# Computing Eigenvalues and Eigenvectors

Jonathan D. Hauenstein

For a matrix $A \in \mathbb{C}^{N\times N}$, $\lambda \in \mathbb{C}$ is an *eigenvalue* of $A$ with corresponding *eigenvector* $v \in \mathbb{C}^N$ if

$$Av - \lambda v = 0 \tag{1}$$

and $v \neq 0$. Even though there are more efficient specialized methods for computing eigenvalues and eigenvectors, we utilize this example to demonstrate using `Bertini` to solve systems defined on a product of affine and projective space. Since (1) is homogeneous with respect to $v$, i.e., if $v$ is an eigenvector of $A$ corresponding to $\lambda$, then $\alpha v$ is also an eigenvector of $A$ corresponding to $\lambda$ for any $\alpha \neq 0$, we should naturally treat (1) as a system of $N$ equations defined on the product space $\mathbb{C} \times \mathbb{P}^{N-1}$. In particular, since each equation in (1) is linear in both $\lambda$ and $v$, a homotopy on $\mathbb{C} \times \mathbb{P}^{N-1}$ to solve (1) requires tracking $\binom{N}{1} = N$ solution paths (see [2, § 5.1]), which is equal to the generic number of distinct eigenvalue and eigenvector pairs.

Executing `Eigenvalues.sh` calls `Bertini` using the input file `inputEigenvalues` which solves (1) for a matrix whose Jordan canonical form has the following structure:

$$\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & 1 & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & \\ & & & & \lambda_3 \end{bmatrix}$$

for distinct $\lambda_1, \lambda_2, \lambda_3$. That is,

- $\lambda_1$ is an eigenvalue of algebraic and geometric multiplicity $1$;
- $\lambda_2$ is an eigenvalue of algebraic multiplicity $2$ and geometric multiplicity $1$;
- $\lambda_3$ is an eigenvalue of algebraic and geometric multiplicity $2$.

Let $v_1 \in \mathbb{P}^4$ and $v_2 \in \mathbb{P}^4$ be the unique eigenvectors corresponding to $\lambda_1$ and $\lambda_2$, respectively. In terms of the polynomial system (1), we have the corresponding solutions:

- $(\lambda_1, v_1)$ is a nonsingular solution of multiplicity $1$;
- $(\lambda_1, v_2)$ is a singular solution of multiplicity $2$;
- $(\lambda_3, v_3)$ and $(\lambda_3, v_3')$ are singular solutions of a positive-dimensional component, where $v_3$ and $v_3'$ are some eigenvectors associated with $\lambda_3$.

This is confirmed by viewing the output of `Bertini`. For example, the following is a portion of the output to the screen:

```
                | Number of real solns |  Number of non-real solns   |  Total
------------------------------------------------------------------------------
Non-singular    |        1             |             0               |   1
Singular        |        3             |             0               |   3
------------------------------------------------------------------------------
Total           |        4             |             0               |   4


Finite Multiplicity Summary

  Multiplicity  |  Number of real solns |  Number of non-real solns
---------------------------------------------------------------------
        1       |        3              |       0
        2       |        1              |       0
```

This shows the following:

- non-singular solution is $(\lambda_1, v_1)$;
- the singular solution of multiplicity $2$ is $(\lambda_2, v_2)$;
- the other $2$ solutions listed as "multiplicity 1" (i.e., each is an endpoint of $1$ path) that are singular correspond with $(\lambda_3, v_3)$ and $(\lambda_3, v_3')$. Since they are singular but the endpoint of $1$ path, the local dimension test [1] yields that they lie on a positive-dimensional component.

The values for $\lambda_i$ can, for example, be obtained as the first coordinate of the points listed in `finite_soluions`, namely:

$$\lambda_1 \approx -1.14286,\ \lambda_2 \approx 1.42857,\ \lambda_3 \approx -0.28571.$$

## References

[1] D.J. Bates, J.D. Hauenstein, C. Peterson, and A.J. Sommese, A numerical local dimension test for points on the solution set of a system of polynomial equations. *SIAM J. Num. Anal.*, 47(5), 3608–3623, 2009.

[2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. *Numerically Solving Polynomial Systems with Bertini*, Volume 25 of Software, Environments, and Tools, SIAM, Philadelphia, 2013.