6 Sets and relations

A *set*, which is nothing more than a collection of objects, is one of the most basic notions in mathematics. The objects belonging to the set are called its *elements*. We write '$x \in A$' to indicate that x is an element of A.

The most basic of all sets is the *empty set* $\emptyset$. That is, $\emptyset$ is the unique set which contains no elements. The following definition presents a variety of other basic terminology connected with sets.

Definition 6.1 *Let A and B be sets.*

- *The* union *of A and B is the set*

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

- *The* intersection *of A and B is the set*

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

- *The* difference *between A and B is the set*

$$A - B := \{x \in A : x \notin B\}.$$

- *B is a* subset *of A if every element of B is also an element of A. When B is a subset of A, we call $A - B$ the* complement *of B in A, and when the set A can be understood from context, we write B^c for $A - B$.*

- *We say that A is a* subset *of B if for every $x \in A$, we also have $x \in B$. In this case, we write $A \subset B$.*

- *We say that $A = B$ if $A \subset B$ and $B \subset A$.*

- *We say that A and B are* disjoint *if $A \cap B = \emptyset$.*

Many assertions in mathematics boil down to statements about the relationship between two sets. For instance, the assertion *the solutions of $x^2 = 1$ are 1 and -1* can be rephrased as an equality between two sets

$$\{x \in \mathbf{R} : x^2 = 1\} = \{-1, 1\}.$$

Proving that two sets are equal, or that one is a subset of another is therefore an important skill. Fortunately, it's not a difficult one as long as you remember what you're up to. Let us give an example here.

Proposition 6.2 *For any sets A, B, C, we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Before beginning, we point out the basic strategy. By definition, showing two sets are equal means showing that one is a subset of the other and vice versa. And to show that one set is a subset of another, we must show that any element in the first is an element of the second.

Proof. To show that the left set is a subset of the right, let $x \in A \cap (B \cup C)$ be given. Then on the one hand $x \in A$, and on the other hand $x \in B$ or $x \in C$. If $x \in B$, then it follows that $x \in A \cap B$. Likewise, if $x \in C$, then it follows that $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. This proves

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C).$$

To show the right set is a subset of the left set, let $x \in (A \cap B) \cup (A \cap C)$ be given. Then either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so certainly $x \in B \cup C$, too. Hence $x \in A \cap (B \cup C)$. If, on the other hand, $x \in A \cap C$, then we similarly see that $x \in A \cap (B \cup C)$. So in either case, we see that $x \in A \cap (B \cup C)$. This proves

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C).$$

Putting the results together, we conclude that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

□

There is one other way to combine two sets. In some sense, it's the *largest* possible way to combine two sets.

Definition 6.3 *The* cartesion product *of two sets A and B is the set*

$$A \times B := \{(a, b) : a \in A, b \in B\}$$

comprising all ordered pairs whose first element lies in A and whose second element lies in B.

So if A is the set of all U.S. presidents and B is the set of all species of trees, then (Woodrow Wilson, weeping willow) is an example of an element of $A \times B$. Any kind of 'connection' between the elements of A set with the elements of B can be described as a subset of $A \times B$.

Definition 6.4 *A subset $R \subset A \times B$ is called a* relation *from A to B. If $A = B$, then we say simply that R is a relation on A.*

So if A is the set of all readers of these notes and B is the set of all flavors of ice cream, then

$$R = \{(a, b) \in A \times B : a \text{ likes } b\text{-flavored ice cream}\}$$

is a relation from A to B. One element in R is $(\text{Diller}, \text{strawberry})$. This is not the only element in R, since the author of these notes enjoys several flavors of ice cream. However, (Diller, Chunky Monkey) is certainly not in R, even though it is a well-documented element of $A \times B$.

An example of a relation on $\mathbf{Z}$ is the *order* relation

$$R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : a < b\}.$$

So $a < b$ means exactly the same thing as $(a, b) \in R$. In fact, one often writes aRb ('a is related to b') instead of $(a, b) \in R$ ('(a, b) belongs to R'), but keep in mind that the two pieces of notation mean exactly the same thing. Concerning the example in the previous paragraph, I might equally well have said *Diller R strawberry* (or better yet, *Diller ♡ strawberry*!)

Definition 6.5 *A relation R on a set A is called*

- Reflexive *if xRx for every $x \in A$;*
- Symmetric *if xRy implies yRx for every $x, y \in A$;*
- Transitive *if xRy and yRz imply that xRz for every $x, y, z \in A$.*

We call R an equivalence relation *if R enjoys all three of these properties.*

So the order relation $<$ is transitive but not symmetric or reflexive. In particular, it is not an equivalence relation. Consider on the other hand the following relation on the set A of all people

$$R = \{(x, y) \in A \times A : x \text{ and } y \text{ have the same gender}\}.$$

Then R is certainly reflexive, symmetric, and transitive. Hence R is an equivalence relation. More generally and speaking loosely, an equivalence relation on a set A is a relation that ties together elements that have some property in common.

Definition 6.6 *Let R be an equivalence relation on a set A and $x \in A$ be any element. The* equivalence class of *x is the set*

$$[x] = \{y \in A : xRy\}.$$

In the preceding example, there are only two different equivalence classes: the set of all men, and the set of all women.

Theorem 6.7 *Let R be an equivalence relation on a set A. Then each $x \in A$ belongs to its own equivalence class $[x]$, and if $y \in A$ is another element, we have either*

- *xRy, in which case $[x] = [y]$; or*
- *R does not relate x and y, in which case $[x] \cap [y] = \emptyset$.*

Proof. Let $x \in A$ be given. Then xRx because R is reflexive. Hence $x \in [x]$.

Now let $y \in A$ be another element. Suppose first that xRy. I must show that $[x] = [y]$. To do this, let $z \in [y]$ be any element. Then yRz by definition of equivalence class. Since R is transitive and we are assuming that xRy, it follows that xRz. Hence $z \in [x]$. This proves that $[y] \subset [x]$. To prove that $[x] \subset [y]$, I note that by symmetry of R, xRy implies that yRx. So if $z \in [y]$, I can repeat the previous argument with the roles of x and y reversed, to conclude that $[x] \subset [y]$. I conclude that $[x] = [y]$.

It remains to consider the case where x and y are not related by R. In this case, I must show that $[x] \cap [y] = \emptyset$. Suppose in order to get a contradiction that $z \in [x] \cap [y]$. Then by definition of equivalence class, xRz and yRz. Since R is symmetric, it follows that zRy, and since R is transitive it further follows that xRy, contradicting the fact that x and y are not related. I conclude that there is no element z in $[x] \cap [y]$. $\square$