Assignment 3
Math 60380, Winter '10
Due Friday, March 5

On this homework it will suffice if you only do one of the two problems 3 and 4. Which one you do is up to you.

Problem 1. Let $\gamma : [0,1] \to \mathbf{C}$ be a path with $\gamma(0) = z_0$. Suppose that $f : D(z_0, R) \to \mathbf{C}$ is holomorphic, and let $[f]_{z_0} \in \mathcal{O}_{z_0}$ be the germ of f at z_0. Verify that the following are equivalent:

- f can be analytically continued along γ;
- there exists a lift $\tilde{\gamma}$ of γ to $|\mathcal{O}_{\mathbf{C}}|$ satisfying $\tilde{\gamma}(z_0) = [f]_{z_0}$.

Mostly, this amounts to unraveling and comparing definitions.

Solution. That f can be analytically continued along γ means there exists a family of disks $\{D_t = D(\gamma(t), r(t)) : t \in [0,1]\}$ and holomorphic functions $f_t \in \mathcal{O}(D_t)$ such that

- $D_0 = D(z_0, R)$ and $f_0 = f$;
- For each $t \in [0,1]$ there exists $\epsilon > 0$ such that $|t' - t| < \epsilon$ implies $\gamma(t') \in D_t$ and $f_{t'} \equiv f_t$ on $D_{t'} \cap D_t$.

From this data, I define $\tilde{\gamma} : [0,1] \to |\mathcal{O}_{\mathbf{C}}|$ by $\tilde{\gamma}(t) = [f_t]_{\gamma(t)}$. Then $\tilde{\gamma}(0) = [f]_{z_0}$ by definition. Likewise, if $\pi : |\mathcal{O}_{\mathbf{C}}| \to \mathbf{C}$ is the natural projection, then $\pi \circ \tilde{\gamma} = \gamma$. To verify that $\tilde{\gamma}$ is continuous, fix $t \in [0,1]$, let $\epsilon > 0$ be as a above, and fix a neighborhood $U \subset |\mathcal{O}_{\mathbf{C}}|$ of $\tilde{\gamma}(t)$. By definition of the topology on $|\mathcal{O}_{\mathbf{C}}|$, we may shrink U if necessary and assume that $U = \{[f_t]_z : z \in D(\gamma(t), \epsilon'\}$ for some $\epsilon' < \epsilon$. If (t_j) is a sequence converging to t, then there exists $J \in \mathbf{N}$ such that $j > J$ implies that $|\gamma(t_j) - \gamma(t)| < \epsilon'$. In particular, $\gamma(t_j) \in D(\gamma(t), \epsilon')$ and $f_t \equiv f_{t_j}$ near $z := \gamma(t_j)$. Thus $\tilde{\gamma}(t_j) = [f_{t_j}]_z = [f_t]_z \in U$. Hence $\tilde{\gamma}$ is continuous and therefore a lift of γ.

Starting over, we begin with a lift $\tilde{\gamma} : [0,1] \to |\mathcal{O}_{\mathbf{C}}|$ satisfying $\tilde{\gamma}(0) = [f]_{z_0}$. For each $t \in [0,1]$, the germ $\tilde{\gamma}(t) \in \mathcal{O}_{\gamma(t)}$ is given by $[f_t]_{\gamma(t)}$ where f_t is holomorphic on a neighborhood U of $\gamma(t)$. Shrinking U, we may assume $U = D_t$ is a disk. Let $\tilde{U} = \{[f_t]_z : z \in D_t\}$ be the corresponding open set in $|\mathcal{O}_{\mathbf{C}}|$. Then by continuity of $\tilde{\gamma}$, we have $\epsilon > 0$ such that $\tilde{\gamma}(t') \in \tilde{U}$ when $|t' - t| < \epsilon$. That is, $f_t \equiv f_{t'}$ on some open subset of $\gamma(t')$. This means first of all that $\gamma(t') \in D_t$, and secondly that $f_t \equiv f_t'$ on $D_{t'} \cap D_t$ (because the intersection is connected). Since $f_0 = \tilde{\gamma}(0) = [f]_{z_0}$ by definition, it follows that the family $\{f_t \in \mathcal{O}(D_t)\}$ is an analytic continuation of f along γ. $\square$

Problem 2. Let $\Omega = \{z \in \mathbf{H} : 0 < \operatorname{Re} z < 1 \text{ and } |z - 1/2| > 1/2\}$ be the maximal hyperbolic triangle we used to construct the function λ; let R denote reflection about the bottom side of T, and let $\tilde{R}$ denote reflection about the image $R(\gamma)$ where γ is the left side of Ω. Verify that $\tilde{R} \circ R = \frac{z}{2z+1}$.

Solution. Let me first work out the formula for reflection about an arbitrary circle $S := bD(P, R)$. The affine transformation $T(z) = \frac{z-P}{R}$ maps S to

$bD(0,1)$. Hence reflection about S is given by $R_S = T^{-1} \circ R_{bD(0,1)} \circ T$, where $R_{bD(0,1)}(z) = \frac{1}{\bar{z}}$ is reflection about $bD(0,1)$. Thus

$$R_S(z) = T^{-1}(R/(\overline{z-P})) = \frac{R^2}{\overline{z-P}} + P.$$

Now I specialize this formula to the reflections R and $\tilde{R}$ of interest in this problem. It follows from the previous formula that

$$R(z) = \frac{1/4}{\bar{z} - 1/2} + 1/2 = \frac{\bar{z}}{2\bar{z} - 1}.$$

In particular, applying R to the endpoints of γ, we find $R(0) = 0$ and $R(\infty) = \frac{1}{2}$. Hence $\tilde{R}$ is reflection about $bD(1/4, 1/4)$, which is given by

$$\tilde{R}(z) = \frac{1/16}{\bar{z} - 1/4} + 1/4 = \frac{\bar{z}}{4\bar{z} - 1}.$$

It follows that

$$\tilde{R} \circ R(z) = \frac{z}{2z+1},$$

as desired.

Problem 3. Let $\Omega \subset \mathbf{C}$ be open and $c_0, \ldots, c_{n-1} : \Omega \to \mathbf{C}$ be continuous functions. Consider the 'continuous' family of polynomials $P(z, w) = z^n + c_{n-1}(w) z^{n-1} + \cdots + c_0(w)$. Suppose for some $w_0 \in \Omega$ that the polynomial $P(z, w_0)$ has n distinct roots $z_0, \ldots, z_{n-1}$.

(a) Fix $\epsilon > 0$ such that $\epsilon < \frac{1}{2} \min_{i \neq j} |z_i - z_j|$. Show that there exists $\delta > 0$ such that for all $w \in D(w_0, \delta)$, the polynomial $P(z, w)$ has (distinct) roots $z_1(w), \ldots, z_n(w)$ such that $|z_j(w) - z_j| < \epsilon$. (Hint: use Rouché's Theorem or the argument principle).

Solution. With this choice of ϵ, the disks $D(z_j, \epsilon)$ have disjoint closures. In particular z_j is the only root of $P(z, 0)$ in $D(z_j, \epsilon)$ and there exists $m > 0$ such that $|P(z, 0)| > m$ for all $z \in bD(z_j, \epsilon)$ and all $1 \leq j \leq n$. Since the functions c_j are continuous, the function $P(z, w)$ is continuous as a function of both variables z and w. So we further have $\delta > 0$ such that $|P(z, 0) - P(z, w)| < m/2$ for all $|w| < \delta$ and $z \in bD(z_j, \epsilon)$. This means that when $|w| < \delta$, the hypothesis of Rouché's Theorem is satisfied by $P(z, 0)$ and $P(z, w)$:

$$|P(z, 0) - P(z, w)| < m/2 < |P(z, 0)| \leq |P(z, 0)| + |P(z, w)|$$

for all $z \in bD(z_j, \epsilon)$. Hence $P(z, w)$ has the same number of zeroes in $D(z_j, \epsilon)$ as $P(z, 0)$ does—that is, exactly one. Call this zero $z_j(w)$. Since the disks $D(z_j, \epsilon)$ are mutually disjoint, the zeroes $z_1(w), \ldots, z_n(w)$ of $P(z, w)$ are all distinct. □

(b) Show that if $\gamma \subset \mathbf{C}$ is a simple closed curve enclosing exactly one zero (counting multiplicity) z_0 of a holomorphic function $f : \mathbf{C} \to \mathbf{C}$, then $z_0 = \frac{1}{2\pi} \int_\gamma \frac{zf'(z)}{f(z)}\, dz$.

Solution. The value of the given integral is the sum of the residues of the function $h := zf'/f$ over all points inside γ. Now h has poles only where f vanishes, so by hypothesis, h has exactly one pole (at the point z_0) inside γ. Since $f(z)$ has a simple zero at z_0, I can write

$f(z) = (z - z_0)g(z)$ where g is a non-vanishing holomorphic function. From this, I compute

$$h(z) = \frac{z}{(z - z_0)} + \frac{zg'}{g},$$

so that h has (at worst) a simple pole at z_0. I can therefore compute the residue of h at z_0 by

$$\operatorname{res}_h(z_0) = \lim_{z\to z_0} (z - z_0)h(z) = \lim_{z\to z_0} z + z(z - z_0)g'(z)/g(z) = z_0.$$

I conclude that

$$\frac{1}{2\pi} \int_\gamma \frac{zf'(z)}{f(z)}\, dz = \frac{1}{2\pi} \int_\gamma h(z)\, dz = \operatorname{res}_h(z_0) = z_0.$$

(c) Now assume that the functions c_j are holomorphic on Ω. Show that the root functions $z_j(w)$ are also holomorphic on $D(w_0, \delta)$.

Solution. From the previous two parts of the problem, I have that

$$z_j(w) = \frac{1}{2\pi i} \int_{bD(z_j,\epsilon)} z\frac{P'(z,w)}{P(z,w)}\, dw.$$

In particular, $z_j(w)$ depends smoothly on w since the integrand varies smoothly with w at all points $z \in bD(z_j, \epsilon)$. To check that $z_j(w)$ is holomorphic, I differentiate

$$\frac{\partial}{\partial \bar{w}} z_j(w) = \frac{1}{2\pi i} \int_{bD(z_j,\epsilon)} \frac{\partial}{\partial \bar{w}} \left(z\frac{P'(z,w)}{P(z,w)} \right) dw = 0$$

since the integrand is a holomorphic function of w for each $z \in bD(z_j, \epsilon)$.
$\square$

Problem 4. Let $\Omega \subset \mathbf{C}$ be a domain and $A : \Omega \to M_{n\times n}(\mathbf{C})$ be a holomorphically varying $n \times n$ matrix and $z_0 \in \Omega$ be a given point. The existence and uniqueness theorem for linear ordinary differential equations says that there exists $\delta > 0$ such that for any vector $\mathbf{w}_0 \in \mathbf{C}^n$ there exists a unique holomorphic mapping $F : D(z_0, \delta) \to \mathbf{C}^n$ satisfying $F(z_0) = \mathbf{w}_0$ and $F'(z) = A(z)F(z)$ for all $z \in D(z_0, \delta)$. Here you are asked to prove this theorem as follows:

(a) Show for any matrix $n \times n$ matrix M and any vector $\mathbf{x} \in \mathbf{C}^n$, one has $\|M\mathbf{x}\| \leq n \|M\| \|\mathbf{x}\|$, where $\|\mathbf{x}\|$ and $\|M\|$ both denote the magnitude of the largest entry.

Solution. Let $M_{ij}, x_j \in \mathbf{C}$ denote the ij entry of M and jth entry of $\mathbf{x}$ respectively. Then

$$\|M\mathbf{x}\| = \max_i |\sum_j M_{ij}x_j| \leq \max_i \sum_j |M_{ij}x_j| \leq \sum_j \max_i |M_{ij}x_j| \leq \sum_{j=1}^n \|M\| \|\mathbf{x}\| = n \|M\| \|\mathbf{x}\|.$$

(b) Let $\mathcal{F}$ be the set of bounded holomorphic mappings $F : D(z_0, \delta) \to \mathbf{C}^n$ satisfying $F(z_0) = \mathbf{w}_0$. 'Bounded' here means that $\|F\| := \max\{\|F(z)\| : z \in D(z_0, \delta)\} < \infty$. Show that if we set $d(F, G) = \|F - G\|$, then $\mathcal{F}$ is a complete metric space.

Solution. Suppose that the sequence $(F_n) \subset \mathcal{F}$ is Cauchy with respect to the given metric. Let $F_{nj} : D(z_0, \delta) \to \mathbf{C}$ denote the jth entry in F_n. By definition, we have $|F_{nj}(z) - F_{mj}(z)| \le \|F_n(z) - F_m(z)\| \le \|F - G\|$, so it follows that $(F_{nj}(z))_{n \ge 0} \subset \mathbf{C}$ is a Cauchy sequence for each $1 \le j \le n$ and each $z \in D(z_0, \delta)$. Since $\mathbf{C}$ is a complete metric space, it follows that this last sequence converges. We call the limit $F_j(z)$. I claim that $F_{nj} \to F_j$ uniformly on $D(z_0, \delta)$. To see that this is so, fix $\epsilon > 0$ and let $N \in \mathbf{N}$ be large enough that $m, n \ge N$ and $z \in D(z_0, \delta)$ implies $|F_n(z) - F_m(z)| < \epsilon/2$. Then

$$|F_{nj}(z) - F_j(z)| \le |F_{nj}(z) - F_{mj}(z)| + |F_{mj}(z) - F_j(z)| < \epsilon/2 + |F_{mj}(z) - F_j(z)|$$

for all $m \ge N$. Letting m tend to infinity, we see that $|F_n(z) - F(z)| \le \epsilon/2 < \epsilon$. for all $n \ge N$, establishing my claim. It follows that F_j is holomorphic on $D(z_0, \delta)$. Letting $F = (F_1, \dots, F_n) : D(z_0, \delta) \to \mathbf{C}$ denote the mapping obtained by collecting these component functions and increasing N above so that $|F_j(z) - F(z)| < \epsilon$ for *all* $1 \le j \le n$ and all $z \in D(z_0, \delta)$, we conclude that

$$d(F_n, F) = \max_{z,j} |F_{nj} z - F_j(z)| < \epsilon.$$

This proves that $F_n \to F$; i.e. Cauchy sequences in $\mathcal{F}$ converge. □

(c) Given $F \in \mathcal{F}$, let $H(z) = \mathbf{w}_0 + \int_{z_0}^{z} A(z)F(z)\, dz$; that is, H is the unique holomoprhic antiderivative of AF satisfying $H(z_0) = \mathbf{w}_0$. Show that $H \in \mathcal{F}$ and that the function $T : \mathcal{F} \to \mathcal{F}$, given by $T(F) = H$ is a contraction mapping if $\delta > 0$ is set small enough.

Solution. First pick δ_0 so that $\overline{D(z_0, \delta_0)} \subset \Omega$. Let $M = \max_{z \in \overline{D(z_0,\delta_0)}} \|A(z)\|$. Using the straight line path from z_0 to z in the integral above, we find for any $F, G \in \mathcal{F}$ that

$$\|T(F) - T(G)\| = \left\| \int_{z_0}^{z} (AF(z) - AG(z))\, dz \right\| \le |z - z_0| \, \|A(F - G)\| < \delta M n \, \|F - G\| \,.$$

So if we set $\delta < (Mn)^{-1}$, we obtain that $d(T(F), T(G)) \le C d(F, G)$ where $C = \delta M n < 1$. So T is a contraction mapping. □

(d) Conclude.

Solution. With δ set as in the previous item, the contraction mapping theorem tells us that there is a unique $F \in \mathcal{F}$ satisfying $F = T(F)$; i.e. F is the unique holomorphic antiderivative of AF satisfying $F(z_0) = \mathbf{w}_0$. Rephrasing one more time, we conclude that $F : D(z_0, \delta) \to \mathbf{C}$ is the unique holomorphic mapping satisfying $F(z_0) = \mathbf{w}_0$ and $F'(z) = A(z)F(z)$ for all z. □

Problem 5. Let $f : D^*(0, 1) \to \mathbf{C}$ be a holomorphic function with an isolated singularity at 0. Suppose that the image $f(D^*(0, 1))$ omits two distinct points in $\mathbf{C}$. Complete the following outline to show that f has either a removable singularity or a pole at 0.

(a) For any $\epsilon \in (0,1]$, let $f_\epsilon : D^*(0,1) \to \mathbf{C}$ be given by $f_\epsilon(z) = f(\epsilon z)$. Show that we may choose a strictly decreasing sequence $\epsilon_j \to 0$ such that $f_j := f_{\epsilon_j}$ converges normally on $D^*(0,1)$. In particular, f_j converges uniformly on the circle $bD(0,1/2)$.

> *Solution.* By hypothesis, $f_\epsilon(D^*(0,1)) = f(D^*(0,\epsilon)) \subset f(D^*(0,1)) \subset \mathbf{C} \setminus \{z_0, z_1\}$ for two distinct points $z_0, z_1 \in \mathbf{C}$. Hence by Montel's Theorem $\mathcal{F} = \{f_\epsilon : \epsilon \in (0,1]\}$, is a normal family. So if $(\epsilon_j) \subset (0,1)$ is *any* sequence, we may pass to a subsequence and assume that (f_{ϵ_j}) converges normally on $D^*(0,1)$. In particular, starting with e.g. $\epsilon_j = 1/j$ and refining gives us a sequence (ϵ_j) that decreases to zero and for which $f_j = f_{\epsilon_j}$ converges normally to some function g on $D^*(0,1)$ that is either holomorphic or everywhere infinite. Since $bD(0,1/2)$ is a compact subset of $D^*(0,1)$, the convergence is uniform on $bD(0,1/2)$.

(b) Suppose first that f_j does not converge normally to ∞; i.e. $f_j \to g$ for some holomorphic function $g : D^*(0,1) \to \mathbf{C}$. Let M be the maximum of $|g|$ on $bD(0,1/2)$. Let $A_j = \{\epsilon_j/2 < |z| < \epsilon_{j+1}/2\}$, and show that for j large enough $|f| < M+1$ on $\overline{A_j}$.

> *Solution.* Since g is finite, and the convergence of f_j to g is uniform on $bD(0,1/2)$, there is $J \in \mathbf{N}$ such that $j \geq J$ implies that $|f_j - g| < 1$ on $bD(0,1/2)$. In particular $|f_j| < M+1$ on $bD(0,1/2)$ for all $j \geq J$. Since $f_j(z) = f(\epsilon_j z)$, it follows that $|f| < M+1$ on $bD(0,\epsilon_j/2)$ for all $j \geq J$. Since $j+1 \geq J$ if $j \geq J$, it follows that $|f_j| < M+1$ on bA_j. By the maximum principle, we see that $|f_j| < M+1$ on all of $\overline{A_j}$.

(c) Conclude that in this case f has a removable singularity at 0.

> *Solution.* We have $|f_j| < M+1$ on $\bigcup_{j\geq J} \overline{A_j} \supset D^*(0,\epsilon_J/2)$. That is f is bounded on a punctured neighborhood of zero and therefore has a removable singularity there. □

(d) Supposing instead that $f_j \to \infty$ normally. Use similar reasoning (presented in abbrieviated form) to show that f has a pole at 0.

> *Solution.* In this case, we have for any $M > 0$ that there exists $J \in \mathbf{N}$ such that $j \geq J$ implies $|f_j| \geq M$ on $bD(0,1/2)$. From this it follows as above that $|f| \geq M$ on $\overline{A_j}$ for all $j \geq J$; i.e. $|f| \geq M$ on $D^*(0,\epsilon_J/2)$. Since this is true for every M (for some $\epsilon_J > 0$ which depends on M), I conclude that $\lim_{z\to 0} f(z) = \infty$. So f has a pole at 0. □