1. Berman-Boucksom formula for the derivative of the modified energy functional

Let X be a compact complex manifold with Kähler form ω. Let $E : \mathrm{PSH}(\omega) \to [-\infty, \infty)$ be the Aubin-Mabuchi energy functional discussed in class, and $\mathcal{E}^1(\omega)$ denote the subset of $\mathrm{PSH}(\omega)$ on which E is finite. From concavity and upper-semicontinuity of E, it follows that $\mathcal{E}^1$ is convex and closed (in the L^1 topology).

Given $\varphi \in \mathcal{E}^1(\omega)$ and $\psi \in C(X)$, we set $\varphi_t = \varphi + t\psi$ and $e(t) = E(\varphi_t)$. We showed in class that $e'(0) = \int \psi\, \omega_\varphi^n$, which is very useful for understanding critical points of E. However, a big drawback in the definition of e is that it concerns values of the energy functional at functions φ_t which are not necessarily ω-psh. To fix this problem, we replace E with the modified energy functional $E \circ P$ where, for any usc function ϕ we define $P(\phi)$ to be an upper envelope

$$P(\phi) := \sup\{u \in \mathrm{PSH}(\omega) : u \leq \phi\}.$$

If the set on the right side is non-empty, then $P(\phi) \in \mathrm{PSH}(\omega)$ (note that the upper envelope $P(\phi)$ is automatically usc since ϕ is). If not, we set $P(\phi) \equiv -\infty$. Several properties of P are immediate consequences of the definition:

- $P(\phi) \leq \phi$ with equality everywhere if and only if $\varphi \in \mathrm{PSH}(\omega)$;
- $P(\phi_1 + \phi_2) \geq P(\phi_1) + P(\phi_2)$;
- $\phi_1 \leq \phi_2$ implies $P(\phi_1) \leq P(\phi_2)$;
- $P(\varphi_t) \geq \varphi + t\,\|\psi\|_\infty$.

A less obvious property that will be important to us is

Theorem 1.1. *For any continuous function ϕ on X, we have $P(\phi) = \phi$ a.e. with respect to $\omega_{P(\phi)}$.*

Proof. Suppose $P(\phi) < \phi$ at z_0. We work in local coordinates about z_0. Since $P(\phi) - \phi$ is usc, there exist $\epsilon, r > 0$ such that $P(\phi) < \phi(z_0) - \epsilon < \phi$ on $\overline{B_r(z_0)}$. Let h be a local potential for ω on a neighborhood of $\overline{B_r(z_0)}$. Let $u : \overline{B_r(z_0)} \to \mathbf{R}$ be the maximal psh function such that $u \equiv P(\phi) + h$ on $bB_r(z_0)$. Then

$$\tilde{\phi} := \begin{cases} u - h & \text{on} \quad B_r(z_0) \\ P(\phi) & \text{on} \quad X - B_r(z_0). \end{cases}$$

is an ω-psh function satisfying $P(\phi) \leq \tilde{\phi} \leq \phi$. Therefore $P(\phi) = \tilde{\phi}$ and $\omega^n_{P(\phi)} = (dd^c u)^n \equiv 0$ on $B_r(z_0)$. It follows that $z_0 \notin \operatorname{supp} \omega^n_{P(\phi)}$. $\square$

Let $\tilde{e}(t) = E \circ P(\varphi_t)$. Berman and Boucksom showed that $\tilde{e}$ has the same derivative as e at $t = 0$.

Theorem 1.2. $\tilde{e}'(0) = \int \psi\, \omega_\varphi^n$

Let us take this theorem for granted momentarily and explain its connection with variational solution of the complex Monge-Ampere equation.

Corollary 1.3. *Let μ be a non-negative Borel measure on X with the same total mass as ω^n. If $\varphi \in \mathcal{E}(\omega)$ maximizes $E_\mu(\varphi) := E(\varphi) - \int \varphi\, \mu$, then $\omega_\varphi^n = \mu$.*

Proof. Note first that since $P(\varphi) = \varphi$, we have that φ maximizes $E_\mu \circ P$ over all usc functions on X. On the other hand, since $P(\phi) \leq \phi$, we have

$$E \circ P(\phi) - \int \phi \, \mu \leq E_\mu \circ P(\phi)$$

with equality at any $\phi \in \mathrm{PSH}(\omega)$. Hence φ also maximizes the left side of this inequality. But the second term on the left is linear in φ, so from Theorem 1.2, we see that the function

$$g(t) = E \circ P(\varphi_t) - \int \varphi_t \, \mu$$

is differentiable at $t = 0$ with

$$g'(0) = \int \psi \, (\omega_\varphi^n - \mu).$$

We infer that the right side is zero for any $\psi \in C(X)$. Hence $\omega_\varphi^n = \mu$ as desired. □

1.1. **Proof of Theorem 1.2, step 1.** We spend the remainder of this section proving Theorem 1.2. First we reduce to the case where both φ and ψ are smooth. Since both functions are at least usc, we can choose sequences $(\varphi_j), (\psi_j) \subset C^\infty(X)$ decreasing to φ and ψ at every point. One can check that then $P(\varphi_j + t\psi_j)$ decreases to $P(\varphi_t)$, too.

Note that by the fundamental theorem of calculus, Theorem 1.2 is equivalent to the statement that

$$\tilde{e}(T) - \tilde{e}(0) = \int_0^T \int_X \psi \, \omega_{P(\varphi_t)}^n \, dt.$$

By continuity of Monge-Ampere under decreasing limits and (on the right side) the dominated convergence theorem, this equation is the limit of

$$E \circ P(\varphi_j + t\psi_j) - E \circ P(\varphi_j) = \int_0^T \int_X \psi \, \omega_{P(\varphi_j + t\psi_j)}^n \, dt.$$

So it suffices to justify this equation, which is (again) equivalent to

$$\frac{d}{dt} E \circ P(\varphi_j + t\psi_j)|_{t=0} = \int \psi_j \, \omega_{P(\varphi_j)}^n.$$

Note that we write $P(\varphi_j)$ instead of φ_j on the right side, because we do not assume that the approximants φ_j are ω-psh (we could do this if we were willing to break down and invoke Demailly's approximation theorem).

1.2. **Proof of Theorem 1.2, step 2.** So from now on, we take φ, ψ to be smooth, but we do assume that φ is necessarily ω-psh. Our next step will be to 'linearize' out the E in $E \circ P$. By concavity of E, we have

$$\tilde{e}(t) - \tilde{e}(0) = E(P(\varphi_t)) - E(P(\varphi)) \leq D_{P(\varphi)} E(P(\varphi_t) - P(\varphi)) = \int (P(\varphi_t) - P(\varphi)) \, \omega_{P(\varphi)}^n.$$

Thus

$$\limsup_{t \to 0} \frac{\tilde{e}(t) - \tilde{e}(0)}{t} \leq \limsup_{t \to 0} \frac{1}{t} \int (P(\varphi_t) - P(\varphi)) \, \omega_{P(\varphi)}^n.$$

With slightly more effort we will show that the reverse inequality holds. For any $t \in \mathbf{R}$ and any $s \in [0,1]$, we have $\varphi_{st} = \varphi(1-s) + s\varphi_t$. Setting $T = st$, we invoke convexity of P to get

$$\tilde{e}(T) = E(P(\varphi_T)) \geq E(P(\varphi) + s(P(\varphi_t) - P(\varphi)).$$

Letting $s \to 0$ while holding t fixed gives

$$\liminf_{T\to 0} \frac{\tilde{e}(T)-\tilde{e}(0)}{T} \geq \frac{1}{t} DE_{P(\varphi)}((P(\varphi_t)-P(\varphi)) = \frac{1}{t}\int (P(\varphi_t)-P(\varphi))\,\omega^n_{P(\varphi)}.$$

Letting $t \to 0$ on the right side, we infer that $\tilde{e}'(0)$ exists and satisfies

$$\tilde{e}'(0) = \limsup_{t\to 0} \frac{1}{t}\int (P(\varphi_t)-P(\varphi))\,\omega^n_{P(\varphi)}.$$

1.3. **Proof of Theorem 1.2, step 3.** We will conclude the proof of Theorem 1.2 by showing for any $\varphi, \psi \in C(X)$.

$$\lim_{t\to 0} \frac{1}{t}\int (P(\varphi_t)-P(\varphi))\,\omega^n_{P(\varphi)} = \int \psi\,\omega^n_{P(\varphi)}.$$

In fact $\leq$ follows from subadditivity of P: namely, $P(\varphi_t) \leq P(\varphi) + tP(\psi) \leq P(\varphi) + t\psi$. So we need only establish that

$$\liminf_{t\to 0} \frac{1}{t}\int (P(\varphi_t)-P(\varphi)-t\psi)\,\omega^n_{P(\varphi)} \geq 0.$$

Since $P(\varphi_t) - P(\varphi) \geq tP(\psi) \geq -t\,\|\psi\|_\infty$, the integrand is bounded below by $-Ct$ and is non-zero only on the open set

$$\tilde{\mathcal{O}}_t := \{P(\varphi_t) < P(\varphi) + t\psi\} \subset \mathcal{O}_t := \{P(\varphi_t) < \varphi_t\}\{P(\varphi_t) < P(\varphi) + t\psi\}$$

It therefore suffices to show that $\omega^n_{P(\varphi)}(\tilde{\mathcal{O}}_t)$ tends to 0 with t.

Scaling ψ appropriately, we may assume that $\omega \geq -dd^c\psi$, i.e. $\psi \in \mathrm{PSH}(\omega)$. Hence φ_t and $\tilde{\varphi}_t := P(\varphi) + t\psi$ are ω_t-psh, where $\omega_t := (1+t)\omega$. We estimate

$$\int_{\tilde{\mathcal{O}}_t} \omega^n_{P(\varphi)} \leq \int_{\tilde{\mathcal{O}}_t} (\omega_{P(\varphi)} + t\omega_\psi)^n = \int_{\tilde{\mathcal{O}}_t} \omega^n_{t,P(\varphi)+t\psi} \leq \int_{\tilde{\mathcal{O}}_t} \omega_{t,P(\varphi_t)} \leq \int_{\mathcal{O}_t} \omega_{t,P(\varphi_t)},$$

the second inequality resulting from the comparison principle and the definition of $\tilde{\mathcal{O}}_t$. The last integral is equal to.

$$\int_{\mathcal{O}_t} \omega^n_{P(\varphi_t)} + \sum_{j=1}^{n} t^j \begin{pmatrix} n \\ j \end{pmatrix} \int_{\mathcal{O}_t} \omega^j \wedge \omega^{n-j}_{P(\varphi_t)}$$

Theorem 1.1 tells us that $P(\varphi_t) = \varphi_t$ a.e. with respect to $\omega^n_{P(\varphi_t)}$, so the first integral is zero. The integrals in the sum are all controlled by expanding the domain of integrationa dn applying Stokes Theorem:

$$\int_{\mathcal{O}_t} \omega^j \wedge \omega^{n-j}_{P(\varphi_t)} \leq \int_X \omega^j \wedge \omega^{n-j}_{P(\varphi_t)} = \int_X \omega^j.$$

Together our estimates show that

$$\int_{\tilde{\mathcal{O}}_t} \omega^n_{P(\varphi)} \leq O(t)$$

so that the left side tends to zero with t as desired. $\square$