Homework 8
(due Monday, March 30)

Turn in answers only.

8.3: 6

Turn in full solutions.

8.3: 5, 8, 10abe, 12, 13, 15, 16a (use Green's Theorem to convert to a double integral; use the hint about symmetry only if it helps you), 20a

Problem 1. Let $\Phi : \mathbf{R}^n \to \mathbf{R}^m$ be a function, and write $y = \Phi(x) = (\Phi_1(x), \ldots, \Phi_m(x))$ so that x denotes a point in the source $\mathbf{R}^n$ and y denotes a point in the target $\mathbf{R}^m$. As you well-know by now, any function $f : \mathbf{R}^m \to \mathbf{R}$ on U can be 'pulled back' to a function $f \circ \Phi : \mathbf{R}^n \to \mathbf{R}$. The composite function $f \circ \Phi$ is also denoted (among mathematicians mostly) by '$\Phi^* f$' and called the <u>pullback</u> of f.

The reason for the extra verbiage/notation is that when Φ is C^1, one can extend the notion of 'pullback' to 1-forms. Namely, if $\omega = \sum_{i=1}^m \omega_i \, dy_i$ is a 1-form on the target $\mathbf{R}^m$, then the <u>pullback</u> of ω by Φ is the one form on the source $\mathbf{R}^n$ denoted/given by

$$\Phi^*\omega := \sum_{i=1}^{m} (\omega_i \circ \Phi) \, d\Phi_i,$$

where Φ_i is the ith component of Φ.

(a) Let $\Phi : \mathbf{R}^2 \to \mathbf{R}^3$ be given by $\Phi(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2, x_2^{-1})$ and $\omega = y_2 \, dy_1 + y_3 \, dy_2 + y_1 \, dy_3$. Compute $\Phi^*\omega$.

(b) The remaining parts of this problem are general facts about pulling back 1-forms, and your job is to verify each. Proofs mostly amount to applying definitions, using previous items, and (especially in part (d)) the chain rule.
For any two 1-forms ω and $\tilde{\omega}$, one has and $\Phi^*(\omega + \tilde{\omega}) = \Phi^*\omega + \Phi^*\tilde{\omega}$.

(c) If $f : \mathbf{R}^m \to \mathbf{R}$ is a function, then $\Phi^*(f\omega) = (f \circ \Phi) \, \Phi^*\omega$. That is, pullback distributes over products.

(d) If $f : \mathbf{R}^m \to \mathbf{R}$ is a C^1 function, then $\Phi^*(df) = d(f \circ \Phi)$ (i.e $d(\Phi^* f) = \Phi^*(df)$, so that pulling back 'commutes' with taking differentials).

(e) If $\Pi : \mathbf{R}^p \to \mathbf{R}^n$ is a(nother) C^1 function, then we have that $(\Phi \circ \Pi)^*\omega = \Pi^*\Phi^*\omega$. (Hint: the previous items are helpful here).

Problem 2. Let $U \subset \mathbf{R}^2$ be open and $f : U \to \mathbf{R}$ be a C^2 function. Then we know from class that (1) df, being exact, is also closed; and (2) for any C^1 1-form ω on $\mathbf{R}^2$, we have $d\omega = 0$ if and only if ω is closed. Hence $d^2 f := d(df) = 0$. Show that $d^2 f = 0$ by everyone's favorite alternative method: direct computation.

Extra Credit: 8.3.26 (I confess I haven't worked this one out for myself, but it sure looks interesting.)