1. Mass and trace measure for positive currents

Let $\{\alpha_j\}$ be a collection of (p,q) forms on Ω giving a basis for (p,q) forms at each point. Then for any test form $\Phi = \sum \phi_j(z)\alpha_j \in \mathcal{D}_{(p,q)}(\Omega)$ and relatively compact $E \subset \Omega$, we define the L^∞ by

$$\|\Phi\|_{\infty,E} = \max_j \|\phi_j\|_{\infty,E}.$$

A different choice of basis will result in an equivalent norm with comparability constant depending on E. A current T is said to be of *order zero* if it is continuous with respect to the topology induced by these norms. That is, for any relatively compact open $U \subset \Omega$, there is a constant C_U such that $|\langle T, \Phi\rangle| \le C_U \|\Phi\|_\infty$ for all test forms Φ supported on U. Hence currents of order 0 extend to continuous linear functionals on test forms with merely continuous coefficients.

Now suppose that T is a (p,p) current and let $\omega_E = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ be the Euclidean Kähler form on $\mathbf{C}^n$. Since T is of order zero, the product $T \wedge \frac{\omega_E^{n-p}}{(n-p)!}$ is also of order zero and therefore equal to a complex Borel measure. This is the *trace measure* of T. The origin of the name can be seen by considering a $(1,1)$ current $T = \frac{i}{2}\sum t_{ij}\, dz_i \wedge d\bar{z}_j$. In this case, the trace measure of T is just $\sum t_{jj}$. If T is (also) positive, then the trace measure is positive and we set $\|T\|_K := \int_K T \wedge \frac{\omega_E^{n-p}}{(n-p)!}$. for any compact $K \subset \Omega$.

Theorem 1.1. *Let T be a positive (p,p) current on a domain $\Omega \subset \mathbf{C}^n$. Then T is of order zero. Moreover, for any compact K there exists a constant C (depending only on the above choice of basis) such that for all continuous $(n-p,n-p)$ forms supported on U,*

$$|\langle T, \Phi\rangle| \le C \|\Phi\|_\infty \|T\|_K.$$

The final assertion says that the mass of T on K is controlled by $\|T\|_K$. For this reason, we will often refer to $\|T\|_K$ as the *mass of T on K* when T is a positive closed currents. Be warned that this is non-standard and for non-positive T just plain wrong.

Proof. Let us begin by fixing constant coefficient $(n-p,n-p)$ forms α_j and (p,p)-forms β_j that form dual bases for the corresponding spaces of test forms at each point in Ω. We may assume that α_j has constant coefficients and is simple positive for each j. Duality means that $\alpha_j \wedge \beta_k$ is the Euclidean volume form if $j = k$ and zero otherwise.

Writing $T = \sum t_j \beta_j$, we have for any non-negative test function $\varphi \in C_0^\infty(\Omega)$, that $\langle t_j, \varphi\rangle = \langle T, \varphi\alpha_j\rangle \ge 0$ by positivity of T. Hence t_j is a non-negative distribution and therefore a Borel measure acting continuously by integration even on continuous test functions. It follows that T has order zero.

For the final conclusion of the proof, note that if $\omega = \frac{i}{2}\sum \omega_{jk} dz_j \wedge d\bar{z}_k$ is any constant real $(1,1)$ form, then $C\omega_E - \omega$ is positive for C large enough simply because if A is an $n \times n$ Hermitian matrix, then $CI - A$ is positive for $C > 0$ large enough. Suppose inductively that for any collection $\omega_1, \dots, \omega_k$ of constant coefficient $(1,1)$ forms, we have C such that $C^{k-1}\omega_E^{k-1} - \omega_1 \wedge \cdots \wedge \omega_{k-1}$ is strongly positive. Then writing

$$C^k\omega_E^k - \omega_1 \wedge \cdots \wedge \omega_k = C\omega_E \wedge (C^{k-1}\omega_E^{k-1} - \omega_2 \wedge \cdots \wedge \omega_k) + (C\omega_E - \omega_1) \wedge \omega_2 \wedge \cdots \wedge \omega_k,$$

shows that $C^k\omega_E^k - \omega_1 \wedge \cdots \wedge \omega_k$ is also strongly positive.

So if $\Phi = \sum \varphi_j \alpha_j$ is a test form, then we have by positivity of T (i.e. of t_j) that

$$|\langle t_j, \varphi_j\rangle| \le \langle t_j, |\varphi_j|\rangle = \langle T, |\varphi|\alpha_j\rangle \le C^{n-p}\langle T, |\varphi_j|\omega^{n-p}\rangle = C^{n-p}\int |\varphi_j| T \wedge \omega^{n-p}.$$

Thus up to multiplicative constants $|\langle T,\Phi\rangle| \leq \sum |\langle t_j,\phi_j\rangle| \leq \sum \int |\varphi_j|\, T\wedge\omega^{n-p} \leq \|\varphi_j\|_\infty \|T\|_K$.
$\square$

2. Integration by parts and the localization trick

Since we are concerned with the operator dd^c on non-smooth functions, it will be useful to know that under some circumstances, we can still integrate by parts relative to this operator. Proving that this is possible is suprisingly tricky.

Proposition 2.1. *Suppose u, v are negative and bounded plurisubharmonic functions on a bounded domain $\Omega \subset \mathbf{C}^n$ and that T is a positive closed $(n-1, n-1)$ current. If $\lim_{z\to b\Omega} u(z) = 0$, then*

$$\int v\, dd^c u \wedge T \leq \int u\, dd^c v \wedge T.$$

Hence if in addition $\lim_{z\to b\Omega} v(z) = 0$, we have equality.

Proof. Let us first consider the alternative situation in which u is replaced by a function $h \in C_0^\infty(\Omega)$. Then by definition of $dd^c v \wedge T$, we have

$$\int h\, dd^c v \wedge T = -\langle dh, d^c(vT)\rangle = -\lim_{j\to\infty} \int dh \wedge d^c S_j,$$

where S_j is a sequence regularizing vT. Now since S_j is an $(n-1, n-1)$ form, we have

$$dh \wedge d^c S_j = \frac{1}{2\pi i}(\bar\partial h \wedge \partial S_j - \partial h \wedge \bar\partial S_j) = -d^c h \wedge dS_j,$$

the other terms being zero since they have bidegrees $(n+1, n-1)$ and $(n-1, n+1)$. Thus

$$\int h\, dd^c v \wedge T = \lim_{j\to\infty} \int d^c h \wedge dS_j = \langle d^c h, d(vT)\rangle = \langle dd^c h, vT\rangle = \int v\, dd^c h \wedge T.$$

So the proposition is true with h in place of u.

Going back to u, we set $u_\epsilon = \max\{u, -\epsilon\}$ and note that $u - u_\epsilon = \min\{0, u+\epsilon\}$ is a compactly supported function decreasing uniformly to u as $\epsilon \to 0$. Hence

$$\int u\, dd^c v = \lim_{\epsilon\to 0} \int (u - u_\epsilon)\, dd^c v \wedge T.$$

Compact support of $u - u_\epsilon$ allows us to regularize and apply the result of the first paragraph in the proof.

$$\int (u-u_\epsilon)\, dd^c v \wedge T = \lim_{j\to\infty} \int (u-u_\epsilon) * \rho_{1/j}\, dd^c v \wedge T = \lim_{j\to\infty} \int v\, dd^c (u-u_\epsilon) * \rho_{1/j} \wedge T.$$

If we fix an open set $\Omega' \subset\subset \Omega$ such that $\{u < -\epsilon\} \subset\subset \Omega'$, then for j large enough the last integrand is supported entirely on Ω'. Thus

$$\int (u-u_\epsilon)\, dd^c v \wedge T = \lim_{j\to\infty} \int_{\Omega'} v\, dd^c (u-u_\epsilon) * \rho_{1/j} \wedge T \geq \overline{\lim_{j\to\infty}} \int_{\Omega'} v\, dd^c u * \rho_{1/j} \wedge T.$$

The last inequality follows from the facts that $v \leq 0$ and $dd^c u_\epsilon \wedge T \geq 0$. Since v is upper-semicontinuous, there are functions $v_k \in C^0(\Omega')$ decreasing to v. So for any k, we can continue our previous estimate as follows

$$\begin{aligned}\int_\Omega (u-u_\epsilon)\, dd^c v\wedge T \quad &\geq \quad \overline{\lim_{j\to\infty}} \int_{\Omega'} v_k\, dd^c u * \rho_{1/j}\wedge T = \int_{\Omega'} v_k\, dd^c u\wedge T \\ &\underset{k\to\infty}{\longrightarrow} \int_{\Omega'} v\, dd^c u\wedge T \underset{\Omega'\nearrow\Omega}{\longrightarrow} \int_\Omega v\, dd^c u\wedge T.\end{aligned}$$

Letting $\epsilon\to 0$ in the first integral concludes the proof. □

In order to employ Proposition 2.1 and to simplify arguments in other ways, we observe in advance that many assertions that we will seek to prove (e.g. weak convergence of currents, mass bounds on compact sets) are 'local' in nature and can be reduced to the following situation via finite covers, partitions of unity, translation and scaling, etc.

(1) The domain of all objects in question is the unit ball $B_1(\mathbf{0})$, and the assertions concern only the restrictions of these objects to a smaller ball, e.g. $B_{1/2}(\mathbf{0})$.
(2) All plurisubharmonic functions involved are bounded above by -1 near $\overline{B_{1/2}(\mathbf{0})}$, and any bounded plurisubharmonic function is bounded below by -2 on the same set.
(3) Replacing each bounded psh u in (2) with $\max\{u, 100(\|z\|^2-1)\}$, all bounded psh functions are smooth and equal near $bB_1(\mathbf{0})$ and tend to 0 as $\|z\|\to 1$.

Following Kolodziej, we will refer to these assumptions collectively as 'the localization trick'. Combining this trick with Proposition 2.1, we can now show that the definition of Monge-Ampere is independent of order.

Corollary 2.2. *If u, v are locally bounded plurisubharmonic functions on a domain Ω and T is a positive closed current, then $dd^c u\wedge dd^c v\wedge T = dd^c v\wedge dd^c u\wedge T$.*

Proof. The assertion is local, so we employ the localization trick. In particular, the previous proposition applies. For any smooth test form β, we have.

$$\langle \beta, v\, dd^c u\wedge T\rangle = \langle dd^c\beta, v\, dd^c u\wedge T\rangle = \int v\, dd^c u\wedge dd^c\beta\wedge T$$

Since β is smooth, we can choose $A>>0$ and express $dd^c\beta\wedge T = (dd^c(\beta + A\|z\|^2)\wedge T - Add^c\|z\|^2\wedge T$ as a difference of positive closed currents. So the previous proposition allows us to continue evaluating

$$\langle \beta, v\, dd^c u\wedge T\rangle = \int u\, dd^c v\wedge dd^c\beta\wedge T = \langle \beta, u\, dd^c v\wedge T\rangle$$

as asserted. □

Theorem 2.3 (Chern-Levine-Nirenberg inequality)**.** . *Let $\Omega\subset\mathbf{C}^n$ be a domain, and $K\subset\Omega$ be compact and $U\subset\Omega$ a relatively compact neighborhood of K. Then there exists a constant $C = C(K,U)$ such that for any positive closed (p,p) current T and locally bounded plurisubharmonic $u_0, u_1, \dots, u_k$ on Ω, we have*

$$\|u_0\, dd^c u_1\wedge\cdots\wedge dd^c u_k\wedge T\|_K \leq C\|u_0\|_{L^\infty,U}\|u_1\|_{L^\infty,U}\cdots\|u_k\|_{L^\infty,U}\|T\|_U\,.$$

The L^∞ norm of u_0 can be replaced by the L^1 norm of u_0 on U with respect to the trace measure of T, i.e. by

$$\int_U |u_0|\,\omega_E^{n-p}\wedge T.$$

Proof. The case $k = 0$ holds more or less tautologically. To establish the result for $k > 0$, it suffices to prove and iterate the case $k = 1$. For this, we employ the localization trick, taking $K \subset B = B_{1/2}(\mathbf{0}) \subset\subset U \subset\subset B_1(\mathbf{0})$. We let χ be a smooth cutoff function for B supported on U and estimate

$$\begin{aligned}\int_K |u_0|\, dd^c u_1 \wedge T \wedge \omega_E^{n-p-1} &\leq -\int_B u_0\, dd^c u_1 \wedge T \wedge \omega_E^{n-p-1} = -\int_B u_1\, dd^c u_0 \wedge T \wedge \omega_E^{n-p-1} \\ &\leq \int_U \chi dd^c u_0 \wedge T \wedge \omega_E^{n-p-1} = \int_U u_0\, dd^c \chi \wedge T \leq C \int_U |u_0|\, T \wedge \omega_E^{n-p}.\end{aligned}$$

The last inequality uses that $C\omega_E - dd^c\chi$ is strongly positive for $C >> 0$ depending only on χ (i.e. on U and K). $\square$

The proof of the CLN inequality shows that one can use the L^∞ norms of u_j, $1 \leq j \leq k$ on just $U \setminus K$ (since that is where the second derivative $dd^c\chi$ is supported) in bounding $\|u_0\, dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge T\|_K$. This is very useful when one wants to relax the local boundedness restriction to allow e.g. $u_j = \log \|z\|$.

Corollary 2.4. *If $u_1, \ldots, u_n$ are locally bounded plurisubharmonic functions on a domain $\Omega \subset \mathbf{C}^n$, then* any $u \in \mathrm{PSH}(\Omega)$ *is locally integrable with respect to the measure $\mu := dd^c u_1 \wedge \cdots \wedge dd^c u_n$. Hence μ does not charge pluripolar sets. In particular μ does not charge analytic subvarieties.*

Proof. Given $u \in \mathrm{PSH}(\Omega)$ and compact $K \subset \Omega$, we may assume that $u < 0$ on a relatively compact neighborhood U of K. Usc functions are Borel measureable, so the integral of u against μ is well-defined. We must show it is not $-\infty$. Applying the CLN estimate with $k = n$, $u_0 = \max\{u, -M\}$ and $T = 1$ we find that

$$\begin{aligned}-\int_K u\, d\mu &= -\lim_{M\to\infty} \int_K \max\{u, -M\}\, d\mu \\ &\leq -C\, \|u_1\|_{L^\infty,U} \cdots \|u_n\|_{L^\infty,U} \lim_{M\to\infty} \int_U \max\{u, -M\}\, d\lambda \\ &\leq C\, \|u_1\|_{L^\infty,U} \cdots \|u_n\|_{L^\infty,U} \int_U u\, d\lambda.\end{aligned}$$

The last quantity is finite because plurisubharmonic functions are always locally integrable with respect to Lebesgue measure. So u is integrable with respect to μ.

Now if S is a pluripolar set and $p \in S$ is any point, we can choose a neighborhood U of p and a plurisubharmonic u on U such that $S \subset \{u = -\infty\}$. As u is integrable with respect to μ, it follows that μ does not charge $S \cap U$. This and the fact that analytic varieties are pluripolar justify the final two assertions. $\square$

It is worth pointing out that the proof *does not* show that μ is absolutely continuous with respect to Lebesgue measure. This will be borne out by examples in later sections.

3. Continuity along decreasing sequences

Now we arrive at the most important continuity property for the Monge-Ampere operator.

Theorem 3.1. *Let $T \geq 0$ be a positive closed (p,p) current on a domain $\Omega \subset \mathbf{C}^n$. For each $0 \leq j \leq k$, let $(u_j^\ell)_{\ell\in\mathbf{N}} \subset (\Omega)$ be a sequence of uniformly locally bounded functions decreasing* *to u_j. Then we have the weak convergence of Monge-Ampere measures*

$$u_0^\ell\, dd^c u_1^\ell \wedge \cdots \wedge dd^c u_k^\ell \wedge T \underset{\ell\to\infty}{\longrightarrow} u_0\, dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge T.$$

Proof. The assertion is local, so we employ the localization trick throughout. For $k = 0$, the assertion follows from the monotone convergence theorem. Given the assertion for $k-1$ in place of k, weak continuity of dd^c implies that

$$S_\ell := dd^c u_1^\ell \wedge \cdots \wedge dd^c u_k^\ell \wedge T \underset{\ell\to\infty}{\longrightarrow} S := dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge T.$$

That is, the assertion holds for $k = k$ when $u_0^\ell \equiv u_0 \equiv 1$. It remains to show that $u_0^\ell S_\ell \to u_0 S$. The CLN estimate and the uniform local boundedness assumption imply that the mass of $u_0^\ell S_\ell$ is uniformly bounded on any compact $K \subset \Omega$. Hence the sequence $\{u_0^\ell S_\ell\}$ is relatively compact in the weak topology on currents, we need to show that uS is the only possible limit point. So refining the sequence, we assume $u_0^j S_j \to \Theta \le 0$ and seek to show $\Theta = uS$.

First we show $\Theta \le u_0 S$. Note that

$$u_0^j S_j \le u_0^\ell S_j \le u S_j$$

for any $j \le \ell$ and any continuous $u \ge u_0^\ell$ with compact support on Ω. Letting $j \to \infty$ shows $\Theta \le uS$. Now letting u decrease to u_ℓ and then $\ell \to \infty$, monotone convergence gives $\Theta \le u_0 S$, as desired.

Now we know that $u_0 S - \Theta$ is actually positive. To see that the difference vanishes, it suffices to show that $\|u_0 S - \Theta\|_K = 0$ for any compact $K \subset \Omega$. But by positivity,

$$\|u_0 S - \Theta\|_K \le C \int_\Omega (u_0 S - \Theta) \wedge \omega_E^{n-p-k},$$

so it suffices to show that the right side is non-positive. To this end, we estimate

$$\begin{aligned}
\int_\Omega u_0^\ell S_\ell \wedge \omega_E^{n-p} &= \int u_0^\ell\, dd^c u_1^\ell \wedge \cdots \wedge dd^c u_k^\ell \wedge T \omega_E^{n-p} \\
&\ge \int u_0\, dd^c u_1^\ell \wedge \cdots \wedge dd^c u_k^\ell \wedge T \omega_E^{n-p} \\
&= \int u_1^\ell\, dd^c u_0 \wedge \cdots \wedge dd^c u_k^\ell \wedge T \omega_E^{n-p} \\
&\ge u_1\, dd^c u_0 \wedge \cdots \wedge dd^c u_k^\ell \wedge T \omega_E^{n-p} \\
&\ge \cdots \ge u_0\, dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge T \omega_E^{n-p} = \int u_0 S \wedge \omega_E^{n-p-k}.
\end{aligned}$$

Letting $\ell \to \infty$ in the first integrand (and using the fact that all functions are the same outside a compact subset of Ω), we see that $\int \Theta \wedge \omega_E^{n-p-k} \ge \int u_0 S \wedge \omega_E^{n-p-k}$ as desired. □

4. Monge-Ampere of radially symmetric functions

Proposition 4.1. *Let $\Omega \subset \mathbf{C}^n$ be a smoothly bounded domain and $f, g : \overline{\Omega} \to \mathbf{R}$ be C^2 functions. If $df \equiv dg$ on $b\Omega$, then $\int_\Omega (dd^c f)^n = \int_\Omega (dd^c g)^n$.*

Proof. By Stokes Theorem

$$\int_\Omega (dd^c f)^n = \int_{b\Omega} d^c f \wedge (dd^c f)^{n-1} = \int_{b\Omega} d^c g \wedge (dd^c f)^{n-1} = \int_\Omega dd^c g \wedge (dd^c f)^{n-1}.$$

Repeating this argument gives

$$\int_\Omega dd^c g \wedge (dd^c f)^{n-1} = \int_\Omega (dd^c g)^2 \wedge (dd^c f)^{n-2} = \cdots = \int_\Omega (dd^c g)^n.$$

$\square$

The proposition permits us to compute the Monge-Ampere of certain plurisubharmonic functions by replacing them with simpler functions.

Theorem 4.2. *Let $u : \mathbf{C}^n \to [-\infty, \infty)$ be a smooth radially symmetric function. Then*

$$\int_{B_r} (dd^c u)^n = C u'(r)^n r^n$$

for some constant $C > 0$ depending only on n.

Under the hypotheses of the theorem, we certainly have that $(dd^c u)^n = f\, d\lambda$ for some radially symmetric, non-negative function f. Hence one can differentiate the formula in the theorem with respect to r to find a formula for f in terms of u. We leave it to the reader to do this.

Proof. Let $v(z) = A \|z\|^2$, where $A = u'(r)/2r$. Then $dv = du$ on bB_r. Hence

$$\int_{B_r} (dd^c u)^n = \int_{B_r} (dd^c v)^n = CA^n \int_{B_r} d\lambda = CA^n r^{2n} = Cu'(r)^n r^n.$$

$\square$

The conclusion of the theorem remains true for bounded *psh* u if we require smoothness only near bB_r. One can see this by regularizing u on B_{2r} and invoking continuity of Monge-Ampere along decreasing sequences. In fact, such u are necessarily continuous, so we only need continuity along uniformly convergent sequences.

Let us consider more closely $u(z) = \max\{\log \|z\|, 0\}$. Clearly $(dd^c u)^n = 0$ inside B_1. Outside $\overline{B_1}$, we still have that $(dd^c u)_L$ vanishes along any complex line through 0. That is, the complex hessian matrix $(u_{i\bar{j}})$ has zero as an eigenvalue. From this we see that $(dd^c u)^n$ also vanishes for $\|z\| > 1$. The measure $(dd^c u)^n$ is certainly radially symmetric, so it follows that $(dd^c u)^n$ is a non-negative multiple of Lebesgue measure on the unit $2n-1$ sphere. To see that the multiple is positive, it suffices to apply the previous theorem on a ball of radius larger than 1. For instance

$$\int_{bB_0} (dd^c u)^n = \int_{B_2} (dd^c u)^n = Cu'(2)2^n = C > 0,$$

where the last constant C is precisely the one in the theorem. We leave it as an exercise to find the exact value of C. Note in particular, that $(dd^c u)^n$ is not absolutely continuous with respect to Lebesgue measure $d\lambda$ on $\mathbf{C}^n$.

ere's a problem e–since $(dd^c)^n$ non-linear, e doesn't have t the spherical erage of $(dd^c u)^n$ equal to $(dd^c)^n$ plied to the herical average of

Theorem 4.2 can sometimes be applied to help compute Monge-Ampere for non-radially symmetric plurisubharmonic functions u. For instance if $u(z) = \log \max\{|z_j|\}$ on $\mathbf{C}^n$, then it is not hard to see that $(dd^c u)^n$ is a non-negative multiple of Lebesgue measure on the unit n-torus $\{|z_1| = \cdots = |z_n| = 1\}$. Replacing u by its average over spheres centered at the origin, one obtains a radially symmetric plurisubharmonic function $\hat{u}$ such that $\hat{u}(z) - \log \|z\|$ is constant for $\|z\|$ large enough. The total mass of $(dd^c \hat{u})^n$ on a large ball will be the same as that of $(dd^c u)^n$, so it then follows from Theorem 4.2 applied to $\hat{u}$ that $(dd^c u)^n$ is a *positive* multiple of Lebesgue measure on the unit n-torus.

5. Discontinuity of the Monge-Ampere operator: an example

The following example is essentially taken from Klimek's book, which attributes it to?? Let $u : \mathbf{C}^2 \to \mathbf{R}$ be the psh function $u(z_1, z_2) = \log \max\{|z_1|, |z_2|, 1\}$. In the last section we showed that the Monge-Ampere measure $(dd^c u)^2$ has strictly positive total mass. Nevertheless, we will see here that there is a sequence of non-negative psh functions $u_k \leq 2u$ converging to u in $L^p_{loc}(\mathbf{C}^2)$ for every $p < \infty$ and such that $(dd^c u_k)^2 = 0$ for every k. In particular, we do *not* have corresponding weak convergence $(dd^c u_k)^2 \to (dd^c u)^2$ of Monge-Ampere measures. So the Monge-Ampere operator is discontinuous in the L^p topology on bounded psh functions.

For any $t \in \mathbf{R}$, let $\log^+ t = \max\{\log t, 0\}$. Let $u_k(z) = \frac{1}{k} \log^+ |z_1^k + z_2^k|$. Then

$$0 \leq u_k(z) \leq \log^+ |z_1| + \log^+ |z_2| \leq 2 \log^+ \max\{|z_1|, |z_2|\} = 2u$$

for all k and z. Moreover, we have that $|z_1^k + z_2^k|^{1/k} \to \max\{|z_1|, |z_2|\}$ uniformly on compact subsets of $\mathbf{C}^2 - \{|z_1| = |z_2|\}$. Since $\{|z_1| = |z_2|\}$ has measure 0, it follows from domintated convergence that $u_k \to u$ in L^p_{loc} for all $p < \infty$.

To complete the example, we observe that each approximant u_k has the form $h_+ := \max\{h, 0\}$ for some pluriharmonic function h. Then we show

Proposition 5.1. *If h is pluriharmonic on Ω, then $(dd^c h_+)^2 \equiv 0$.*

Proof. Let $K \subset\subset \Omega$ be given and $\chi \in C_0^\infty(\Omega)$ be a cutoff for K. Thne

$$\int_K (dd^c h_+)^2 \leq \chi\, (dd^c h_+)^2 \leq h_+\, dd^c \chi \wedge dd^c h_+ \leq C \int_{\operatorname{supp}\chi} h_+ \omega_E \wedge dd^c h_+ \leq C\epsilon \int_{\operatorname{supp}\chi} \omega_E \wedge dd^c h_+,$$

for any $\epsilon > 0$, since $dd^c h_+ = dd^c h = 0$ on $\{h_+ > \epsilon\}$. The last integral is finite, so letting $\epsilon \to 0$ concludes the proof. $\square$

This example readily generalizes to any non-negative plurisubharmonic u of the form $\max\{h_1, \ldots, h_k, 0\}$ where h_j are pluriharmonic. Indeed, it can be shown that any continuous plurisubharmonic function is (after translation) a uniform local limit of such u, so if $(dd^c u)^2 \not\equiv 0$, we can manufacture a sequence $u_j \to u$ in L^p_{loc} converging to u such that $(dd^c u_k)^2 \equiv 0 \not\to (dd^c u)^2$. In the case of strictly convex functions u the approximation can be accomplished via maxima of *affine* pluriharmonic functions.

6. Capacity and quasicontinuity

Definition 6.1. *Let E be a Borel subset of a domain $\Omega \subset \mathbf{C}^n$. The* relative capacity *of E in Ω is*

$$\operatorname{cap}(E, \Omega) := \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega),\ -1 \leq u \leq 0\}.$$

So capacity (a general notion; the definition above is only a particular case) determines the size of set relative to an entire collection of measures. The result is a set-valued function that in many ways resembles a measure but which tends to exaggerate the size of smaller sets. It follows quickly from the definition that

- $E \subset F \subset \Omega$ implies that $\operatorname{cap}(E, \Omega) \leq \operatorname{cap}(F, \Omega)$.
- $\operatorname{cap}(\cup_{j=1}^\infty E_j, \Omega) \leq \sum \operatorname{cap}(E_j, \Omega)$.

However, the (first) subadditivity property of capacity cannot be sharpened to additivity $\operatorname{cap}(E \cup F, \Omega) = \operatorname{cap}(E, \Omega) + \operatorname{cap}(F, \Omega)$ for disjoint Borel sets E and F. In particular, if $U \subset \Omega$ is any open set and M is any given positive number, it is possible to find disjoint compact sets $K_1, \dots, K_N$ such that $\sum \operatorname{cap}(K_j, \Omega) > M$. The Wikipedia entry on capacity makes the nice comment that capacity measures the ability of set to hold charge.

The CLN estimate implies that $\operatorname{cap}(E, \Omega)$ is finite for any relatively compact E. If $\Omega \subset B_r(p)$ is bounded, then $u(z) = \frac{1}{r^2} \|z - p\|^2$ shows that $\operatorname{cap}(E, \Omega) \geq \frac{C}{r^2} \lambda(E)$ for some dimensional constant $C > 0$. That is, relative capacity dominates (a multiple of) Lebesgue measure. We remark in passing that relative capacity can be made 'outer regular' with respect to open sets by fiat, modifying the above definition to $\operatorname{cap}^*(E, \Omega) := \inf_{E \subset U_{open} \subset \Omega} \operatorname{cap}(U, \Omega)$, one can even force cap^* to be outer regular. It follows from a theorem of Choquet that cap^* is also inner regular–i.e. that $\operatorname{cap}^*(E, \Omega) = \sup_{K_{cpct} \subset E} \operatorname{cap}^*(K, \Omega)$.

Definition 6.2. *A sequence of Borel measureable functions u_j on Ω is said to converge to u* in capacity *if*

$$\lim_{j \to \infty} \operatorname{cap}(K \cap \{|u_j - u| > \epsilon\}, \Omega) = 0$$

for any $K \subset\subset \Omega$ and $\epsilon > 0$.

Though more restrictive, convergence in capacity is quite similar in spirit to convergence in measure. For instance, the proof of one of Littlewood's principles carries over directly from measure theory to give

Proposition 6.3. *Suppose that $u_j \to u$ in capacity on Ω. Then for any $\epsilon > 0$, there is a Borel set $E \subset \Omega$ such that $u_j \to u$ uniformly on $\Omega - E$.*

The next two theorems place continuity of Monge-Ampere under decreasing limits in a new light.

Theorem 6.4. *Suppose for $1 \leq j \leq n$ that $(u_j^k)_{k \in \mathbf{N}}$ are sequences of locally uniformly bounded plurisubharmonic functions on Ω converging in measure to $u_j \in \mathrm{PSH}(\Omega)$. Then*

$$dd^c u_1^k \wedge \cdots \wedge dd^c u_n^k \to dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

weakly.

Proof. Observe that

$$dd^c u_1^k \wedge \cdots \wedge dd^c u_n^k - dd^c u_1 \wedge \cdots \wedge dd^c u_n = \sum_{j=1}^n dd^c u_1^k \wedge \cdots \wedge dd^c (u_j^k - u_j) \wedge \cdots \wedge dd^c u_n = \sum dd^c (u_j^k - u_j) \wedge T_j,$$

where the last equality defines T_j. Given any test function φ and any $\epsilon, t > 0$, we have that $E = E(j, k, t) := \operatorname{supp} \varphi \cap \{|u_j^k - u_j| \geq t\}$ has capacity smaller than ϵ for k large enough and all j. Since $dd^c \varphi \leq C \omega_E$, we may estimate

$$\begin{aligned} |\langle \varphi, dd^c (u_j^k - u_j) \wedge T_j \rangle| &= |\langle dd^c \varphi, (u_j^k - u_j) T_j \rangle| \\ &= \int_E (u_j^k - u_j) dd^c \varphi \wedge T + \int_{\operatorname{supp} \varphi - E} (u_j^k - u_j) dd^c \varphi \wedge T_j \\ &\leq C \int_E |u_j^k - u_j| \omega_E \wedge T_j + Ct \|T_j\|_{\operatorname{supp} \varphi}. \end{aligned}$$

We bound the last integral as follows. The psh function $v := \|z\|^2 + \sum_j u_j^k + \sum_j u_j \in \mathrm{PSH}(\Omega)$ is locally bounded and satisfies $(dd^c v)^n \geq dd^c\psi_i \wedge T_j$ for $1 \leq j \leq n$. Hence

$$\int_E |u_j^k - u_j| \omega_E \wedge T \leq \int_E |u_j^k - u_j| \, (dd^c v)^n \leq C \left\| u_j^k - u_j \right\|_{\infty,E} \|v\|_{\infty,E}^n \operatorname{cap}(E,\Omega) = C\epsilon.$$

Similarly,

$$\|T_j\|_{\operatorname{supp}\varphi} \leq C \int_{\operatorname{supp}\varphi} (dd^c v)^n \leq C\operatorname{cap}(\operatorname{supp}\varphi, \Omega).$$

Altogether then

$$|\left\langle \varphi, dd^c(u_j^k - u_j) \wedge T_j \right\rangle| \leq C(\epsilon + t)$$

for all j, all $k \geq K(\epsilon, t)$ large enough. Letting $\epsilon, t \to 0$ concludes the proof. □

Theorem 6.5. *Any decreasing, uniformly locally bounded sequence $(u_j) \subset \mathrm{PSH}(\Omega)$ converges in capacity.*

Before proving this theorem, we establish some important consequences. If the functions u_j in Theorem 6.5 are continuous, then $\{u_j - u > t\}$ is open for any $t \in \mathbf{R}$. Hence the sets E in Proposition 6.3 may be taken to be open. Taking (u_j) to be regularizations of some fixed $u \in \mathrm{PSH}(\Omega)$, we get more or less immediately that

Corollary 6.6. *Any $u \in \mathrm{PSH}(\Omega)$ is quasicontinuous. That is, for any $\epsilon > 0$ there is an open set $U \subset \Omega$ and $v \in C(\Omega)$ such that $\operatorname{cap}(U, \Omega) < \epsilon$ and $u \equiv v$ on $\Omega - U$.*

This allows us to strengthen an assertion used in proving Theorem 3.1.

Corollary 6.7. *Let T_j, T be mixed Monge-Ampere currents associated to uniformly locally bounded psh functions on Ω. Suppose $T_j \to T$ weakly. Then for any locally bounded psh function u on Ω, we have $uT_j \to uT$*

Proof. The result is local, so we may assume that u is bounded and that $T_j = dd^c u_1^j \wedge \cdots \wedge dd^c u_p^j$ for $u_k^j \in \mathrm{PSH}(\Omega)$ that are uniformly bounded in all Ω. Note that $T_j \wedge \omega_E^{n-p} \leq (dd^c\psi_j)^n$ where $\psi_j = \|z\|^2 + \sum_{k=1}^n u_k^j \in \mathrm{PSH}(\Omega)$ has L^∞ norm bounded uniformly in j. Hence for any compact $K \subset \Omega$, we have $\operatorname{cap}(K, \Omega) \geq C \|T_j\|_K$ for all j. The same observation applies with T in place of T_j.

Given $\epsilon > 0$, let $v \in C(\Omega)$ be such that $v \equiv u$ off an open set $U \subset \Omega$ of capacity less than ϵ. Observe that $vT_j \to vT$ since T_j, T are currents of order zero. Now fix a test form Φ. Then

$$|\langle \Phi, u(T_j - T)\rangle| \leq |\langle \Phi, v(T_j - T)\rangle| + |\langle \Phi, (u - v)(T_j - T)\rangle|.$$

The first term tends to 0 as $j \to \infty$. We control the second term as follows.

$$\begin{aligned} |\langle \Phi, (u - v)(T_j - T)\rangle| &= \left| \int_U (u - v)\, \Phi \wedge (T_j - T) \right| \\ &\leq C \int_U \omega_E^{n-p} \wedge (T + T_j) < C\operatorname{cap}(U, \Omega) < C\epsilon, \end{aligned}$$

for $C > 0$ independent of j. All told, we have

$$\overline{\lim_{j\to\infty}} |\langle \Phi, u(T_j - T)\rangle| < C\epsilon,$$

and since $\epsilon > 0$ is arbitrary, the proof is complete. □

Corollary 6.7 together with the rest of the proof of Theorem 3.1 give continuity of Monge-Ampere under *increasing* limits.

Theorem 6.8. *Suppose for $0 \le j \le p$ that $(u_j^k)_{k\in\mathbf{N}}$ are increasing sequences of locally uniformly bounded plurisubharmonic functions on $\Omega \subset \mathbf{C}^n$ with a.e. limits $u_j \in \mathrm{PSH}(\Omega)$. Then*

$$u_0^k \, dd^c u_1^k \wedge \cdots \wedge u_0 \, dd^c u_p^k \to u_0 \, dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

weakly.

Proof. As in the proof of continuity for Monge-Ampere along decreasing sequences, we proceed by induction on p. The key point is the same as before: given positive closed (p,p) currents T_k converging weakly to T, one must show that $u_0^k T_k \to u_0 T$. The CLN estimate tells us that the sequence $(u_0^k T_k)$ is relatively compact in the weak topology, so by refining our sequence we may assume that $u_0^k T_k$ converges to some current Θ and complete the proof by showing that $\Theta = u_0 T$.

By monotonicity we have $u_0^k T_k \le u_0 T_k$ for all $k \in \mathbf{N}$. It follows from Corollary 6.7 that $\Theta \le u_0 T$. Fixing j, we also have $u_0^k T_k \ge u_0^j T_k$ for all k large enough. Hence $\Theta \ge u_0^j T$. The monotone convergence theorem allows us to let $j \to \infty$ here, so we conclude that $\Theta \ge u_0 T$. □

Finally we attend to the proof of Theorem 6.5. First we introduce some new wedge products that will help us control locally bounded psh functions in terms of their gradients. Note that if u is a smooth function and T is a positive closed current, then

$$du \wedge d^c u \wedge T = dd^c u^2 \wedge T - 2u \, dd^c u \wedge T.$$

The right side makes sense (and is positive) even for locally bounded psh functions so for such functions we take the above equation as the *definition* of $du \wedge d^c u \wedge T$. In fact, if v is a second locally bounded psh function then we further define

$$du \wedge d^c v \wedge T := \frac{1}{2}(d(u+v) \wedge d^c(u+v) \wedge T - du \wedge d^c u \wedge T - dv \wedge d^c v \wedge T,$$

which is also consistent with the case of smooth functions. We then extend linearly to the vector space consisting of all differences of locally bounded psh functions. From these definitions and Theorem 3.1, it follows that $du \wedge d^c v \wedge T$ is continuous under regularization of u and v. In particular $du \wedge d^c u \wedge T$ is positive even when u is only a difference of locally bounded psh functions. Hence Schwarz' inequality applies. That is, for any Borel set $E \subset \Omega$

$$\left| \int_E du \wedge d^c v \wedge T \right| \le \left(\int_E du \wedge d^c u \wedge T \right)^{1/2} \left(\int_E dv \wedge d^c v \wedge T \right)^{1/2}.$$

For u smooth, this follows from positivity of T and the fact that $du \wedge d^c$ is a simple positive $(1,1)$ form. For a difference $u - v$ of locally bounded psh functions, the inequality then follows from regularizing u and v

To prove Theorem 6.5 we may employ the localization trick. Thus the sets $E = E(t,j) = \{u_j - u > t\}$ lie in a fixed compact $K \subset \Omega$ independent of t and j. Expanding K if necessary, we may assume that $u_j \equiv u$ are all smooth on $\Omega - K$. We must show $\lim_{j\to\infty} \mathrm{cap}(E(t,j),\Omega) = 0$ for each $t > 0$. Let $\mathcal{F}$ denote the set of all $v \in \mathrm{PSH}(\Omega)$ such that $0 \le v \le 1$. Then

$$\mathrm{cap}(E(t,j),\Omega) \le \sup_{v\in\mathcal{F}} t \int_K (u_j - u)(dd^c v)^n.$$

Let $\chi \in C_0^\infty(\Omega)$ be a cutoff function equal to 1 in a neighborhood of K. Since $u_j - u \equiv 0$ on $\operatorname{supp} d\chi$, we have

$$\begin{aligned}
\int_K (u_j - u)\,(dd^c v)^n &\le \int_\Omega \chi(u_j - u)\,(dd^c v)^n = -\int_\Omega \chi\, d(u_j - u) \wedge d^c v \wedge (dd^c v)^{n-1} \\
&= \int_K d(u_j - u) \wedge d^c v \wedge (dd^c v)^{n-1} \\
&\le \left(\int_K d(u_j - u) \wedge d(u_j - u) \wedge (dd^c v)^{n-1} \right)^{1/2} \left(\int_K dv \wedge d^c v \wedge (dd^c v)^{n-1} \right)^{1/2}.
\end{aligned}$$

Since $v \in \mathcal{F}$, the right integral is controlled by $\operatorname{cap}(K, \Omega)$. We estimate the left integral by reversing the previous integration by parts.

$$0 \le \int_K d(u_j - u) \wedge d^c(u_j - u) \wedge (dd^c v)^{n-1} = \int_K (u_j - u)\, dd^c(u - u_j) \wedge (dd^c v)^{n-1} \le \int_K (u_j - u)\, dd^c u \wedge (dd^c v)^{n-1}.$$

Repeating this process $n - 1$ more times, we arrive at

$$\int_K (u_j - u)(dd^c v)^n \le C \left(\int_K (u_j - u)\,(dd^c u)^n \right)^{1/2^n}.$$

This bound is independent of v and tends to 0 as $j \to \infty$ by the monotone convergence theorem. Hence $\lim_{j\to\infty} \operatorname{cap}(E(t,j), \Omega) = 0$. □

7. Monge-Ampere and sums of psh functions

Proposition 7.1. *Let A be an $n \times n$ Hermitian non-negative matrix. Then*

$$\det^{1/n} A = \frac{1}{n} \inf_{\det B = 1} \operatorname{tr}_B A$$

where $\operatorname{tr}_B A := \operatorname{tr} \bar{B}^T A B$.

Proof. Fix B with determinant 1 and let $\lambda_1, \dots, \lambda_n \ge 0$ be the eigenvalues of $\bar{B}^T A B$. Then from the arithmetic-geometric mean inequality, we obtain

$$\det^{1/n} A = \det^{1/n} \overline{B}^T A B = (\lambda_1 \dots \lambda_n)^{1/n} \le \frac{1}{n} \sum \lambda_j = \frac{1}{n} \operatorname{tr}_B A$$

with equality when all the eigenvalues are the same.

Hence $\det^{1/n} A \le \frac{1}{n} \inf_{\det B=1} \operatorname{tr}_B A$. If A is positive, the Gram-Schmit process gives us B so that all eigenvalues of $\operatorname{tr}_B A$ are the same. This gives us the reverse inequality. Hence the proposition holds and the infimum is actually achieved in this case. If A is indefinite, then we can still choose B so that $\bar{B}^T A B$ is diagonal. One of the diagonal entries, say the first, of $\bar{B}^T A B$ will be zero. So if we let D be the diagonal matrix with 11 entry equal to $t >> 0$ and all other diagonal entries equal to $t^{\frac{1}{1-n}}$, we can make the non-zero diagonal entries of $\overline{BD}^T A(BD)$ as small as we want. Thus $\det^{1/n} A = 0 = \lim_{t\to\infty} \operatorname{tr}_{DB} A$. □

Since $\operatorname{tr}_B A$ is linear in A and linear functions are concave, we immediate infer

Proposition 7.2. *The function $\det^{1/n}$ is 1-homogeneous and concave on the closed convex cone of $n \times n$ non-negative Hermitian matrices. Specifically,*

- $\det^{1/n}(\lambda A) = \lambda \det^{1/n} A$*; and*
- $\det^{1/n}(A + B) \ge \det^{1/n} A + \det^{1/n} B$

for any $\lambda > 0$ and non-negative Hermitian A and B.

In what follows, we let $D\bar{D}u$ denote the (non-negative Hermitian) matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$ of (possibly distributional) mixed second partial derivatives of a given function u. Recall that if u is smooth, then $(dd^c u)^n = C_n \det(D\bar{D}u)\, d\lambda$ where $C_n = n!/\pi^n$. Given $A \in SL(n, \mathbf{C})$, we let $\Delta_A u = \frac{1}{n}\operatorname{tr}_A(D\bar{D}u)$. Note that the operator Δ_A is (up to positive multiple) just the usual laplacian relative to the coordinate $w = Az$ on $\mathbf{C}^n$. Proposition 7.1 tells us that $\det^{1/n} D\bar{D}u = \inf \Delta_B u$ for all smooth u.

Proposition 7.3. *Given a smoothly bounded domain $\Omega \subset\subset \mathbf{C}^n$ and $f \in C(\overline{\Omega})$, suppose that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\Delta v = f^{1/n}$. Suppose $u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ satisfies $(dd^c u)^n \geq C_n f\, d\lambda$. If $u \leq v$ on $b\Omega$ then $u \leq v$ on Ω.*

Proof. Suppose the assertion is false for some u and v. Then it remains false if we replace u with $u + \epsilon(\|z\|^2 - R^2)$ for $R > 0$ large enough and $\epsilon > 0$ small enough. That is, we may assume that u is strictly plurisubharmonic on Ω. Let $K \subset \Omega$ be the compact set of points where $u - v > 0$ is maximal. Let $U \subset \Omega$ be the open set of points where v is strictly plurisubharmonic. Then by Proposition 7.1 $(dd^c v)^n \leq C_n(\Delta v)^n\, d\lambda = C_n f d\lambda \leq (dd^c u)^n$ on U. If $K \subset U$, we contradict the comparison principle for the Monge-Ampere operator. So we can choose $z_0 \in K$ where v is not strictly plurisubharmonic. That is, there is a linear disk D centered at z_0 where $u - v$ is strictly subharmonic. But this contradicts maximality of $u - v$ at z_0. □

Proposition 7.4. *Suppose $u \in C(\Omega) \cap \mathrm{PSH}(\Omega)$ satisfies $(dd^c u)^n \geq C_n f\, d\lambda$ for some $f \in C(\Omega)$, and let $u * \rho$ be a standard smoothing of u. Then $\det^{1/n}(D\bar{D}(u * \rho)) \geq f^{1/n} * \rho$.*

Proof. We claim for any $A \in SL(n, \mathbf{C})$, that (the Borel measure) $\Delta_A u \geq f^{1/n}\, d\lambda$ on Ω. Granting this we have that

$$\det^{1/n}(D\bar{D}(u * \rho)) = \inf_A \Delta_A(u * \rho) \geq f^{1/n} * \rho$$

by linearity of Δ_A.

To prove the claim, we may change coordinates so that $\Delta_A = \Delta$ is the standard Laplacian. We fix $z \in \Omega$ and $r > 0$ small enough and choose $v \in C(\overline{B_r(z)})$ to satisfy $\Delta v = f^{1/n}$ and $v|_{bB_r(z)} = u|_{bB_r(z)}$. Then $u \leq v$ by Proposition 7.3. Since $(u - v)|_{bB_r(z)} \equiv 0$, it follows from Green's identity ($\int_{b\Omega} \frac{dh}{dn} = \int_\Omega \Delta h$) that $\int_{B_r(z)} \Delta(u - v) \geq 0$. Thus $\int_{B_r(z)} \Delta u \geq \int_{B_r(z)} f^{1/n}\, d\lambda$. Since z and r were arbitrary, we have $\Delta u \geq f^{1/n}(z)\, d\lambda$, as asserted. □

Theorem 7.5. *Suppose that $u, v \in \mathrm{PSH}(\Omega)$ satisfy $(dd^c u)^n \geq f\, d\lambda$ and $(dd^c v)^n \geq g\, d\lambda$, where $f, g \in C(\Omega)$. Then*

$$(dd^c(u + v))^n \geq (f^{1/n} + g^{1/n})^n\, d\lambda.$$

Proof. If u and v are smooth, then $(dd^c u)^n = C_n \det(D\bar{D}u)\, d\lambda$, so the Theorem follows directly from Proposition 7.2. In general we apply Propositions 7.2 and 7.4 to the standard regularizations u_j, v_j of u, v, obtaining

$$(dd^c(u_j + v_j))^n \geq (f^{1/n} * \rho_{1/j} + g^{1/n} * \rho_{1/j})^n\, d\lambda.$$

The left side tends weakly to $(dd^c(u + v))^n)$ and the right tends uniformly to $(f^{1/n} + g^{1/n})^n$ as $j \to \infty$, so the theorem follows. □

8. The dirichlet problem

We concern ourselves with the following situation. Let $\Omega \subset \mathbf{C}^n$ be a bounded and strictly pseudoconvex domain. That is, $\Omega = \{\rho < 0\}$, where ρ is a smooth real-valued function on a neighborhood $U \supset\supset \Omega$ satisfying $dd^c\rho \geq \omega_E$ (and therefore $(dd^c\rho)^n \geq d\lambda$). Let $\psi \in C(b\Omega)$ and $f \in C(\overline{\Omega})$ be real-valued functions and assume f is non-negative. Bedford and Taylor proved

Theorem 8.1. *There exists a unique function $u \in C(\overline{\Omega}) \cap \mathrm{PSH}(\Omega)$ satisfying $u|_{b\Omega} = \psi$ and $(dd^c u)^n = f\,d\lambda$.*

Here we present the proof of this theorem, which is rooted in the classical Perron method for constructing solutions of the Laplace equation. Before beginning to elaborate the argument, we make two comments. While the proof is somewhat involved, it only requires pluripotential theory results for *continuous* functions and in particular continuity of Monge-Ampere along *uniformly* convergent sequences. These things are much easier to prove and digest than the more delicate analogues for bounded psh functions and monotone convergence. Moreover, the original proof given by Bedford and Taylor used a construction due to Goffman and Perrin and rooted in Proposition 7.2 to give an alternative definition of the complex Monge-Ampere operator. Subsequent simplifications of the proof allow us to do without the Goffman-Perrin construction and rely completely on the definition of Monge-Ampere (initiated by Bedford-Taylor) that we have discussed above.

Definition 8.2. *We say that $v \in C(\overline{\Omega}) \cap \mathrm{PSH}(\Omega)$ is a* subsolution *for the data ψ, f if $v|_{b\Omega} \leq \psi$ and $(dd^c v)^n \geq f\,d\lambda$.*

We let $\mathcal{F}$ denote the family of all subsolutions and define $u(z) = \sup_{v\in\mathcal{F}} v(z)$ to be the upper envelope of $\mathcal{F}$. The first thing to note is

Proposition 8.3. *$\mathcal{F}$ is non-empty and uniformly bounded above. Hence the upper envelope u is well-defined at every point.*

Proof. Observe that $A\rho + \min\psi \in \mathcal{F}$ for $A >> 0$. So $\mathcal{F} \neq \emptyset$. We also have $v - \max\psi \leq 0$ on $b\Omega$ for all $v \in \mathcal{F}$, so by the maximum principle $v \leq \max\psi$ on all of $\overline{\Omega}$. □

Lemma 8.4. *Given $v_1, v_2 \in \mathcal{F}$, we have $\frac{v_1+v_2}{2} \in \mathcal{F}$ and $\max\{v_1, v_2\} \in \mathcal{F}$.*

Proof. In both cases, the condition on boundary values is clearly satisfied. The correct bound on Monge-Ampere for $\max\{v_1, v_2\}$ follows from

$$(dd^c \max\{v_1, v_2\})^n \geq (dd^c v_1)^n|_{\{v_1 \geq v_2\}} + (dd^c v_2)^n|_{\{v_1 < v_2\}},$$

which we proved in class. The Monge-Ampere bound for $\frac{1}{2}(v_1 + v_2)$ follows from Theorem 7.5. □

Lemma 8.5. *If $(\psi_j) \subset C(b\Omega)$ and $(f_j) \subset C(\overline{\Omega})$ are sequences converging uniformly to ψ, f, then the corresponding upper envelopes u_j converge uniformly to u.*

Proof. Observe that there is a constant $C > 0$ such that if $\|\psi - \psi_j\|, \|f - f_j\| < \epsilon$, then $v \in \mathcal{F}_j$ implies $\tilde{v} := v + C\epsilon^{1/n}(\rho - 1) \in \mathcal{F}$. Indeed $(dd\tilde{v})^n \geq (dd^c v)^n + C\epsilon(dd^c\rho)^n \geq (f_j + \epsilon)\,d\lambda$. Thus $u_j \leq u + C'\epsilon^{1/n}$. The reverse inequality holds for the same reason. So $u_j \to u$ uniformly. □

In light of Lemma 8.5 and continuity of Monge-Ampere along uniformly convergent sequences, we can proceed under the assumption that f and ψ are smooth functions. Taking advantage of this, we extend our given data and assume in what follows that f, ψ, ρ are all smooth and well-defined on an open set $U \supset\supset \overline{\Omega}$. This allows us to extend subsolutions to U, too.

Proposition 8.6. *There exists $A > 0$ such that for all $v \in \mathcal{F}$, the function*

$$\tilde{v}(z) := \begin{cases} \max\{A\rho(z) + \psi(z), v(z)\} & \text{if} \quad z \in \Omega, \\ A\rho(z) + \psi(z) & \text{if} \quad z \notin \Omega \end{cases}$$

belongs to $\mathcal{F} \cap \mathrm{PSH}(U)$ and satisfies $(dd^c\tilde{v})^n \geq f$ on all of U.

Proof. The main thing is to see that $\tilde{v}$ is plurisubharmonic at points $z_0 \in b\Omega$, i.e. that $\tilde{v}$ satisfies the subaveraging property on linear disks centered at z_0. But subaveraging on such disks already holds for v. Since $\tilde{v}(z) \geq v(z)$ with equality at z_0, subaveraging for $\tilde{v}$ follows immediately. □

Lemma 8.7. *For all $z_0 \in b\Omega$ we have $\lim_{z \to z_0} u(z) = \psi(z_0)$.*

Proof. We already know $u \geq A\rho + \psi$ for $A >> 0$. Taking any $v \in \mathcal{F}$ and applying the maximum principle to the plurisubharmonic function $v + A\rho - \psi$, we see that $u \leq \psi - A\rho$ also holds. □

Lemma 8.8. *u is continuous on $\overline{\Omega}$. In particular $u \in \mathrm{PSH}(\Omega)$. If $f^{1/n}$ is Lipschitz, so is u.*

Proof. Given $\epsilon > 0$, choose δ small enough that $\|z_1 - z_2\| < \delta$ implies $|\psi(z_1) - \psi(z_2)| < \epsilon$ and $|f^{1/n}(z_1) - f^{1/n}(z_2)| < \epsilon$ for all $z_1 \in \overline{\Omega}$. If $v = \tilde{v} \in \mathcal{F}$ satisfies the conclusions of Proposition 8.6, then for any $w \in B_\delta(0)$, we claim that

$$v_w(z) := v(z + w) + \epsilon(\rho - 1).$$

belongs to $\mathcal{F}$. That $v_w \leq \psi$ on $b\Omega$ is clear. From Theorem 7.5, we have $(dd^c v_w)^n \geq (f^{1/n}(z + w) + \epsilon)^n \, d\lambda \geq f(z) \, d\lambda$, which verifies our claim.

Now pick two points $z_1, z_2 = z_1 - w \in \Omega$ within distance δ of each other, and choose $v \in \mathcal{F}$ to satisfy $v(z_1) > u(z_1) - \epsilon$. Then if $M = \min_\Omega \rho$, we have

$$u(z_2) \geq v_w(z_2) = v(z_1) + \epsilon(\rho(z_2) - 1) \geq v(z_1) - M\epsilon \geq u(z_1) - (M + 1)\epsilon.$$

Reversing the roles of z_1 and z_2, we infer $|u(z_1) - u(z_2)| \leq (M + 1)\epsilon$ and continuity of u follows.

The second statement in the lemma follows from observing that if $f^{1/n}$ (and ψ) are Lipschitz, then we can take $\delta = C\epsilon$ in the argument above. □

Lemma 8.9. *There is a sequence $(u_j) \subset \mathcal{F}$ converging uniformly to u.*

Proof. Given $j > 0$, we note that the sets $U_v := \{u - v < 1/j\}$, $v \in \mathcal{F}$ form an open cover of $\overline{\Omega}$. By compactness Ω is covered by finitely many such sets $U_{v_1}, \ldots, U_{v_k}$. Then in fact $\overline{\Omega} \subset U_{u_j}$ where $u_j = \max\{v_1, \ldots, v_k\}$. This proves the lemma. □

Corollary 8.10. *$u \in \mathcal{F}$.*

Proof. The fact that $(dd^c u)^n \geq f\, d\lambda$ follows from the preceding lemma and continuity of Monge-Ampere under uniformly convergent sequences. All the other necessary properties of u have already been noted. $\square$

Corollary 8.11. *Theorem 8.1 holds in general provided it holds in the case where $\Omega = B_1(0)$ is the unit ball.*

Proof. For more general domains Ω, we complete the proof of Theorem 8.1 as follows. We already know that the upper envelope u is a subsolution agreeing with ψ on $b\Omega$. Given any ball $B_r(z_0) \subset \Omega$ we let v solve the Dirichlet problem on $B_r(z_0)$ with boundary values given by u and Monge-Ampere given by f. The function $\tilde{u}$ equal to u on $\Omega - B_r(z_0)$ and v on $B_r(z_0)$ is then a subsolution (see the proof of Proposition 8.6). Since $(dd^c\tilde{u})^n = f\, d\lambda \leq (dd^c u)^n$ on $B_r(z_0)$, it follows that $\tilde{u} \geq u$ on $B_r(z_0)$. But u is the upper envelope of subsolutions, so equality holds, and $(dd^c u)^n = f\, d\lambda$ as desired. $\square$

For the remainder of the argument, we assume implicitly that ψ and $f^{1/n}$ are smooth functions and that Ω is the unit ball. Note that if f vanishes somewhere, then smoothness of $f^{1/n}$ is stronger assumption than smoothness of f. The next step is arguably the central one in the argument. Refining the proof of continuity for u gives a one-sided bound for approximations of the second derivatives of u.

Lemma 8.12. *For any $r < 1$, there exists $C > 0$ (depending on r and on the C^2 norms of ψ and $f^{1/n}$) such that*

$$u(z+h) + u(z-h) - 2u(z) \leq C \left\|h\right\|^2.$$

for all $\|z\| < r$ and $\|h\| < \frac{1-r}{2}$.

Proof. We first claim that it suffices to assume that $z = 0$. To see this, suppose the lemma holds for $z = 0$ and consider some other point $z \in B_1(0)$. Since $\operatorname{Aut} B_1(0)$ is transitive, we can find $T \in \operatorname{Aut} B_1$ such that $T(0) = z$. We have $h = T'(0)\tilde{h}$ for some $\tilde{h} \in B_1$. Clearly $u \circ T$ is the upper envelope of subsolutions for the boundary data $\psi \circ T$ and inhomogeneity term $|\det T'|^2 f\, d\lambda$. Since u is Lipschitz on $\overline{B_1}$, our assumption gives us

$$u(z+h) + u(z-h) - 2u(z) = u \circ T(\tilde{h}) + u \circ T(-\tilde{h}) - 2u \circ T(0) + O(\left\|\tilde{h}\right\|^2) \leq C \left\|\tilde{h}\right\|^2, .$$

where C depends only on the C^2 norms of $\psi \circ T$ and $|\det T'|^2 f^{1/n}$. But for $z \in B_r$, $r < 1$, these are uniformly comparable to the C^2 norms of ψ and f, because the C^2 norms of T and T^{-1} are uniformly bounded above on $\overline{B_1(0)}$ indepedent of $z \in B_r$. Likewise $\left\|\tilde{h}\right\|$ is comparable to $\|h\|$, so our claim is proved. From now on we assume $z = 0$.

Note that by composing with a unitary transformation, we can further assume $h = (h_1, 0, \ldots, 0)$ for some $\delta > h_1 \geq 0$. Let $T_h \in \operatorname{Aut} B_1$ be given by

$$T_h(z) = T_h(z_1, z') = \left(\frac{z_1 + h_1}{1 + z_1\bar{h}_1}, \frac{\sqrt{1 - |h_1|^2}}{1 + z_1\bar{h}_1}\right),$$

Then $T_h(0) = h$, and one can verify the following estimates on T_h and its derivative.

- $T_h(z) = (1 - \bar{h}_1 z_1)z + h + O(|h|^2)$;
- $|\det T_h'(z)|^2 = 1 - 2(n+1)\operatorname{Re} z_1\bar{h}_1 + O(\|h\|^2)$.

The error terms are uniform for $z \in \overline{B_1}$. We then consider $v(z) = \frac{1}{2}(u(z+h) + u(z-h))$.

For $\|z\| = 1$, we have

$$\begin{aligned} v(z) &= \frac{1}{2}(\psi \circ T_h(z) + \psi \circ T_{-h}(z)) \\ &= \psi(z) + \psi'(z)(T_h(z) - z) + \psi'(z)(T_{-h}(z) - z) + O(\|T(z) - z\|^2) \\ &= \psi(z) + O(\|h\|^2). \end{aligned}$$

Note that in the last equality we are using our estimate above to infer that $T_h(z)+T_{-h}(z)-2z$ vanishes to second order in h. At any rate, our estimate shows that in terms of boundary values, v is within multiple of $\|h\|^2$ of being a subsolution.

To check the Monge-Ampere of v, we observe that approximating u with smooth psh functions gives

$$(dd^c(u \circ T_h))^n \geq (f \circ T_h)|\det T_h'|^{2n}\, d\lambda.$$

Hence by Theorem 7.5

$$\begin{aligned} (dd^c v)^n &\geq ((f^{1/n} \circ T_h)|\det T_h'|^2 + (f^{1/n} \circ T_{-h})|\det T_{-h}'|^2)^n \\ &= (f^{1/n}(z) + O(\|h\|^2))^n = f(z) + O(\|h\|^2). \end{aligned}$$

In light of these estimates, we may slightly modify v to get an actual subsolution

$$v(z) + C\,\|h\|^2\,(2 - \|z\|^2)$$

for our Dirichlet problem. Thus

$$v(z) - u(z) \leq C\,\|h\|^2$$

which, on setting $z = 0$, gives the bound we seek. □

Lemma 8.13. *The upper envelope u is $C^{1,1}$. The second partial derivatives of u therefore exist pointwise a.e. and are locally bounded functions agreeing with the corresponding distributional derivatives of u.*

Proof. Let $u_j = u \star \rho_{1/j} \in \mathrm{PSH}(B_{1-1/j})$ be a standard regularizing sequence for u. The inequality in Lemma 8.12 is preserved by convolution (with perhaps weaker constants) and therefore holds uniformly for u_j. Given $z \in B_1(0)$ and a unit vector $h \in \mathbf{C}^n$, we consider the subharmonic function $g(t) = u_j(z + th)$ for small $t = x + iy \in \mathbf{C}$.

$$g(t) + g(-t) - 2g(0) \leq C|t|^2.$$

for small enough $t \in \mathbf{C}$. Letting $t \to 0$ along real and imaginary directions gives us that $g_{xx}(0), g_{yy}(0) \leq C$, where the constant C is uniform in h and also for z in any compact subset of $B_0(1)$. But $g_{xx} + g_{yy} \geq 0$ by subharmonicity, so in fact $|g_{xx}(0)|, |g_{yy}(0)| \leq C$.

This implies that any second partial derivative of u_j is locally uniformly bounded in j. It follows that if D_k is any first partial derivative, then the sequence $(D_k u_j)$ is uniformly (in j) Lipschitz and therefore equicontinuous on compact sets. Since $D_k u_j \to D_k u$ distributionally, we may refine our regularizing sequence so that $D_k u_j \to D_k u$ uniformly locally. Hence $D_k u$ is Lipschitz. By the Rademacher theorem, the first partial derivatives of u are themselves differentiable a.e. and their derivatives are equal to the distributional second partial derivatives of u. Since $u_j, D_k u_j \to u, D_k u$ uniformly locally, we have $u_j \to u$ in C^1. Hence u is $C^{1,1}$. □

Thus the final step in establishing that u solves the Dirichlet problem is accomplished by

Lemma 8.14. *We have*

$$(dd^c u)^n = \frac{n!}{\pi^n} \det\left(\frac{\partial^2 u}{\partial z_k \partial \bar z_\ell}(z)\right) d\lambda \le f(z)\, d\lambda,$$

the (first) equality holding in the sense of measures and the (second) inequality holding pointwise a.e.

Proof. The first equality certainly holds for the regularizing sequence u_j. Since second partial derivatives of u are L^∞_{loc} functions, it follows (general facts about regularizing by convolution) that any second partial derivative of u_j converges pointwise a.e. to that of u. Hence $\det\left(\frac{\partial^2 u_j}{\partial z_k \partial \bar z_\ell}(z)\right)$ is a uniformly locally bounded sequence converging pointwise a.e. to $\det\left(\frac{\partial^2 u}{\partial z_k \partial \bar z_\ell}(z)\right)$. By the bounded convergence theorem, the convergence takes place in L^1. Since we also have $(dd^c u_j)^n \to (dd^c u)^n$ weakly, the first equality in the Lemma holds for u.

Now suppose to obtain a contradiction that $(dd^c u)^k(z) > f(z)\, d\lambda$ at some point $z \in \Omega$ where the first equality holds. By continuity of f, there exist $\delta > 0$ and $t < 1$ such that $t(dd^c u)^k(z) > f(z+h)\, d\lambda$ for all $h < |\delta|$. Taylor's theorem tells us that also

$$u(z+h) = u(z) + \operatorname{Re} P(h) + \sum_{k,\ell} \frac{\partial^2 u}{\partial z_k \partial \bar z_\ell} h_k \bar h_\ell + o(\|h\|^2),$$

where P is a holomorphic polynomial of degree two and, by supposition, $\left(\frac{\partial^2 u}{\partial z_k \partial \bar z_\ell}(z)\right)$ is positive definite. Thus we further have, on shrinking δ, that

$$u(z+h) > v(z+h) := u(z) + \operatorname{Re} P(h) + t \sum_{j,k} u_{j\bar k} h_j \bar h_k + \epsilon$$

for some $\epsilon > 0$ and all $|h| = \delta$. It follows that the function

$$\tilde u(w) := \begin{cases} u(w) & \text{if} \quad |w - z| > \delta \\ \max\{u(w), v(w) + \epsilon\} & \text{if} \quad |w - z| \le \delta \end{cases}$$

is a subsolution. Since $\tilde u(z) = u(z) + \epsilon > u(z)$, we have our contradiction. We conclude that $(dd^c u)^k(z) \le f(z)\, d\lambda$ for a.e. $z \in \Omega$. □

9. Maximal plurisubharmonic functions

Definition 9.1. *We call $u \in \mathrm{PSH}(\Omega)$ maximal if for any open $U \subset \Omega$, compact $K \subset U$, and $v \in \mathrm{PSH}(U)$ we have that $v \le u$ on $U - K$ implies $v \le u$ on U.*

Note that if u is maximal in Ω, $U \subset\subset \Omega$ is open and $v \in \mathrm{PSH}(U)$ satisfies $\limsup_{z \to z_0} v(z) \le u(z)$ for all $z_0 \in bU$, then $v \le u$ on U. This follows from the fact that v is upper semicontinuous and therefore $u + \epsilon - v > 0$ near bU for all $\epsilon > 0$.

We remark that pluriharmonic functions are maximal but not vice versa. In fact maximal plurisubharmonic functions are not necessarily even continuous. Nevertheless Theorem 8.1 allows us to characterize maximality for bounded psh functions in PDE terms.

Theorem 9.2. *$u \in \mathrm{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ is maximal if and only if $(dd^c u)^n \equiv 0$.*

Proof. Suppose $(dd^c u)^n \equiv 0$ on Ω. Let $U \subset \Omega$, $K \subset\subset U$ and $v \in \mathrm{PSH}(U)$ be as in the definition of maximality. Shrinking U so that $\overline{U} \subset \Omega$, we can replace v with $\max\{v, -M\}$ where $-M$ is a lower bound for u on $\overline{U}$. So without loss of generality v is locally bounded on U. It follows immediately from the comparison principle that $v \leq u$ on all of U. So u is maximal.

Now begin again, suppposing u is maximal. Let U be a ball with $\overline{U} \subset \Omega$. By upper semicontinuity of u, we can choose a sequence of smooth functions ψ_j decreasing to $u|_{bU}$. From Theorem 8.1 we have $u_j \in C(\overline{U}) \cap \mathrm{PSH}(U)$ satisfying $(dd^c u_j)^n \equiv 0$ on U and $u_j|_{bU} \equiv \psi_j$. Then $v := \lim u_j$ is plurisubharmonic and locally bounded on U. By continuity of Monge-Ampere along decreasing sequences $(dd^c v)^n \equiv 0$. So from the previous paragraph we infer that v is maximal on U. Finally, we have $\lim_{z \to z_0} v(z) - u(z) = 0$ for all $z_0 \in bU$. So in fact $v \equiv u$ on U, since both functions are maximal. In particular $(dd^c u)^n \equiv 0$. $\square$