Review Sheet for 20860 final

Standard disclaimer: The following represents a sincere effort to help you prepare for our exam. It is not guaranteed to be perfect. There might well be minor errors or (especially) omissions. These will not, however, absolve you of the responsibility to be fully prepared for the exam. If you suspect a problem with this review sheet, please bring it to my attention.

Format: The exam will take place Wednesday 5/7 from 4:15-6:15 PM in DBRT 316. I'll hold a review session Monday 3/17 from 3-4(ish) PM in 117 HH. The exam will be comprehensive with greater stress on material not covered on the midterm. In particular, I plan to ask several questions about differential forms/Stokes Theorem, so it would behoove you to work through and understand the problems on the last (i.e. uncollected) homework assignment. Types of exam questions will be similar to those on the midterm: short answer questions of the 'state the following definition/theorem/etc' variety; computational and proof questions with longer solutions. No true/false questions.

Definitions and Statements. Here is a brief glossary of definitions and statements that I will expect you to know. While there aren't many entries, most are substantial, and several are from last semester! These are things that I consider quite central to vector calculus, and anyhow, I'll ask you reproduce many of them on the final exam. Note that in statements that are local in nature, i.e. statements which concern only the behavior of a function (say f) near some given point (say p), I have simplified the statement so that the domain of f is all of \mathbb{R}^n rather than a neighborhood U of p.

• Definition. A mapping $F : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable at a point $p \in U$ if there is a linear transformation $Df(p) : \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h \to 0} \frac{\|F(p+h) - F(p) - T(h)\|}{\|h\|} = 0.$$

We then call the transformation DF(p) the *derivative* of F at p.

• Chain Rule. Let $F : \mathbf{R}^n \to \mathbf{R}^m$, $G : \mathbf{R}^m \to \mathbf{R}^\ell$ be differentiable mappings. Then $G \circ F : \mathbf{R}^n \to \mathbf{R}^\ell$ is a also a differentiable mapping with derivative given by

$$D(G \circ F)(p) = DG(F(p)) \circ DF(p).$$

for any $p \in \mathbf{R}^n$.

- Inverse Function Theorem. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping. Suppose that $p \in \mathbb{R}^n$ is a point at which the derivative DF(p) is invertible. Then there exist neighborhoods V of p and W of F(p) such that $F : V \to W$ is invertible and the inverse function $F^{-1} : W \to V$ is C^1 with derivative $DF^{-1}(F(p)) = DF(p)^{-1}$.
- Taylor's Theorem (2nd order version). Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^2 scalar-valued function and $p \in \mathbf{R}^n$ be a given point. Then for any displacement $h \in \mathbf{R}^n$ we have

$$f(p+h) = f(p) + Df(p)h + \frac{1}{2}h^{T}H(p)h + E(h),$$

where $H(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ is the Hessian matrix at p and the error term E satisfies

$$\lim_{h \to 0} \frac{\|E(h)\|}{\|h\|^2} = 0.$$

- Second Derivative Test. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a C^2 scalar-valued function and $p \in \mathbf{R}^n$ is a critical point of f. Let $H := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ be the Hessian matrix at p. Then
 - if H is positive definite, then f has a local minimum at p;
 - if H is negative definite, then f has a local maximum at p; and
 - if H is strictly indefinite, then f does not have a local maximum or minimum at p.
- Definition. Let $I \subset \mathbf{R}^n$ be a rectangl and $f : I \to \mathbf{R}$ a bounded, scalar-value function. The *upper integral* of f on I is the quantity

$$\overline{\int_{I}} f := \inf\{\int \tau : \tau \text{ is step-function on } I \text{ such that } \tau \ge f\}.$$

The *lower integral* of f on I is the quantity

$$\underline{\int_{I}} f := \sup\{\int \sigma : \sigma \text{ is step-function on } I \text{ such that } \sigma \leq f\}.$$

If $\overline{\int_I} f = \underline{\int_I} f$, then we say that f is *Riemann integrable* on I and denote both lower and upper integrals by $\int_I f$.

• Change of Variables Theorem Let $U, V \subset \mathbb{R}^n$ be open sets and $\Phi : U \to V$ be a diffeomorphism. Then for any bounded function $f : V \to \mathbb{R}$, we have

$$\int_{U} (f \circ \Phi) |DetJ\Phi| = \int_{V} f$$

(provided the Riemann integral on each side is actually defined).

• Definition. A linear k-form on \mathbf{R}^n is a function $\omega : (\mathbf{R}^n)^k \to \mathbf{R}$ that is multilinear and alternating. That is, $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is linear in each argument \mathbf{v}_j ; and

 $\omega(\ldots,\mathbf{v}_i,\ldots,\mathbf{v}_j,\ldots)=-\omega(\ldots,\mathbf{v}_j,\ldots,\mathbf{v}_i,\ldots)$

for any $1 \leq i < j \leq n$.

• Stokes Theorem. Let $M \subset \mathbf{R}^n$ be a compact oriented manifold with boundary and ω be a C^1 k - 1-form \mathbf{R}^n of M. Then

$$\int_{\partial M} \omega = \int_M d\omega.$$

Concerning computational problems: Your homework problems will be a good guide. Make sure you understand the solutions to these. Solutions through homework 7 are currently posted on the course webpage.