Integrating over Curves

(*by Christian Gorski*)

Definition 1. *A **curve** is a continuous function $\gamma : [a, b] \to \mathbb{R}^n$ defined on a closed interval $[a, b] \subset \mathbb{R}$.*

- *If the derivative γ' of γ exists and is continuous, then we say that γ is $\mathcal{C}^1$. We may refer to $\gamma'(t)$ as the **tangent vector** to γ at a the point $\gamma(t)$.*
- *γ is **piecewise $\mathcal{C}^1$** if there exist finitely many parameters $a = a_0 < a_1 < \cdots < a_k = b \in \mathbb{R}$ such that γ is $\mathcal{C}^1$ on $[a_{i-1}, a_i]$ for all $1 \leq i \leq k$.*
- *γ is **simple** if $(\gamma(s) = \gamma(t)) \Rightarrow (s = t$ or $\{s, t\} = \{a, b\})$ (that is, γ is one-to-one, except that its endpoints may be equal).*
- *γ is **closed** if $\gamma(a) = \gamma(b)$.*

Definition 2. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ curve. We say $\tilde{\gamma}$ is a **reparametrization of** γ if there exists a diffeomorphism $\Phi : [c, d] \to [a, b]$ such that $\tilde{\gamma} = \gamma \circ \Phi$. We say that Φ is **orientation-preserving** if it is increasing, and we say that it is **orientation reversing** if it is decreasing.*

Now we introduce the first of two ways of integrating over curves: integration with respect to arc-length.

Definition 3. *Given a $\mathcal{C}^1$ curve*

$$\gamma : [a, b] \to \mathbb{R}^n$$

and a continuous function

$$f : \mathbb{R}^n \to \mathbb{R},$$

*the **integral of f with respect to arc-length** on γ is the quantity*

$$\int_\gamma f ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt.$$

Remark. Recall that

$$\text{Length}(\gamma) = \int_a^b \|\gamma'(t)\| dt = \int_C 1 ds.$$

Theorem 4. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ curve and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Let $\tilde{\gamma}$ be a reparametrization of γ. Then*

$$\int_\gamma f ds = \int_{\tilde{\gamma}} f ds$$

Note that the arc-length integral is the same, regardless of whether the reparametrization reverses or preserves orientation. This will not be the case for the next type of integration we consider.

Before we introduce the second (and more important) type of integration, we must introduce the concept of a covector.

Covectors

Definition 5. *A* ***covector*** *on* $\mathbb{R}^n$ *is a linear function* $\omega : \mathbb{R}^n \to \mathbb{R}$.

Suppose $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ is a vector. Then for each $1 \leq j \leq n$ we have a covector $dx_j : \mathbb{R}^n \to \mathbb{R}$ given by

$$dx_j(x_1, \ldots, x_n) = x_j$$

We will call $dx_1, \ldots, dx_n$ the "standard basis covectors."

The set of all covectors $\omega : \mathbb{R}^n \to \mathbb{R}$ is denoted $(\mathbb{R}^n)^*$.

Proposition 6. *Any covector* $\omega : \mathbb{R}^n \to \mathbb{R}$ *may be written uniquely as*

$$\omega = \sum_{i=1}^{n} \omega_i dx_i$$

where each $\omega_i \in \mathbb{R}$ *is a scalar.*

Remark.

- Covectors are also called "linear forms" or "linear functionals."
- For $\omega : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \in \mathbb{R}^n$ one often writes $\langle \omega | \mathbf{x} \rangle := \omega(\mathbf{x})$.
- We can alternatively think of a covector $\omega = \sum \omega_i dx_i : \mathbb{R}^n \to \mathbb{R}$ as the $1 \times n$ matrix $[\omega_1, \ldots, \omega_n]$ (a "row vector").
- Using the dot product can confuse a covector with a vector;

$$\omega(\mathbf{x}) = \sum \omega_i x_i = \left[\begin{array}{ccc} \omega_1 & \cdots & \omega_n \end{array} \right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right] = \left[\begin{array}{c} \omega_1 \\ \vdots \\ \omega_n \end{array} \right] \cdot \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right]$$

Definition 7. *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation and* $\omega : \mathbb{R}^m \to \mathbb{R}$ *be a covector. Then the* ***pullback*** *of* ω *is the covector* $T^* : \mathbb{R}^n \to \mathbb{R}$ *given by*

$$T^*(\omega)(\mathbf{x}) = \omega(T(\mathbf{x}))$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Line Integrals

Definition 8. *Let* $U \subset \mathbb{R}^n$ *be open. A* ***differential 1-form*** *is a function*

$$\omega : U \to (\mathbb{R}^n)^*.$$

Remark. More concretely, $\forall \mathbf{x} \in U$,

$$\omega(\mathbf{x}) = \sum \omega_i(\mathbf{x}) dx_i$$

where $\omega_i : U \to \mathbf{R}$ are scalar-valued functions. We will often refer to differential 1-forms as simply 1-forms. We will always assume ω is continuous(i.e. that each ω_i is continuous), and often assume that ω is $\mathcal{C}^1$ (i.e. that each ω_i is $\mathcal{C}^1$).

Definition 9. *Let $U \subset \mathbf{R}^n$ be open, ω be a continuous 1-form on U and $\gamma : [a,b] \to U$ be a $\mathcal{C}^1$ parametrized curve. Then **the line integral of** ω **over** γ is the quantity*

$$\int_\gamma \omega := \int_a^b \langle \omega(\gamma(t)), \gamma'(t) \rangle dt$$

Theorem 10. *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ curve and $\tilde{\gamma} : [c,d] \to \mathbb{R}^n$ be a reparametrization. Then, for any continuous 1-form ω on $\mathbb{R}^n$, we have*

$$\int_{\tilde{\gamma}} \omega = \int_\gamma \omega$$

if $\tilde{\gamma}$ preserves orientation, and

$$\int_{\tilde{\gamma}} \omega = -\int_\gamma \omega$$

if $\tilde{\gamma}$ reverses orientation.

Note that, unlike in the case of arc length integrals, the orientation of the parametrization *does* matter to the value of the integral.

Closed and Exact 1-Forms

Definition 11. *If $f : U \to \mathbb{R}$ is a $\mathcal{C}^1$ function on an open set $U \subset \mathbb{R}^n$, then the 1-form*

$$df := \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

*is called the **differential** of f.*

Definition 12. *A 1-form $\omega : U \to (\mathbb{R}^n)^*$ on an open set $U \subset \mathbb{R}^n$ is called **exact** if there exists a $\mathcal{C}^1$ function $f : U \to \mathbb{R}$ such that $\omega = df$. In this case, the function f is called a **potential** for ω.*

Note that potentials are not unique; in fact, given a potential function f for some 1-form ω, we can obtain another potential $\tilde{f}$ for ω by simply adding a constant—i.e. by defining

$$\tilde{f}(x) = f(x) + c$$

for some constant $c \in \mathbb{R}$.

Theorem 13. *(The Fundamental Theorem of Calculus for Curves) Let $U \subset \mathbb{R}^n$ be an open set, and let $[a,b] \subset \mathbb{R}$ be a compact interval. If $\gamma : [a,b] \to U$ is a $\mathcal{C}^1$ curve and $f : U \to \mathbb{R}$ is a $\mathcal{C}^1$ function, then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Note that this theorem implies that the line integral of a differential over a curve is *path-independent*; that is, the integral depends only on the endpoints and orientation of the curve, and not the path itself.

Note also that this theorem implies that the integral of an exact 1-form over a closed curve is zero. In fact, we have the following set of equivalences:

Theorem 14. *Let ω be a continuous 1-form defined on an open set $U \subset \mathbb{R}^n$. Then the following statements are equivalent:*

1. *ω is exact.*
2. *$\int_\gamma \omega$ is path independent for any curve γ.*
3. *$\int_\gamma \omega = 0$ for any closed curve γ.*

A related notion is that of a *closed* 1-form:

Definition 15. *A $\mathcal{C}^1$ 1-form $\omega = \sum \omega_i dx_i$ defined on $U \subset \mathbb{R}^n$ is **closed** if*

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i} \text{ for all } 1 \leq i, j \leq n.$$

Note that, since partial derivatives commute, a C^1 and exact 1-form is always closed. The next theorem gives a condition under which exactness is equivalent to being closed; to state it, we need one more definition:

Definition 16. *A subset $U \subset \mathbb{R}^n$ is **star-shaped** if there exists $\mathbf{x} \in U$ such that, for any $\mathbf{y} \in U$, the line segment from $\mathbf{x}$ to $\mathbf{y}$ is contained in U.*

Theorem 17. *Let $U \subset \mathbb{R}^n$ be star-shaped, and let ω be a $\mathcal{C}^1$ 1-form on U. Then ω is exact if and only if ω is closed.*

$\mathbb{R}^n$, rectangles and convex sets are examples of star-shaped sets. However, the set $\mathbb{R}^2 - \{\mathbf{0}\}$ is not star-shaped; this allows for the interesting 1-form utilized in the next definition:

Definition 18. *Let $\gamma : [a,b] \to \mathbb{R}^2 - \{\mathbf{0}\}$ be a $\mathcal{C}^1$ closed curve, and set*

$$\omega = \frac{-y dx + x dy}{x^2 + y^2}.$$

*Then the **winding number** (about the origin) of γ is the quantity $\frac{1}{2\pi} \int_\gamma \omega$.*

Note that ω is closed, but not exact, and so can have a nonzero line integral over a closed curve. As we saw in class, the 1-form ω measures the angle a curve turns through; the winding number is an integer which gives the number of times the curve winds about the origin.

Pullbacks of 1-forms

Definition 19. *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^n$ be open. Given a $\mathcal{C}^1$ function $\Phi : U \to V$ and a 1-form $\omega = \sum_{i=1}^n \omega_i d_i$ on V, the* ***pullback*** *$\Phi^*\omega$ of ω by Φ is the 1-form on U given by*

$$\Phi^*\omega := \sum_{i=1}^{n} (\omega_i \circ \Phi) d\Phi_i$$

where Φ_i is the i^{th} component of Φ.

Suppose that $\gamma : [a, b] \to \mathbb{R}^n$ is a $\mathcal{C}^1$ curve and $\omega = \sum \omega_i$ is a $\mathcal{C}^1$ 1-form on $\mathbb{R}^n$. Then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

and so

$$\gamma^*\omega = \sum_i (\omega_i \circ \gamma) d\gamma_i = \sum_i (\omega_i \circ \gamma)\gamma_i'(t)dt = \langle \omega(\gamma(t)), \gamma'(t) \rangle dt$$

and therefore $\int_\gamma \omega = \int_a^b \gamma^*\omega$.

Green's Theorem

Definition 20. *Let $\omega = \omega_x dx + \omega_y dy$ be a $\mathcal{C}^1$ 1-form on an open subset U of $\mathbb{R}^2$. Then the* ***exterior derivative*** *of ω is the expression*

$$d\omega := \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dxdy.$$

Theorem 21. *(Green's Theorem) Let $\Omega \subset \mathbb{R}^2$ be a region such that $\partial\Omega$ is a finite union of piecewise $\mathcal{C}^1$ simple closed curves; let ω be a $\mathcal{C}^1$ 1-form on an open set $U \supset \Omega$. Then*

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega$$

where $\partial\Omega$ is oriented positively relative to ω—i.e. each curve in $\partial\Omega$ is parametrized so that Ω lies counterclockwise from (i.e. to the left of) the parametrization's tangent vector at each point in the curve.

Cross Products

Now we turn to the task of defining the integral over a surface. Our first definition will be analogous to the arc-length integral over a curve, and will be based on the idea of surface area. However, to develop a concept of surface area, we must first define the cross product.

Definition 22. *The* ***cross product*** *of two vectors* $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ *is the vector in* $\mathbb{R}^3$ *given by*

$$\mathbf{x} \times \mathbf{y} := \mathbf{e_1} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \mathbf{e_2} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \mathbf{e_3} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Remark. Consider the formal determinant

$$\begin{vmatrix} \mathbf{e_1} & x_1 & y_1 \\ \mathbf{e_2} & x_2 & y_2 \\ \mathbf{e_3} & x_3 & y_3 \end{vmatrix}.$$

Although this is not the determinant of a real matrix, it is a helpful mnemonic device; if we perform "cofactor expansion" on the first column, we obtain the formula for the cross product of two vectors $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$.

Proposition 23. *(Basic Properties of Cross Products) Given* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, c \in \mathbb{R}$, *we have:*

1. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$
2. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
3. $\mathbf{v} \times \mathbf{w} \perp \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$
4. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
5. $(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$
6. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} + \mathbf{v}) \times (\mathbf{u} + \mathbf{w})$
7. $\|\mathbf{v} \times \mathbf{w}\|^2 + (\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$

Remark. In the above proposition, (2) - (6) follow fairly directly from (1). (7) can be shown by brute force calculation, but its geometrical motivation will be discussed shortly.

Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is the three-dimensional area of the parallelotope determined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$. If $\mathbf{u}$ is a unit vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$, then the parallelotope is a prism whose base is the 2-dimensional parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ and whose height is 1. We would then expect the volume of this parallelotope to be the area of the base parallelogram. But since $\mathbf{v} \times \mathbf{w}$ is also orthogonal to $\mathbf{v}$ and $\mathbf{w}$, it is parallel to the unit vector $\mathbf{u}$, and we have

$$\det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \|\mathbf{u}\| \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v} \times \mathbf{w}\|.$$

This motivates the following definition:

Definition 24. *The two-dimensional area of the parallelogram* $P(\mathbf{v}, \mathbf{w})$ *determined by the vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *is the quantity* $\|\mathbf{v} \times \mathbf{w}\|$.

Remark. Recall that we defined the cosine of the angle θ between two vectors $\mathbf{v}$ and $\mathbf{w}$ by the equation

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$

Moreover, we know from geometry that the area of the parallelogram spanned by $\mathbf{v}, \mathbf{w}$ should be $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$; combining with Definition 24, we should then have

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta.$$

Substituting these equations into (7) from Proposition 23, we get

$$\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \sin^2\theta + \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \cos^2\theta = \|\mathbf{v}\|^2\|\mathbf{w}\|^2,$$

which is consistent with a familiar and fundamental trigonometric identity. This justifies our defining $\sin\theta$ and $\cos\theta$ in this way.

Now that we have a convenient formula for the 2-dimensional area of a parallelogram in $\mathbb{R}^3$, we can define the first notion of integration over a surface.

Integrating over Surfaces

Definition 25. *Let $U \subset \mathbb{R}^2$ be open and let $G : U \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function such that the following hold:*

- *G is one-to-one (i.e. $G(a) = G(b) \Rightarrow a = b \; \forall a, b \in U$).*
- *For all $a \in U$, $DG(a)$ has rank 2 (i.e. the columns of $DG(a)$ are linearly independent).*

Then the image $G(U) \subset \mathbb{R}^n$ is called a ***(smooth) parametrized surface****. If $\tilde{U} \in \mathbb{R}^2$ is another open set and $\Phi : \tilde{U} \to U$ is a diffeomorphism, then we call $\tilde{G} = G \circ \Phi : \tilde{U} \to \mathbb{R}^3$ a* ***reparametrization*** *of $G(U)$.*

Remark. For the purposes of this class, we will only consider surfaces $G(U) \subset \mathbb{R}^3$.

Definition 26. *Let $U \subset \mathbb{R}^2$ be open, and let $G : U \to \mathbb{R}^3, (u, v) \mapsto (x, y, z)$ be a parametrized surface. If $\Omega \subset U$ is a region, we define the* ***surface area*** *of $G(\Omega)$ to be the quantity*

$$\int_{G(\Omega)} dA := \int_\Omega \left\| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right\| dV_2.$$

Moreover, if $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous, then we define the ***integral of f over Ω with respect to surface area*** *to be the quantity*

$$\int_{G(\Omega)} f dA := \int_\Omega f \circ G \left\| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right\| dV_2.$$

Theorem 27. *Let $U, \tilde{U} \subset \mathbb{R}^2$ be open, with a diffeomorphism $\Phi : \tilde{U} \to U$. Let $\Omega \subset U$, $\tilde{\Omega} \subset \tilde{U}$ be regions such that $\Omega = \Phi(\tilde{\Omega})$. If $G : U \to \mathbb{R}^3$ is a parametrized smooth surface and $\tilde{G} = G \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a reparametrization of $G(U)$, then*

$$\int_{\tilde{\Omega}} \left\| \frac{\partial \tilde{G}}{\partial \tilde{u}} \times \frac{\partial \tilde{G}}{\partial \tilde{v}} \right\| dV_2 = \int_\Omega \left\| \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v} \right\| dV_2.$$

Note that Theorem 27 simply states that surface area is independent of parametrization, as we would hope.