv \) is holomorphic, it follows from the Cauchy-Riemann equations that

\[
dv = v_x \, dx + v_y \, dy = -u_y \, dx + u_x \, dy = \ast du.
\]

Now \( v \) is a harmonic conjugate for \( \log |z| \) in some domain if and only if \( v(z) = \theta + C \) where \( \theta \) is the argument of \( z \) and \( C \) is a complex constant. Hence \( d \log |z| = dv = d\theta \).

(d) Let \( \gamma \) be a \( C^1 \) curve in \( \Omega \). Show that in more classical language, one has \( \int_{\gamma} \ast du = \int_{\gamma} \frac{du}{dn} |d\gamma| \) where \( n \) is the righthand normal vector to \( \gamma \).

Solution. The unit tangent vector to \( \gamma \) is given by \( (\gamma_x', \gamma_y')/|\gamma'| \). The right hand normal \( n \) to \( \gamma \) is therefore obtained by rotating this vector \( \pi/2 \).
radialy clockwise. Thus \( n = (\gamma_2', -\gamma_1')/|\gamma'| \) and \( \frac{\partial u}{\partial n} = \nabla u \cdot n = \frac{-u_x \gamma_2' + u_y \gamma_1'}{|\gamma'|} \).

From this, I infer
\[
\int_{\gamma}^* du = \int_{\gamma}^* -u_y \, dx + u_x \, dy = \int (\cdot u_y \gamma_1' + u_x \gamma_2') \, dt = \int \frac{\partial u}{\partial n} |\gamma'(t)| \, dt = \int \frac{\partial u}{\partial n} |d\gamma|.
\]

(e) Show that if \( \Omega' \subset \Omega \) is a bounded open subset with smooth boundary \( b\Omega' \subset \Omega \), and if \( u, v : \Omega \to \mathbb{R} \) are \( C^2 \) functions, then
\[
\int_{\Omega'} u * dv - v * du = \int_{\Omega'} (u \Delta v - v \Delta u) \, dx \, dy.
\]

**Solution.** Green’s/Stokes’ Theorem gives me that
\[
\int_{\Omega'} u * dv - v * du = \int_{\Omega'} d(u * dv - v * du) = \int_{\Omega'} d(u \Delta v - v \Delta u) = \int_{\Omega'} (u \Delta v - v \Delta u) \, dx \, dy.
\]

From the first two parts of this problem, I see that the last integral is the same as \( \int_{\Omega'} (u \Delta v - v \Delta u) \, dx \, dy \).

Problem to be continued on next assignment...

**Problem 2.** Let \( \Omega \subset \mathbb{R}^2 = \mathbb{C} \) be open. As with functions on the real line, one calls a function \( \psi : \Omega \to \mathbb{R} \) of two real variables convex if \( \psi(\frac{z + w}{2}) \leq \frac{1}{2}(\psi(z) + \psi(w)) \) for all \( z, w \in \mathbb{R}^2 \). One can show (and you can take for granted) that convex functions are automatically continuous. Given this, show that a convex function is subharmonic. Show by example that a subharmonic function need not be convex.

**Solution.** If \( \psi \) is convex and \( \overline{D(P, R)} \subset \mathbb{C} \), then for any \( \theta \in \mathbb{R} \), we have
\[
\psi(P) \leq \frac{1}{2}(\psi(P + Re^{i\theta}) + \psi(P - Re^{i\theta})) = \frac{1}{2}(\psi(P + Re^{i\theta}) + \psi(P + Re^{i(\pi + \theta)})).
\]

Hence
\[
\frac{1}{2\pi} \int_0^{2\pi} \psi(P + Re^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{\pi} (\psi(P + Re^{i\theta}) + \psi(P + Re^{i(\pi + \theta)})) \, d\theta \geq \frac{1}{\pi} \int_0^{\pi} \psi(P) \, d\theta = \psi(P).
\]

So \( \psi \) satisfies the subaveraging property and is therefore subharmonic. \( \Box \)

To see that a subharmonic function need not be convex, consider \( \log |z| \) which is subharmonic on \( \mathbb{C} \). However, the restriction of this function to the positive real axis is \( \log x \), which is actually strictly concave down everywhere. So \( \log(\frac{z + w}{2}) > \frac{\log x + \log y}{2} \) for all \( x, y > 0 \).

**Problem 3.** Let \( \Omega = (a, b) \times \mathbb{R} \subset \mathbb{C} \) be an open vertical strip and \( u : \Omega \to [-\infty, \infty) \) be given by \( u(x, y) = \psi(x) \) (i.e. \( u \) is really a function of only one variable). Show that \( u \) is subharmonic if and only if \( \psi \) is convex. (Hint: show that if \( u \) is not convex, then after subtracting the right harmonic function from \( u \), the difference violates the maximum principle).
Solution. If \( \psi \) is convex, then so is \( u \). Hence, from the previous problem, it follows that \( u \) is subharmonic.

Suppose, on the other hand, that \( \psi \) is not convex. Then there exist real numbers \( a < b < c \) such that \( \psi(b) > \ell(b) \), where \( \ell : \mathbb{R} \to \mathbb{R} \) is the affine function agreeing with \( \psi \) at \( a \) and \( c \). Since \( \psi - \ell \) is continuous, we may choose \( x_0 \in (a, c) \) such that \( \psi(x_0) - \ell(x_0) > 0 \) is maximal. Note that \( h(x, y) = \ell(x) \) is a harmonic function on \( \mathbb{C} \). So \( u \) is subharmonic if and only if \( u - h \) is. On the other hand, \( u - h \) is a non-constant function (it’s equal to zero at any point \( a + iy \) but positive at any point \( x_0 + iy \)) with an interior local maximum at any point of the form \( x_0 + iy \). That is, \( u - h \) does not satisfy the maximum principle and is therefore not subharmonic. \( \square \)

**Problem 4.** (‘Radial’ subharmonic functions) Let \( \Omega = \{ R_1 < |z| < R_2 \} \) be an annulus and \( u : \Omega \to [-\infty, \infty) \) be given by \( u(re^{i\theta}) = f(r) \) for all points \( re^{i\theta} \in \Omega \) (i.e. \( u \) is a ‘radial’ function, with \( u(z) \) depending only on the distance of \( z \) from 0). Show that \( u \) is subharmonic if and only if \( f \) is a convex function of \( \log r \) (i.e. \( f(e^z) \) is a convex function of \( x = \log r \)). (Hint: reduce to the previous problem.) From this, give an explicit description (i.e. a formula) for any radial harmonic function on \( \Omega \).

**Solution.** Note that the function \( f(z) = e^z \) maps the open vertical strip \( \Omega' = (\log R_1, \log R_2) \times \mathbb{R} \) onto \( \Omega \). While \( f \) is not \((1, 1)\), we see that \( f'(z) = e^z \) never vanishes. Hence \( f \) is at least locally invertible. Since subharmonicity is a local property, it follows that \( u \) is subharmonic on \( \Omega \) if and only if \( u \circ f \) is subharmonic on \( \Omega' \). Also, since \( u(re^{i\theta}) = f(r) \) is a radial function, we have that \( u \circ f(z) = u(e^z e^{i\theta}) = f(e^z) \) is a function of \( x = \Re z \) only. Hence by the previous problem, \( u \) is subharmonic if and only if \( f(e^z) \) is a convex function of \( x \).

A radial function \( h(re^{i\theta}) = f(r) \) is harmonic if and only if \( h \) and \( -h \) are both subharmonic. By the first part of the problem, this is true if and only if \( f(e^z) \) and \( -f(e^z) \) are both convex, which is to say that \( f \) is an affine function of \( x = \log r \). So \( h \) is harmonic if and only if there exist real constants \( \alpha, \beta \) such that \( h(re^{i\theta}) = \alpha \log r + \beta \) for all \( re^{i\theta} \in \Omega \).

**Problem 5.** Show that if \( f : \Omega' \to \Omega \) is holomorphic and \( u : \Omega \to [-\infty, \infty) \) is subharmonic, then \( u \circ f \) is subharmonic. (Hint: since we showed in class that this is true when \( f \) is injective, and since being subharmonic is a local property, it more or less suffices to establish the subaveraging property about points \( P \) at which \( f'(P) = 0 \). For the latter, it might help to use problem 1 on homework 9 from last semester.)

**Solution.** If \( f \) is constant, the assertion is clear, so suppose that \( f \) is not constant.

As noted in class, it suffices to show for each \( P \in \Omega' \) that \( u \circ f \) is subharmonic on a neighborhood of \( P \), now if \( f'(P) \neq 0 \), it follows that there is a neighborhood \( V \ni P \) such that \( f : V \to f(V) \subset \Omega' \) is invertible. Hence, as we showed in class \( u \circ f \) is subharmonic on \( V \). If \( f'(P) = 0 \), on the other hand, then we showed in homework last semester that there exists a neighborhood \( V \ni P, R > 0, \) and
$k \geq 2$ such that $f = g^k$ where $g$ maps $V$ conformally onto $D(0, R)$. Hence $u \circ f$ is subharmonic on $V$ if and only if $u(w^k)$ is subharmonic on $D(0, R)$. To see that $u(w^k)$ is subharmonic, fix any $r > 0$ smaller than $R$. Then
\[
\int_0^{2\pi} (u(re^{i\theta})^k) d\theta = \int_0^{2\pi} u(re^{ik\theta}) d\theta = \int_0^{2\pi} \frac{u(e^{i\phi})}{k} d\phi = \int_0^{2\pi} u(e^{i\phi}) d\phi \geq u(0).
\]
That is, $u(w^k)$ has the subaveraging property for small enough disks centered at 0. It follows that $u(w^k)$ is subharmonic on $D(0, R)$ and therefore that $u \circ f$ is subharmonic everywhere on $\Omega'$. $\square$