Exponential of a sum of matrices

I'm going to do this a little differently than I did in class. It brings out the role of commutativity more clearly.

**Theorem 0.1.** Let \( A, B \in M_{n \times n}(\mathbb{C}) \) be commuting (real or) complex matrices. Then \( e^{A+B} = e^A e^B \).

**Proof.** I begin by observing that \( B e^A = \sum_{j=0}^{\infty} \frac{B^j A^j}{j!} = \sum_{j=0}^{\infty} \frac{A^j}{j!} = e^A B. \)

Let \( v \in \mathbb{C}^n \) be any vector and define \( y(t) = e^{(A+B)t} v, \) \( z(t) = e^{At} e^{Bt} v. \) Then \( y(0) = z(0) = v \) and \( y'(t) = (A + B)e^{(A+B)t} v = (A + B)y(t). \) From the product rule and my initial observation, I further find
\[
z'(t) = Ae^{At} e^{Bt} v + e^{At} Be^{Bt} v = Ae^{At} e^{Bt} v + Be^{At} e^{Bt} v = (A + B)z(t).
\]
Hence \( y \) and \( z \) solve exactly the same initial value problem. By uniqueness of solutions, I conclude \( y(t) = z(t) \) for all \( t \in \mathbb{R}. \) Taking \( t = 1 \) concludes the proof. \( \square \)

Now here's a point I didn't make in class. The above theorem allows to give elementary instances of subspaces of \( n \times n \) matrices that exponentiate to (multiplicative) subgroups.

**Corollary 0.2.** Suppose that \( H \subset M_{n \times n}(\mathbb{C}) \) is a subspace consisting entirely of matrices that commute with each other. Then
\[
e^H := \{ e^A : A \in H \}
\]
is an abelian subgroup of the (multiplicative) group \( GL(n, \mathbb{C}) \) of invertible \( n \times n \) matrices. That is,
- any \( M \in e^H \) is invertible with \( M^{-1} \in e^H; \)
- given \( M_1, M_2 \in e^H, \) we also have \( M_1 M_2 = M_2 M_1 \in e^H. \)

**Proof.** Given \( A \in H, \) I note that \( -A \in H \) always commutes with \( A. \) Hence \( e^A e^{-A} = e^{A-A} = e^0 = I. \) Hence \( e^A \) is invertible with inverse in \( e^{-A} \in e^H. \)

Given \( A_1, A_2 \in H, \) I have \( A_1 + A_2 \in H. \) Since \( A_1 \) and \( A_2 \) commute by hypothesis, I also have \( e^{A_1} e^{A_2} = e^{A_1 + A_2} \in e^H. \) \( \square \)

Given any fixed \( n \times n \) matrix \( A, \) the subspace \( H \) spanned by \( I, A, A^2, \ldots \) the non-negative powers of \( A \) satisfies the hypothesis of this theorem. Indeed, elements of \( H \) all have the form \( p(A) \) for some polynomial \( p(x), \) and I observed last term that for any two polynomials \( p, q, \) we have \( p(A)q(A) = (pq)(A) = (qp)(A) = q(A)p(A). \) So any two elements of \( H \) commute. The Cayley-Hamilton theorem says \( p(A) = 0 \) when \( p \) is the characteristic polynomial of \( A, \) which implies that \( \dim H \) cannot exceed \( n = \deg p \) even though the full vector space of \( n \times n \) matrices has dimension \( n^2. \) As it turns out, no subspace of \( M_{n \times n} \) that consists of mutually commuting matrices can have dimension larger that \( n. \)

A further interesting problem is to determine which subspaces of \( M_{n \times n} \) exponentiate to “reasonable” (and possibly non-abelian) subgroups. It turns out that the most important condition here is that for all \( A, B \in H \) one has \( AB - BA \in H \)—i.e. \( H \) is closed with respect to commutators. This is the beginning of the story of the correspondence between Lie groups and Lie algebras.