Weakly Infeasible Semidefinite Program

Jonathan D. Hauenstein

In optimization, the Farkas' lemma can be used to create certificates for infeasible linear programs, e.g, see [1, § 3.4]. Weakly infeasible semidefinite programs are precisely the semidefinite programs for which a generalization of the Farkas' lemma does not provide a certificate of infeasibility. In this example, we will apply the homotopy continuation approach developed in [2] using `Bertini` to show weak infeasibility of a given problem.

For symmetric matrices $A_1, \ldots, A_m, C \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^m$, we consider a semidefinite program (SDP) of the form

$$\text{(SDP)} \qquad \begin{aligned} \underset{y}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i A_i + S = C, \\ & S \succeq 0 \end{aligned}$$

where $S \succeq 0$ means that S is a positive semidefinite matrix, i.e., $S \in \mathbb{R}^{n\times n}$ is symmetric with nonnegative eigenvalues. The problem (SDP) is *weakly infeasible* if (SDP) is infeasible, i.e.,

$$\left\{ (S, y) \;\middle|\; \sum_{i=1}^{m} y_i A_i + S = C,\ y \in \mathbb{R}^m,\ S \succeq 0 \right\} = \emptyset,$$

but, for every $\epsilon > 0$, there exists an (S, y) with $S \succeq 0$ such that

$$\left\| C - \sum_{i=1}^{m} y_i A_i - S \right\| \leq \epsilon.$$

For any $M > 0$, the approach in [2] is based on an equivalent description of weak infeasibility, namely that the optimal value of

$$\text{(1)} \qquad \begin{aligned} \underset{y}{\text{maximize}} \quad & \lambda \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i A_i + S + \lambda I = C, \\ & S \succeq 0, \\ & M - \lambda \geq 0 \end{aligned}$$

is 0 but is not attained, i.e., supremum but not maximum. Hence, [2] shows that this can be observed by tracking a solution path that converges to the solution "at infinity" via projective space.

Specifically, we consider the following weakly infeasible dual problem from [2, Ex. 22]:

$$\text{(2)} \qquad \begin{aligned} \text{maximize} \quad & y_1 \\ \text{subject to} \quad & \begin{bmatrix} -y_1 & 1 \\ 1 & 0 \end{bmatrix} \succeq 0 \end{aligned}$$

which correspond to (SDP) with $n = 2$, $m = 1$, $b_1 = 1$,

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Executing `SDP.sh` calls `Bertini` using the input file `inputSDP` which describes a user-defined homotopy for solving the corresponding convex optimization problem (1) with start point provided in `startSDP`. In our test, the optimal value for the corresponding problem (1) was listed in `finite_solutions` as

```
-0.2445702296757684721963589418950279479506e-22 0.3653431852837309498585163934304979240182e-22
```

which is numerically zero. The corresponding projective point $[s_0, s_{11}, s_{12}, s_{22}, y_1, \lambda] \in \mathbb{P}^5$, where s_0 is the homogenizing coordinate, also listed in `finite_solutions` was

```
0.1420070460664174067897746451440804620169e-22 0.2716396090825499892177910018151099374657e-22
-0.9999999999999999999999809212362127033653e0 0.1455368815474751575296648535118111906290e-22
0.1420070460664174067897746451440804620169e-22 0.2716396090825499892177910018151099374657e-22
0.1907876289541177621636074426719895984223e-22 0.1455367866824031885270725179756176428056e-22
0.1000000000000000000000000000000000000000e1 0.0000000000000000000000000000000000000000e0
-0.8675774237068931461938116210113334916410e-23 -0.6697978245640461650248622491859873332321e-23
```

Since $s_0 = 0$, this shows that the solution the corresponding problem (1) is "at infinity," i.e., the optimal value of 0 is not achieved. Therefore, this computation confirms the weak infeasibility of (2).

Following a similar strategy, `Bertini` was applied to the "messy" (i.e., structure was scrambled under row operations and rotations) weakly infeasible cases in the test suite developed in [3] involving SDPs on 10×10 matrices. Table 1 shows how this approach in `Bertini` was able to identify all of these cases while standard software in optimization failed to identify any of them as weakly infeasible.

	$m = 20$	$m = 10$
`SeDuMi` [6]	0	0
`SDPT3` [7]	0	0
`MOSEK` [4]	0	0
Preprocess [5] + `SeDuMi` [6]	0	0
`Bertini`	100	100

TABLE 1. Number of successes (out of 100) of identifying "messy" weakly infeasible SDPs from [3]

References

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