Illustrative Example

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For an illustrative first example of using `Bertini`, we solve the following quintic polynomial equation which can not be solved in terms of radicals:

$$f(x) = x^5 - x + 1 = 0. \tag{1}$$

Executing `Illustrative.sh` calls `Bertini` using the input file `inputIllustrative` which solves (1). Since $\deg f = 5$, `Bertini` tracks 5 paths to compute the 5 solutions to (1). A portion of the screen output is:

```
  Multiplicity  |  Number of real solns |  Number of non-real solns
---------------------------------------------------------------------
        1       |       1               |       4
```

showing that (1) has 1 real and 4 non-real solutions. This can be proven using the numerical approximations computed by `Bertini` via the software `alphaCertified` [1].

In our test, the following is the output file `finite_solutions`:

```
5

7.648844336005846e-01 3.524715460317263e-01

-1.812324444698754e-01 1.083954101317711e+00

-1.167303978261419e+00 -1.665334536937735e-16

7.648844336005847e-01 -3.524715460317264e-01

-1.812324444698754e-01 -1.083954101317711e+00
```

which first lists the number of solutions (5) and then numerical approximations of the real and imaginary coordinates of the solutions, i.e., the line

```
7.648844336005846e-01 3.524715460317263e-01
```

corresponds with

$$0.7648844336005846 + 0.3524715460317263 \cdot \sqrt{-1}.$$

Hence, the third point listed above in `finite_solutions` is a numerical approximation of the real solution of (1). The file `main_data` provides information about the quality of the numerical approximations of each solution. For example, here is a portion of `main_data` from our test:

```
Solution 3 (path number 0)
Estimated condition number: 5.940070457750704e+01
Function residual: 3.972054645195637e-15
Latest Newton residual: 4.405327952961146e-16
T value at final sample point: 3.906250000000000e-04
Maximum precision utilized: 52
T value of first precision increase: 0.000000000000000e+00
Accuracy estimate, internal coordinates (difference of last two endpoint estimates):  5.083739718695013e-13
Accuracy estimate, user's coordinates (after dehomogenization, if applicable):  5.679652284963224e-13
Cycle number: 1
7.648844336005846e-01 3.524715460317263e-01
Paths with the same endpoint, to the prescribed tolerance:
Multiplicity: 1
```

References

[1] J.D. Hauenstein and F. Sottile. Algorithm 921: alphaCertified: certifying solutions to polynomial systems. *ACM Trans. Math. Software*, 38(4), 28, 2012.