Wilkinson's polynomial

Jonathan D. Hauenstein

James H. Wilkinson said "*the most traumatic experience in my career as a numerical analyst*" [5] was when he observed the behavior of numerically solving the following perturbed equation [4, 6]

$$W_1(x) = \prod_{j=1}^{20}(x - j) - 2^{-23}x^{19} = 0 \tag{1}$$

on a Pilot ACE computer with single-precision arithmetic (machine epsilon is 2^{-23}). In `Bertini`, we trivially scale the equation $W_1(x) = 0$ so that the largest coefficient has unit magnitude.

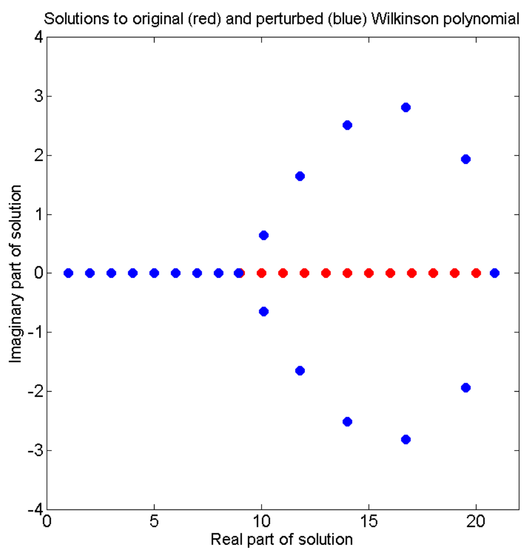
By executing `Wilkinson.sh`, `Bertini` is called using the input file `inputWilkinson` which solves (1). Although there are more efficient specialized methods for solving univarite equations than homotopy continuation, we utilize this example to demonstrate adaptive precision algorithms [1, 2, 3] in `Bertini`. For example, the file `output` presents information about the path tracking. In our test, the following is a portion of the file `output`:

```
numSteps = 10
numSteps = 3
numSteps = 1
numSteps = 1
Increase Prec (success): 52 to 64 (0)
numSteps = 2
Change Prec (success): 64 to 96 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 96 to 128 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 128 to 160 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 160 to 192 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 192 to 224 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 224 to 256 (0)
numSteps = 1
numSteps = 1
numSteps = 1
```

```
Change Prec (success): 256 to 288 (0)
numSteps = 1
numSteps = 1
numSteps = 1
Change Prec (success): 288 to 320 (0)
numSteps = 1
numSteps = 1
numSteps = 1
numSteps = 1
prec: 352 newton: 9.829718e-17 func: 3.102517e-53
prec: 384 newton: 8.047924e-33 func: 3.795543e-83
prec: 416 newton: 9.845633e-63 func: 5.680568e-143
min sv: 3.178497e-21
```

This shows that the adaptive precision algorithm steadily increased the precision up to 320 bits during the path tracking and up to 416 bits to verify convergence of Newton's method. Moreover, the minimum singular value for the internal system at the numerical approximation of the solution is approximately $3 \cdot 10^{-21}$.

The MATLAB script `plotWilkinson.m` can be used to create the following plot of the 10 real and 10 non-real solutions computed by `Bertini` from `finite_solutions`:



REFERENCES

[1] D.J. Bates, J.D. Hauenstein, and A.J. Sommese. Efficient path tracking methods. *Numer. Algor.*, 58(4), 451–459, 2011.

[2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Adaptive multiprecision path tracking. *SIAM J. Numer. Anal.*, 46(2), 722–746, 2008.

[3] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Stepsize control for adaptive multiprecision path tracking. *Contemp. Math.*, 496, 21–31, 2009.

[4] J.H. Wilkinson. The evaluation of the zeros of ill-conditioned polynomials. I, II. *Numer. Math.*, 1, 150–180, 1959.

[5] J.H. Wilkinson. The perfidious polynomial. In *Studies in numerical analysis*, volume 24 of MAA Stud. Math., Math. Assoc. America, Washington, DC, 1984, pp. 1–28.

[6] J.H. Wilkinson. *Rounding errors in algebraic processes*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.