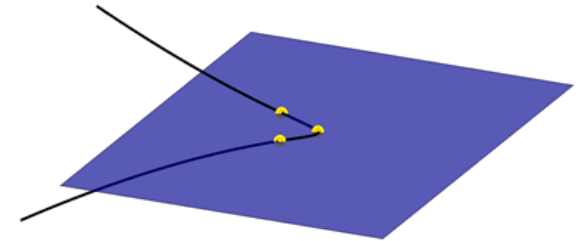
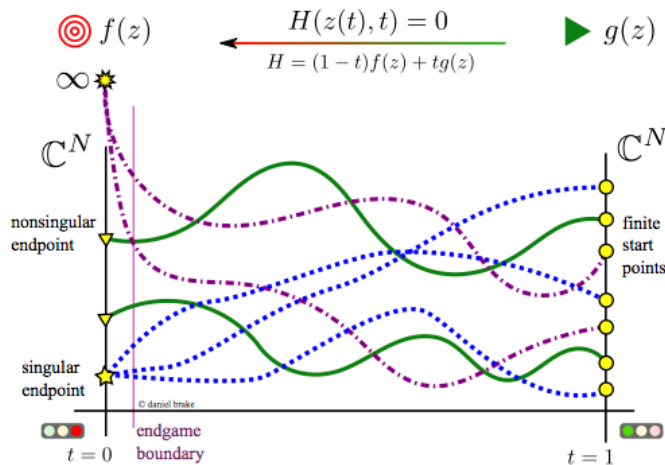
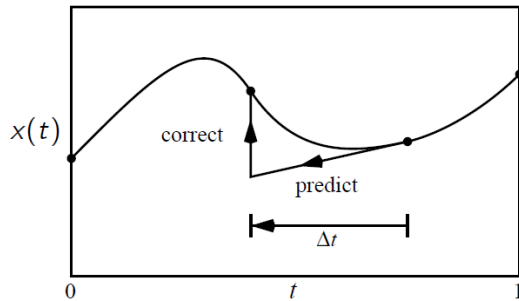


Overview: Numerically solving systems of polynomial equations



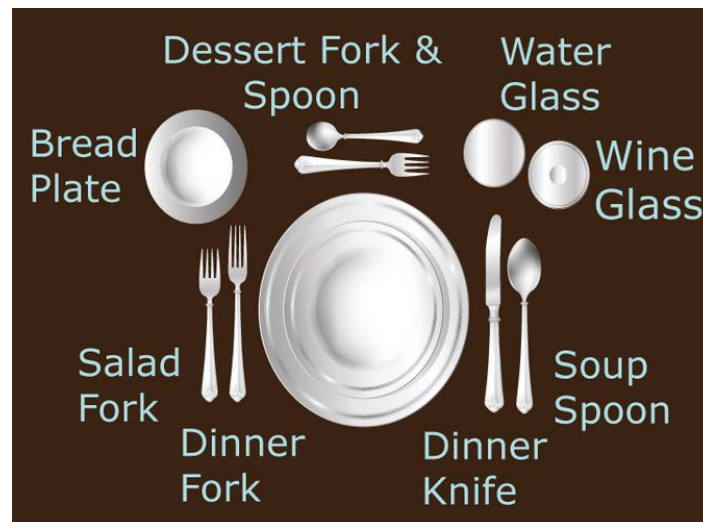
Jonathan Hauenstein

Summer School on Numerical Computing in Algebraic Geometry

August 13, 2018

Setting the Table

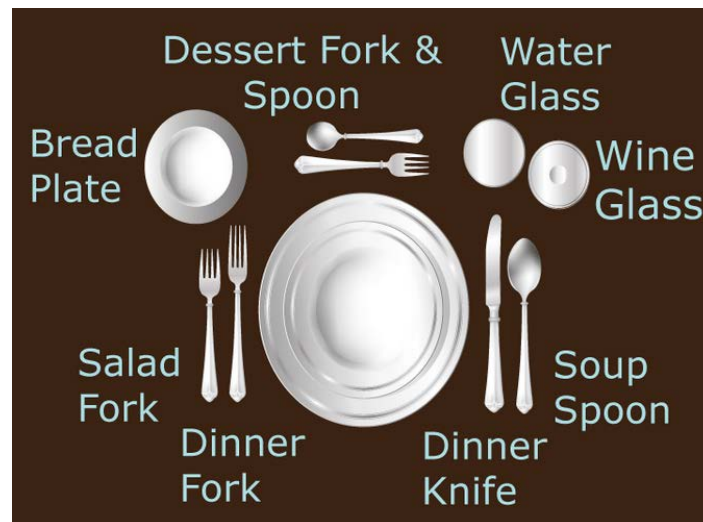
For a polynomial system $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$, **solve** $f(x) = 0$.



Setting the Table

For a polynomial system $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$, **solve** $f(x) = 0$.

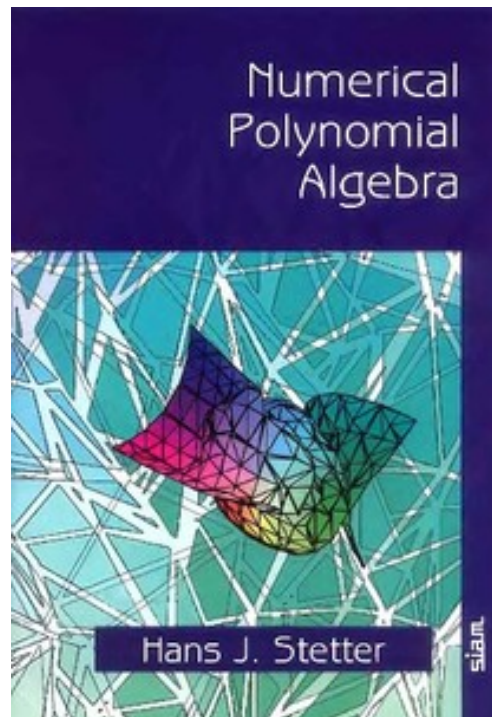
- ▶ **Overview** of homotopy continuation and num. alg. geom.
- ▶ Historical perspective
- ▶ Utilize Bertini but there are many other packages, e.g.:
 - ▶ PHCpack, Hom4PS, NAG4M2, HomotopyContinuation.jl



Algebra vs. Geometry

Algebra:

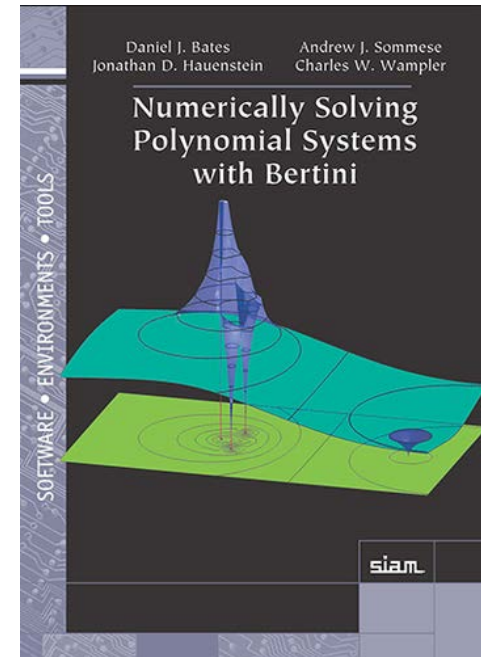
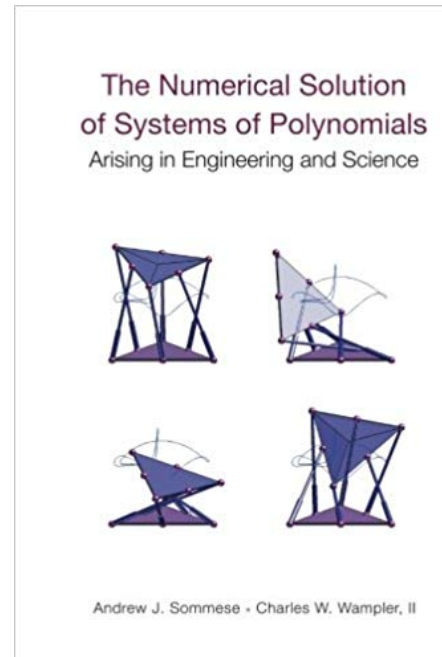
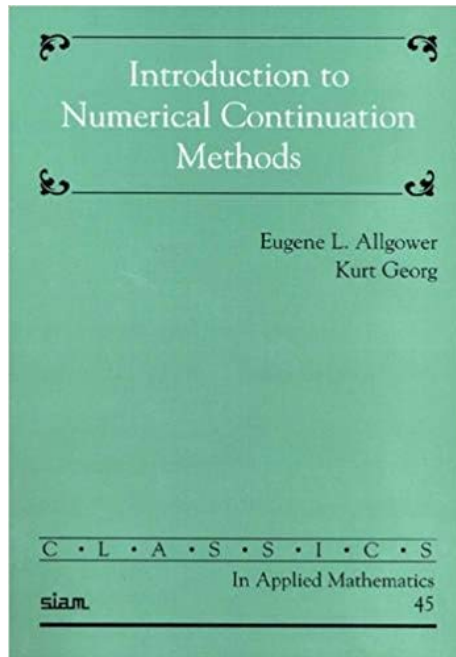
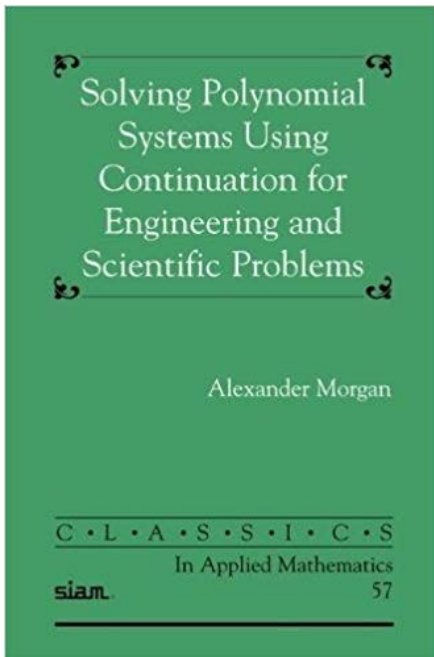
- ▶ “Numerical Polynomial Algebra” by Hans Stetter
- ▶ Normal forms, eigenvectors/eigenvalues, border basis, ...
- ▶ K. Batselier, B. De Moor, P. Dreesen, B. Mourrain, S. Telen, M. Van Barel, ...



Algebra vs. Geometry

Geometry:

- ▶ Homotopy continuation and numerical algebraic geometry
- ▶ Morgan (1987), Allgower-Georg (1990), Sommese-Wampler (2005)
Bates-H.-Sommese-Wampler (2013)



Algebra vs. Geometry

Generally speaking:

- ▶ Algebraic methods prefer vastly over-determined systems
 - ▶ fewer “new” polynomials to compute
 - ▶ Bardet-Faugere-Salvy (2004)

- ▶ Numerical algebraic geometry prefers well-constrained systems of low degrees with coefficients of roughly unit magnitude
 - ▶ codimension = \neq equations
 - ▶ stable under perturbations



Early History of Solving



Early History

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_dx^d = 0$$

▶ $d = 1: x = \frac{-a_0}{a_1}$

▶ $d = 2: x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$

▶ $d = 3:$

$$x_1 = -\frac{b}{3a} - \frac{1}{3a} \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

$$x_2 = -\frac{b}{3a} - \frac{1 + i\sqrt{3}}{6a} \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

$$x_3 = -\frac{b}{3a} - \frac{1 - i\sqrt{3}}{6a} \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

▶ $d = 4:$

$$x_1 = -\frac{b}{4a} + \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3}} + \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3} + \frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

$$x_2 = -\frac{b}{4a} + \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3}} - \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3} + \frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

$$x_3 = -\frac{b}{4a} - \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3}} + \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3} - \frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$

$$x_4 = -\frac{b}{4a} - \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3}} - \frac{1}{4a} \sqrt{\frac{2b^2 - 3ac + 12a^2d}{3} - \frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^3 - 3ac)^3}}{4}}$$



Early History

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d = 0$$

- ▶ Abel-Ruffini Theorem (1824)
 - ▶ No algebraic solution (using radicals) to general polynomial equations of degree 5 or higher with arbitrary coefficients
 - ▶ What does it mean to “solve $x^5 - x + 1 = 0$ ”?

Early History

- ▶ What does it mean to “solve $x^5 - x + 1 = 0$ ”?

Maple

```
> solve(x5 - x + 1);  
RootOf(_Z5 - _Z + 1, index = 1), RootOf(_Z5 - _Z + 1, index = 2), RootOf(_Z5 - _Z + 1, index = 3),  
RootOf(_Z5 - _Z + 1, index = 4), RootOf(_Z5 - _Z + 1, index = 5)
```

Early History

- ▶ What does it mean to “solve $x^5 - x + 1 = 0$ ”?

Maple

```
> solve(x5 - x + 1);  
RootOf(_Z5 - _Z + 1, index = 1), RootOf(_Z5 - _Z + 1, index = 2), RootOf(_Z5 - _Z + 1, index = 3),  
RootOf(_Z5 - _Z + 1, index = 4), RootOf(_Z5 - _Z + 1, index = 5)
```

```
> fsolve(x5 - x + 1);  
-1.167303978
```

Early History

- ▶ What does it mean to “solve $x^5 - x + 1 = 0$ ”?

Maple

```
> solve(x5 - x + 1);  
RootOf(_Z5 - _Z + 1, index = 1), RootOf(_Z5 - _Z + 1, index = 2), RootOf(_Z5 - _Z + 1, index = 3),  
RootOf(_Z5 - _Z + 1, index = 4), RootOf(_Z5 - _Z + 1, index = 5)
```

```
> fsolve(x5 - x + 1);  
-1.167303978
```

```
> evalf(solve(x5 - x + 1));  
0.764884433600585 + 0.352471546031726 I, -0.181232444469875 + 1.08395410131771 I,  
-1.16730397826142, -0.181232444469875 - 1.08395410131771 I, 0.764884433600585  
- 0.352471546031726 I
```

Early History

- ▶ What does it mean to “solve $x^5 - x + 1 = 0$ ”?

Maple

```
> evalf(solve(x^5 - x + 1));  
0.764884433600585 + 0.352471546031726 I, -0.181232444469875 + 1.08395410131771 I,  
-1.16730397826142, -0.181232444469875 - 1.08395410131771 I, 0.764884433600585  
- 0.352471546031726 I
```

Bertini

input

```
variable_group x;  
function f;  
f = x^5 - x + 1;
```

finite_solutions

5

7.648844336005847e-01 -3.524715460317264e-01

7.648844336005849e-01 3.524715460317262e-01

-1.812324444698754e-01 1.083954101317711e+00

-1.167303978261419e+00 -2.220446049250313e-16

-1.812324444698754e-01 -1.083954101317711e+00



Early History

Vast generalization of the meaning of “solve”:

- ▶ Early history: find a solution and study local properties
- ▶ Late 20th century: find all isolated solutions
- ▶ Early 21st century: describe all solutions
 - ▶ isolated and positive-dimensional components

Early History

Lack of exact formula for solutions \longrightarrow iterative refinement

- ▶ Newton (1643-1727), Raphson (1648-1715), Simpson (1710-1761)
- ▶ compute solution to arbitrary accuracy given approximation

Early History

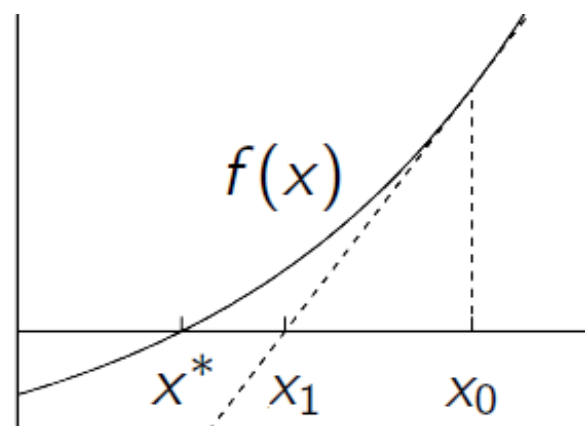
Lack of exact formula for solutions \longrightarrow iterative refinement

- ▶ Newton (1643-1727), Raphson (1648-1715), Simpson (1710-1761)
- ▶ compute solution to arbitrary accuracy given approximation

Newton's method:

- ▶ $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with Jacobian $Jf : \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$
- ▶ Given approximation x_0 , compute x_1, x_2, x_3, \dots via

$$x_{k+1} = x_k - Jf(x_k)^{-1} f(x_k)$$



Early History

Lack of exact formula for solutions \longrightarrow iterative refinement

- ▶ Newton (1643-1727), Raphson (1648-1715), Simpson (1710-1761)
- ▶ compute solution to arbitrary accuracy given approximation

Newton's method:

- ▶ $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with Jacobian $Jf : \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$
- ▶ Given approximation x_0 , compute x_1, x_2, x_3, \dots via

$$x_{k+1} = x_k - Jf(x_k)^{-1} f(x_k)$$

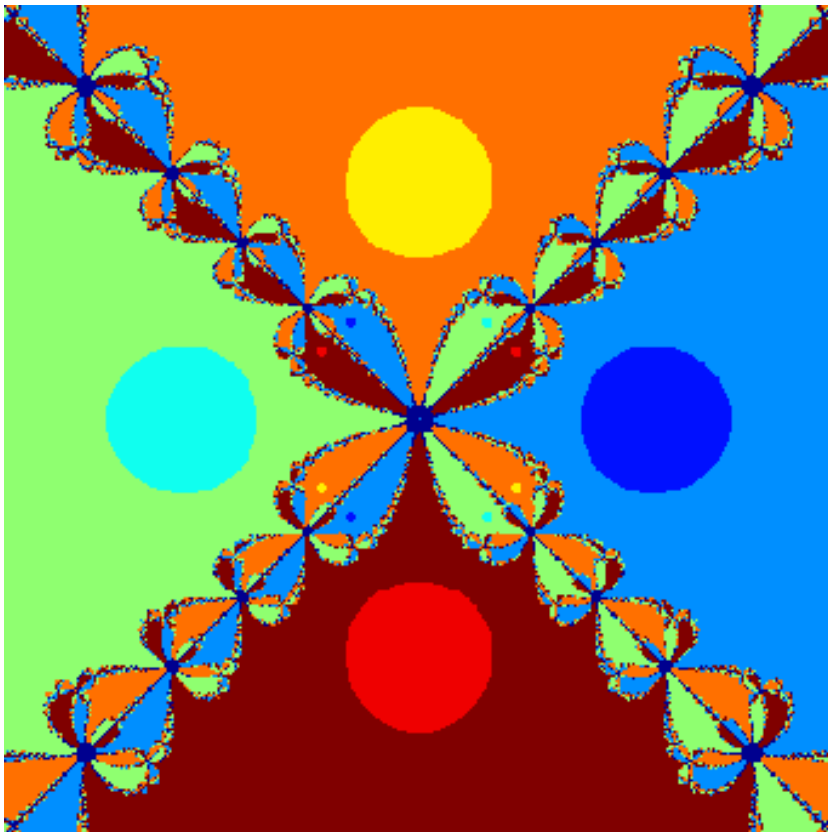
If $f(x^*) = 0$, $Jf(x^*)^{-1}$ exists (**nonsingular**), and $\|x_0 - x^*\|$ small,

$x_k \rightarrow x^*$ quadratically.

Early History

Double-edged sword of Newton's method:

- ▶ Quadratic convergence near nonsingular solutions
- ▶ Slow convergence or divergence near singular solutions
- ▶ Difficulty away solutions (chaos, limit cycles, etc)



$$f(x) = x^4 - 1$$

Early History

Double-edged sword of Newton's method:

- ▶ Quadratic convergence near nonsingular solutions
- ▶ Slow convergence or divergence near singular solutions
- ▶ Difficulty away solutions (chaos, limit cycles, etc)

Goal

- ▶ Use continuation methods to stay near solutions
- ▶ Use deflation to restore quadratic convergence for sing. solns.
 - ▶ Ojika-Watanabe-Mitsui (1983), Ojika (1987),
Leykin-Verschelde-Zhao (2006,2008), Dayton-Zeng (2005),
Mantzaflaris-Mourrain (2011), Guisti-Yakoubsohn (2013),
H.-Wampler (2013), H.-Mourrain-Szanto (2017), ...

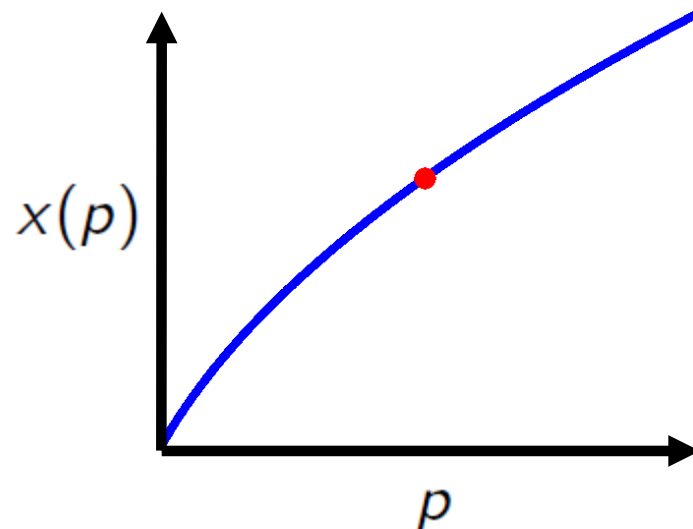
Early History

Continuation from complex analysis:

- ▶ Cauchy (1789-1857), Riemann (1826-1866), Mittag-Leffler (1846-1927)
- ▶ Implicit function theorem
- ▶ Analytic extension of functions (analytic continuation)

Big picture idea:

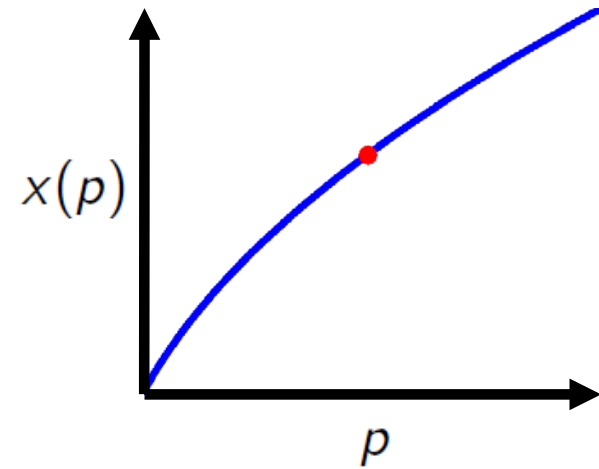
- ▶ solutions “continue” locally under small parameter changes



Example

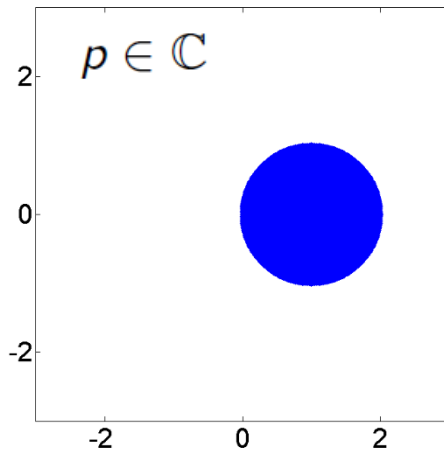
Early History

$$f(x; p) = x^2 - p = 0$$



- ▶ Starting at $(x, p) = (1, 1)$, IFT provides that there is an analytic function $x(p)$ with $x(1) = 1$ such that $f(x(p), p) = 0$.

$$x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n)(n!)^2} (p-1)^n$$



- ▶ converges for $|p - 1| \leq 1$

Example

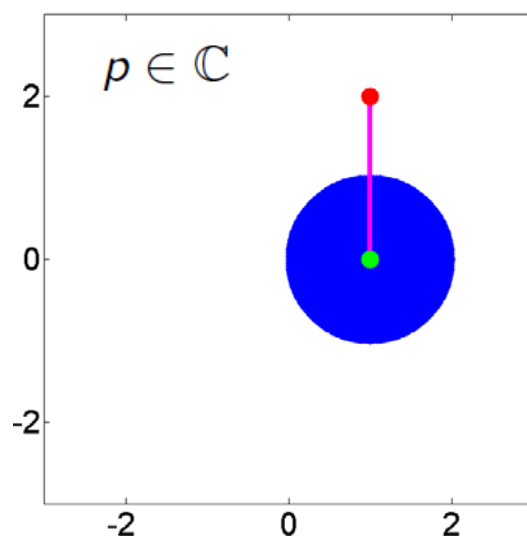
Early History

$$f(x; p) = x^2 - p = 0$$

- ▶ Extend beyond original domain using continuation

Compute $x(1 + 2i) = \sqrt{1 + 2i}$ via the path $x(1 + (1 - t) \cdot 2i)$:

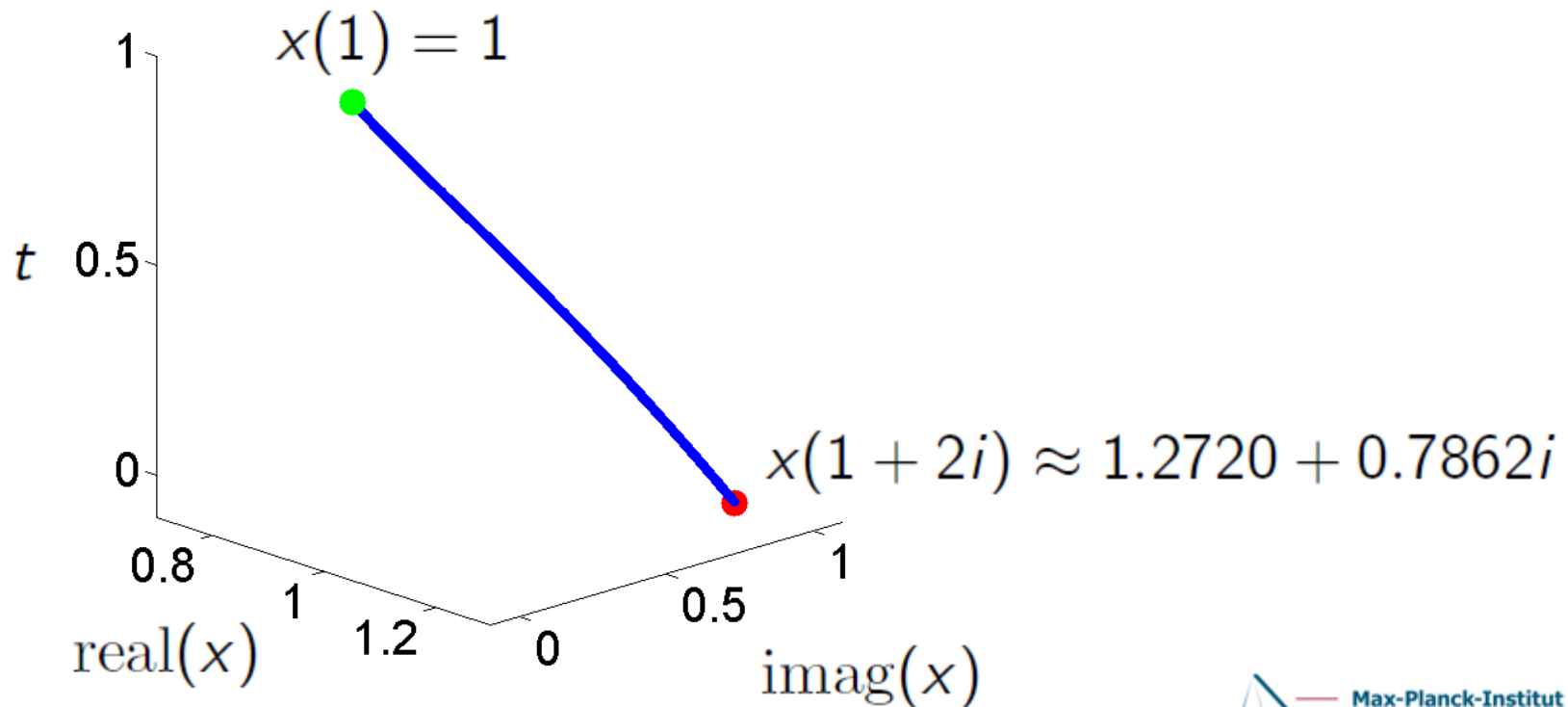
- ▶ $t = 1$: $x(1) = 1$ is known
- ▶ $t = 0$: $x(1 + 2i)$ is what we want to compute



Early History

Compute $x(1 + 2i) = \sqrt{1 + 2i}$ via the path $x(1 + (1 - t) \cdot 2i)$:

- ▶ $t = 1$: $x(1) = 1$ is known
- ▶ $t = 0$: $x(1 + 2i)$ is what we want to compute



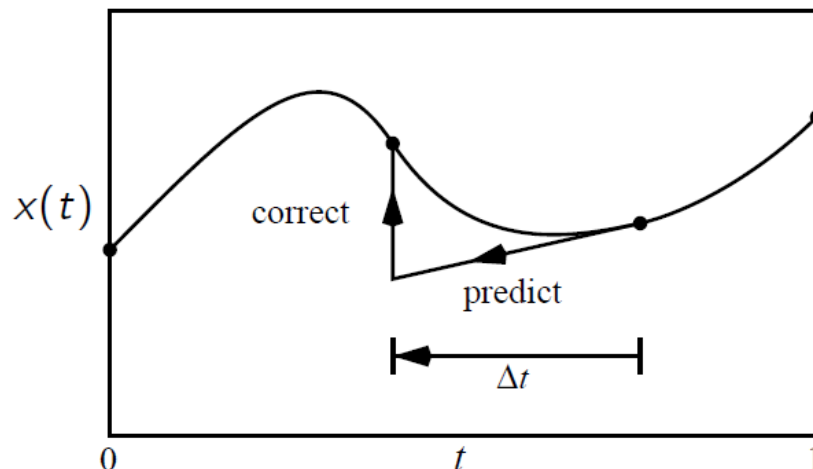
Early History

Numerically track along the path $x(t)$ satisfying $f(x(t), t) = 0$:

- ▶ (Predictor) Estimate $x(t + \Delta t)$ from $x(t)$ by discretizing using the Davidenko differential equation (1953):

$$f = 0 \quad \longrightarrow \quad \frac{d}{dt}f = 0 \quad \longrightarrow \quad \dot{x}(t) = -J_x f(x(t), t)^{-1} J_t f(x(t), t)$$

- ▶ Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg,



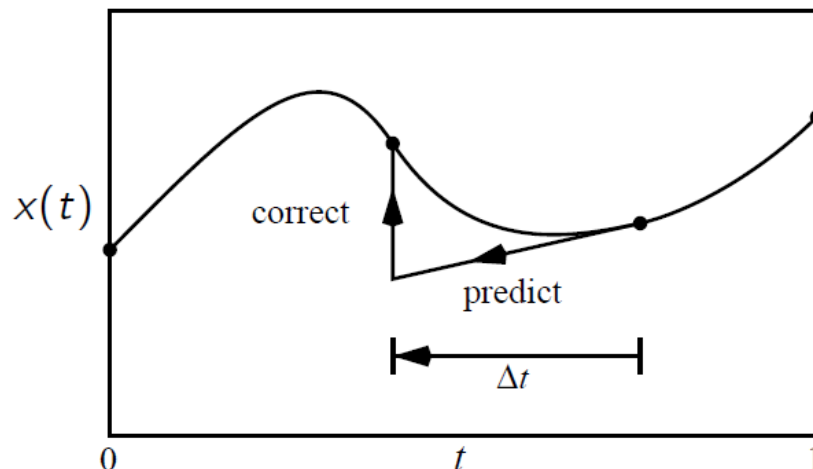
Early History

Numerically track along the path $x(t)$ satisfying $f(x(t), t) = 0$:

- ▶ (Predictor) Estimate $x(t + \Delta t)$ from $x(t)$ by discretizing using the Davidenko differential equation (1953):

$$f = 0 \quad \longrightarrow \quad \frac{d}{dt}f = 0 \quad \longrightarrow \quad \dot{x}(t) = -J_x f(x(t), t)^{-1} J_t f(x(t), t)$$

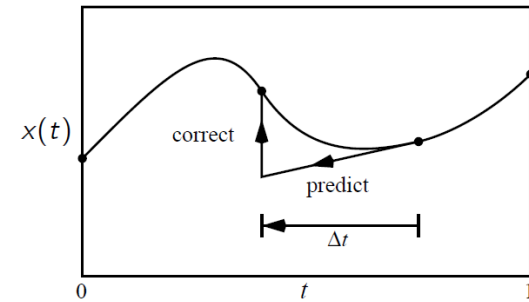
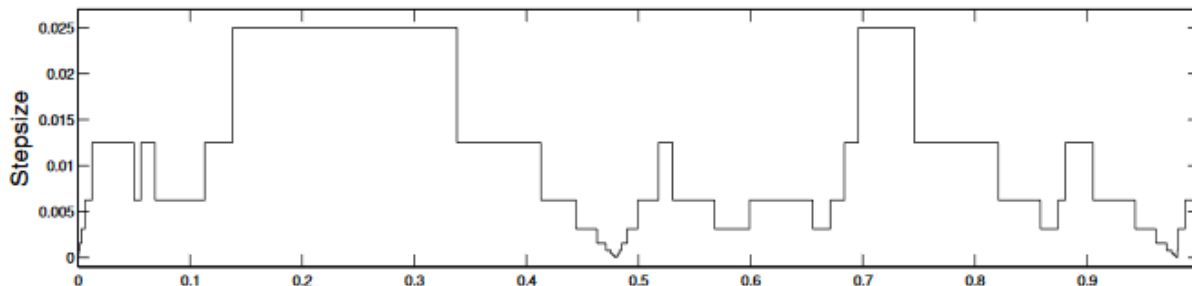
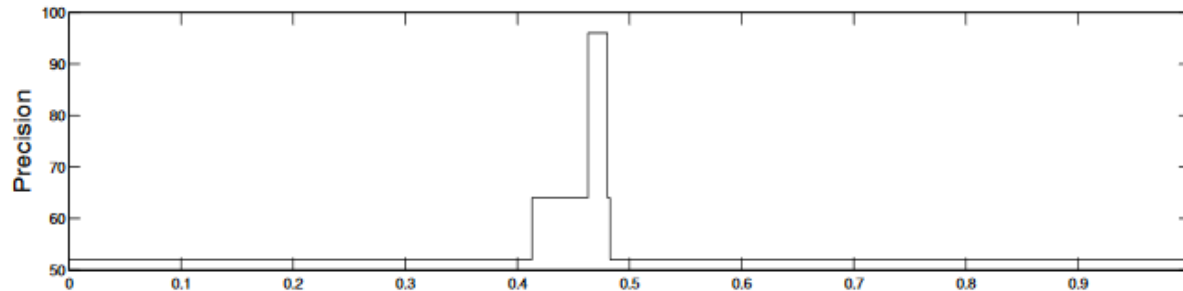
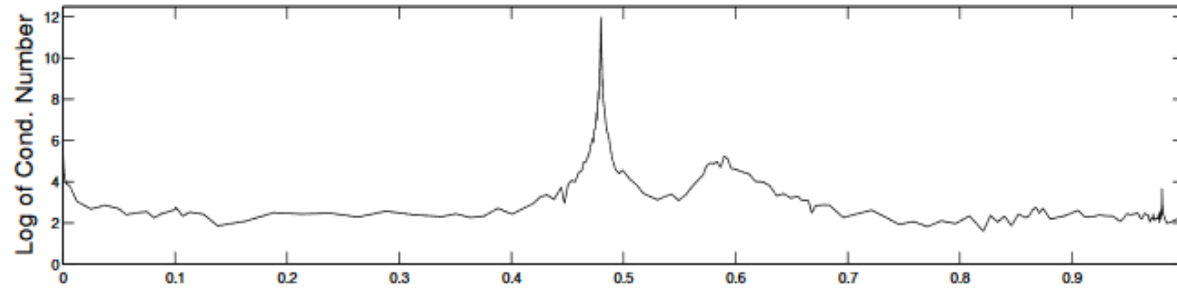
- ▶ Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg,
- ▶ (Corrector) for each t , apply Newton's method to $f(\bullet, t) = 0$



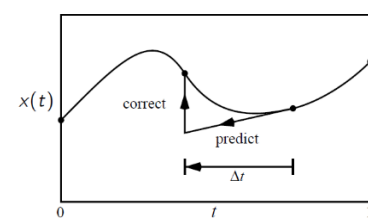
Early History

Locally adapt both stepsize and floating-point precision:

- ▶ Bates-H.-Sommese-Wampler (2008,2009), Bates-H.-Sommese (2011)



Early History



Certified tracking (select stepsize to guarantee to track path):

- ▶ Shub-Smale (“Bézout series” 1990s), Beltran-Leykin (2011,2012), H.-Liddell (2016), Xu-Burr-Yap (2018), ...

Smale’s 17th problem: polynomial time to compute a root

- ▶ Beltran-Pardo (2009, 2011), Cucker-Bürgisser (2011), Lairez (2017)



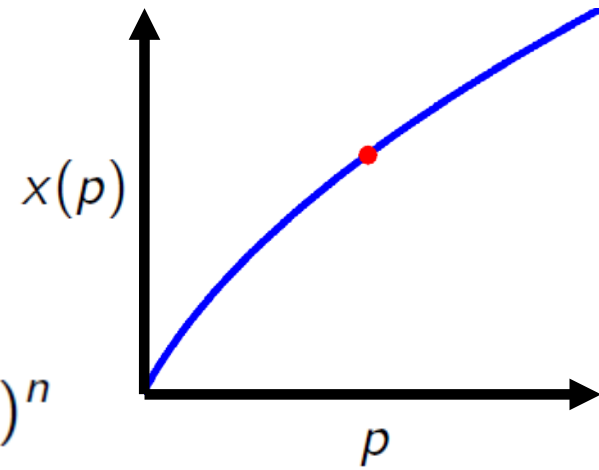
Pierre Lairez: 2017 SIAG/AG Early Career Prize

Example

Early History

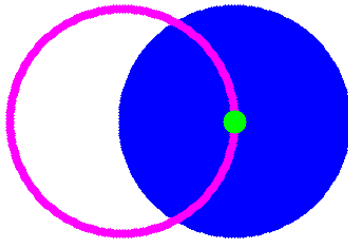
$$f(x; p) = x^2 - p = 0$$

$$x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n)(n!)^2} (p-1)^n$$



Track around a loop: $x(e^{i\theta})$

$$p \in \mathbb{C}$$



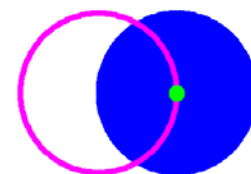
Example

Early History

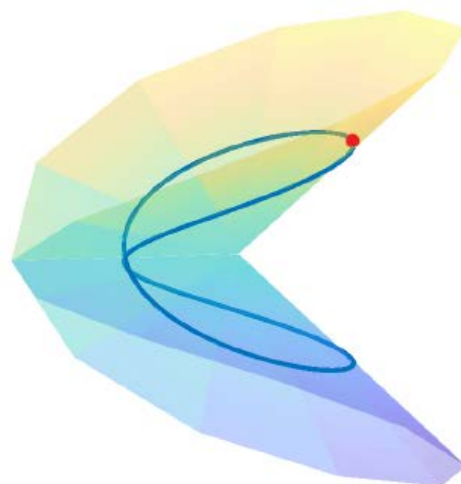
$p \in \mathbb{C}$

$$f(x; p) = x^2 - p = 0$$

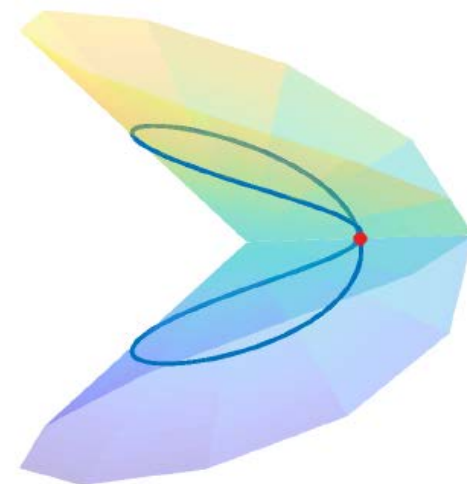
Track around a loop: $x(e^{i\theta})$



- ▶ $\theta = 0: x = 1$
- ▶ $\theta = 2\pi: x = -1$
- ▶ $\theta = 4\pi: x = 1$



real(x)



imag(x)

cycle number = winding number = 2

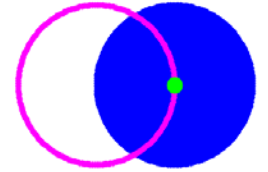
Example

Early History

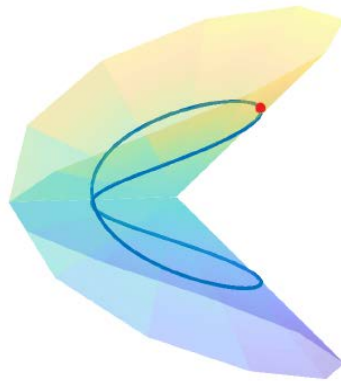
$p \in \mathbb{C}$

$$f(x; p) = x^2 - p = 0$$

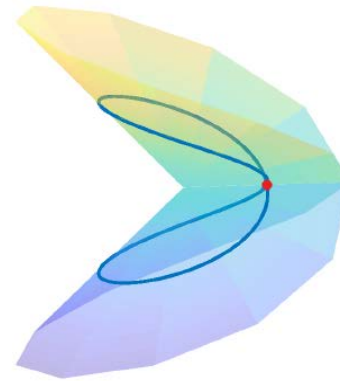
Track around a loop: $x(e^{i\theta})$



- ▶ monodromy action: permutation of solutions along loop
 - ▶ compute other solutions
 - ▶ decompose solution sets



real(x)



imag(x)

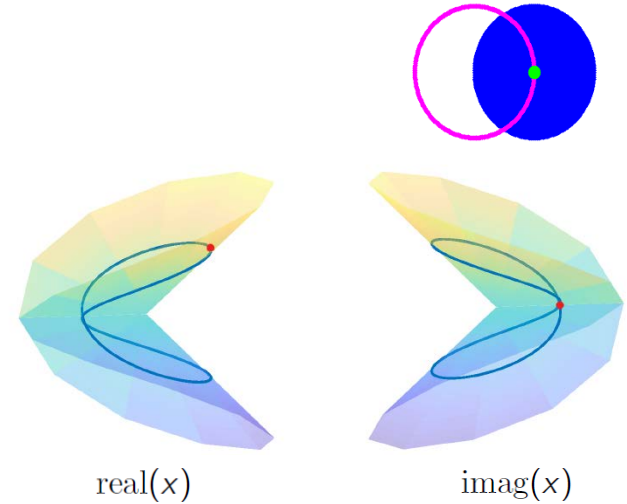
Example

Early History

 $p \in \mathbb{C}$

$$f(x; p) = x^2 - p = 0$$

Track around a loop: $x(e^{i\theta})$



- ▶ Cauchy integral theorem: computing singular endpoints
 - ▶ cycle number c
 - ▶ sufficiently small radius $r > 0$

$$x(0) = \frac{1}{2\pi c} \int_0^{2\pi c} x(re^{i\theta}) d\theta$$

- ▶ Cauchy endgame: Morgan-Sommese-Wampler (1991)

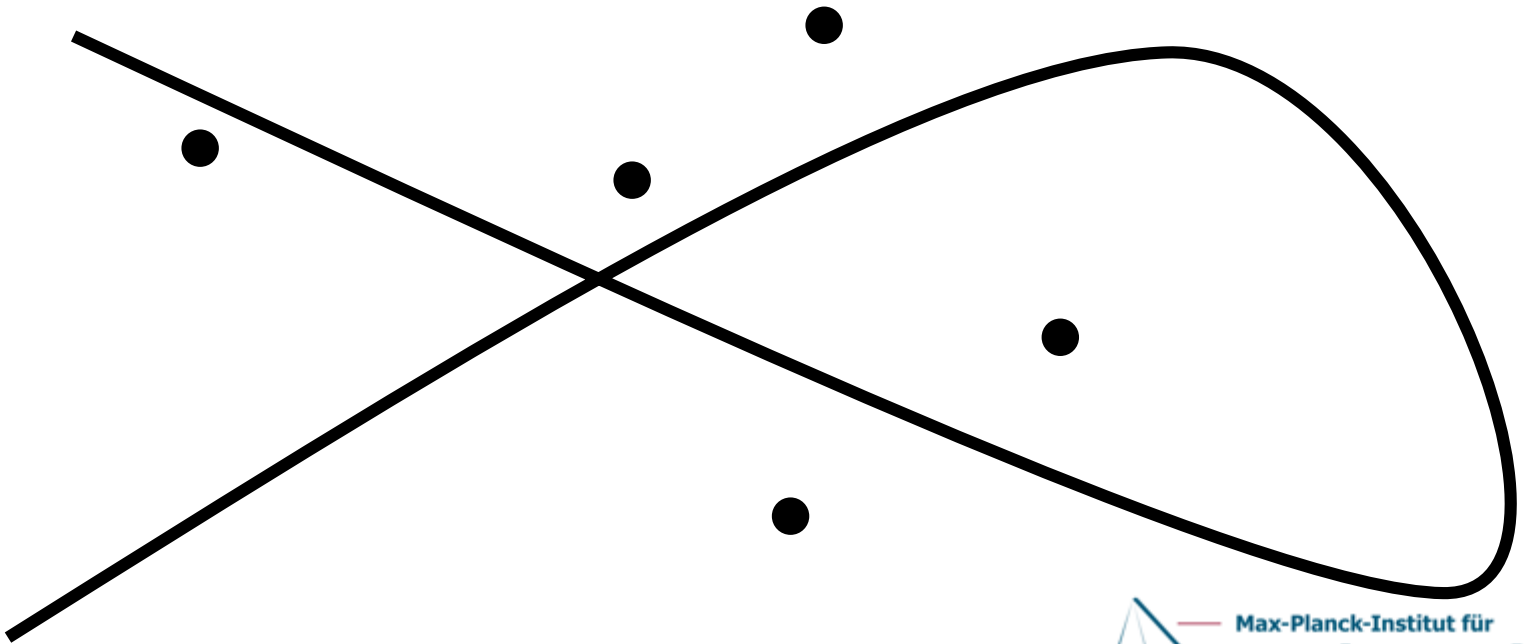
Late 20th Century 1970s – 1990s



Isolated Solutions

Find all isolated solutions of

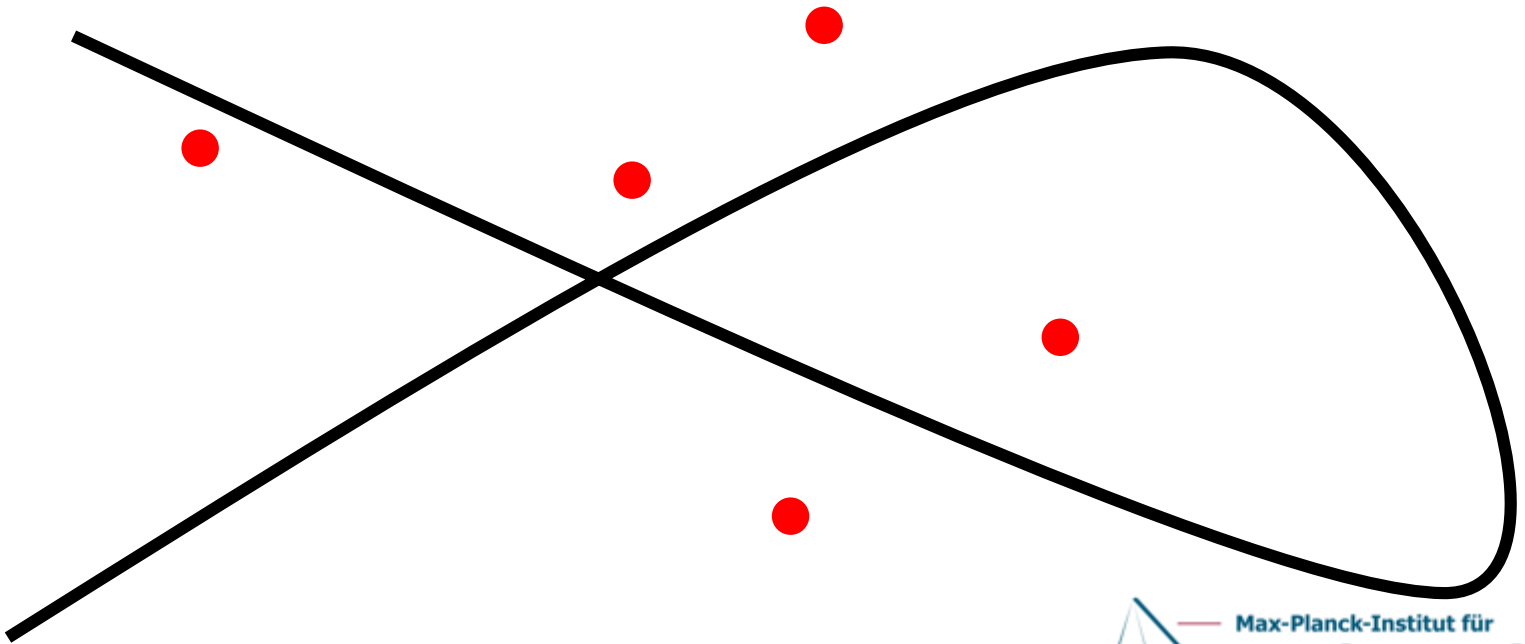
$$f(x) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} = 0$$



Isolated Solutions

Find all isolated solutions of

$$f(x) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} = 0$$



Isolated Solutions

$$f(x) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} = 0$$

Homotopy continuation requires (Morgan-Sommese (1989)):

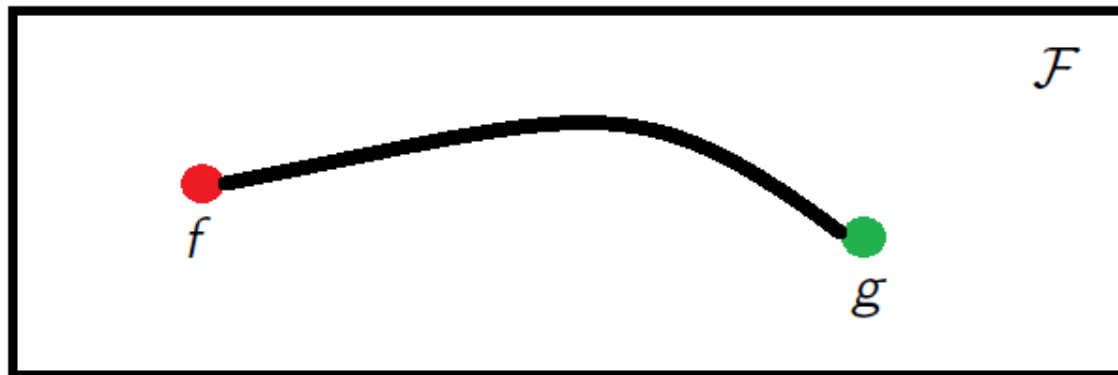
1. parameters to “continue”
 - ▶ think of f as a member of a family \mathcal{F}



Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

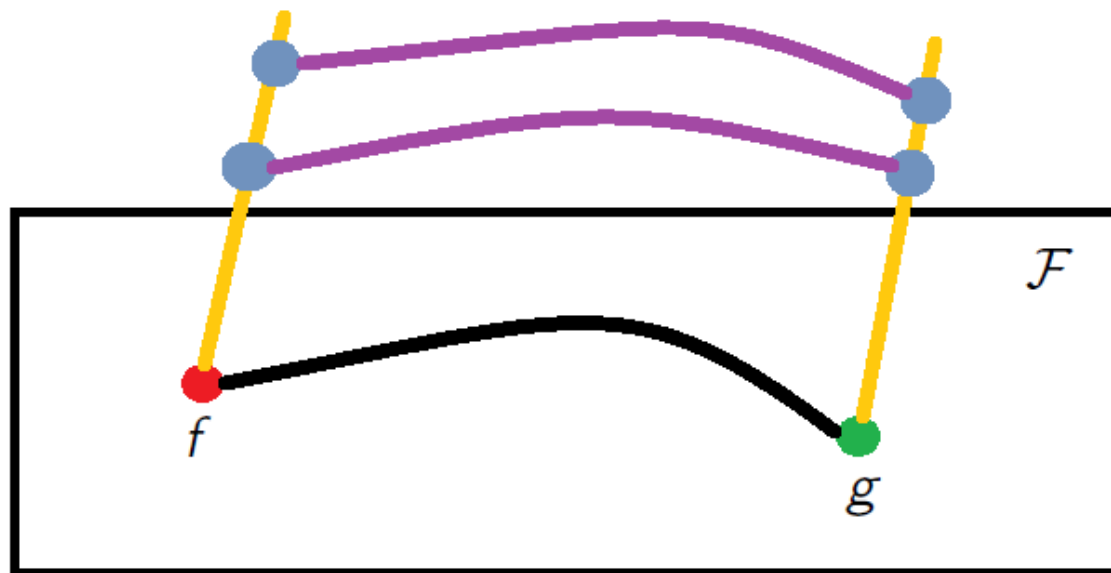
1. parameters to “continue”
 - ▶ think of f as a member of a family \mathcal{F}
2. homotopy that describes the deformation of the parameters
 - ▶ construct a deformation inside of \mathcal{F} that ends at f



Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to “continue”
 - ▶ think of f as a member of a family \mathcal{F}
2. homotopy that describes the deformation of the parameters
 - ▶ construct a deformation inside of \mathcal{F} that ends at f
3. start points to track along paths as parameters deform
 - ▶ parallelize computation – track each path independently

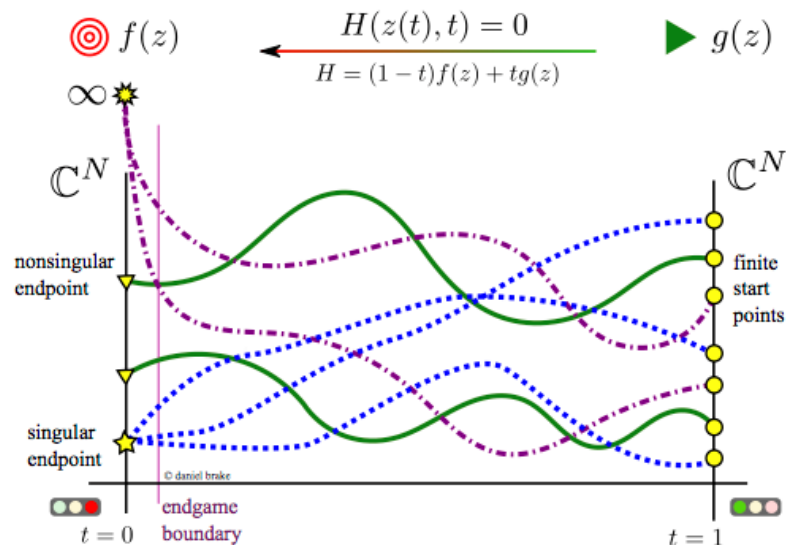


Isolated Solutions

Theorem

For properly constructed homotopies, with finite endpoints $S \subset \mathbb{C}^n$:

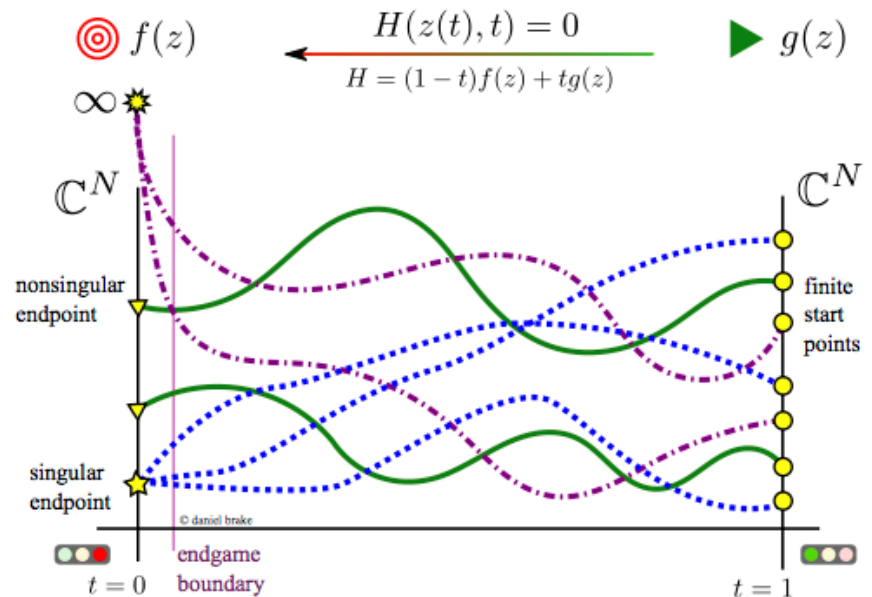
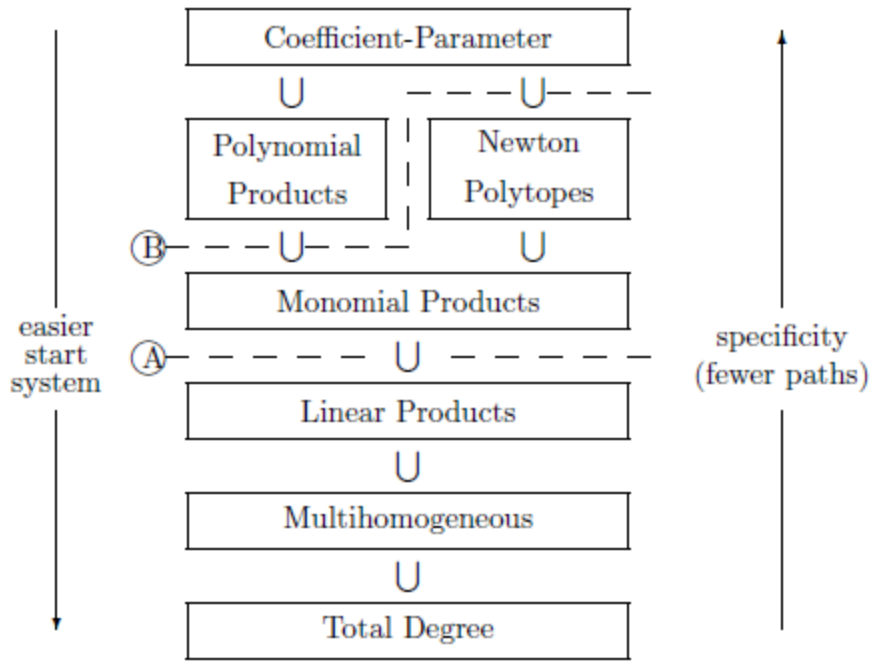
- ▶ each isolated solution is contained in S
 - ▶ in fact, S contains a point on every connected component
- ▶ for square systems, multiplicity = number of paths if isolated.
 - ▶ Local dimension test to identify nonisolated solutions (Bates-H.-Peterson-Sommese (2009))



Isolated Solutions

Art in the construction of family \mathcal{F} :

- ▶ number of start points
- ▶ ease to compute start points



Each method is sharp for generic members of \mathcal{F} .

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- ▶ Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x, y) \\ g_2(x, y) \end{bmatrix} : \deg g_i = 2 \right\} \quad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g : 4

$$H = (1 - t) \cdot f + \gamma t \cdot g$$

- ▶ $\gamma \in \mathbb{C}$ is used to create a general deformation
 - ▶ avoid singularities that arise from tracking over real numbers

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

► Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x, y) \\ g_2(x, y) \end{bmatrix} : \deg g_i = 2 \right\} \quad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g : 4

Bertini

`finite_solutions`

`input`

```
variable_group x,y;
```

```
function f1,f2;
```

```
f1 = x^2 + 2*x - 8;
```

```
f2 = x*y + 2*x + 4*y - 3;
```

1

2.0000000000000000e+00 0.0000000000000000e+00

-1.6666666666666667e-01 0.0000000000000000e+00

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- ▶ Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x, y) \end{bmatrix} : \begin{array}{l} \deg_x g_1 = 2, \\ \deg_x g_2 = \deg_y g_2 = 1 \end{array} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 2)(y - 1) \end{bmatrix} \quad H = (1 - t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g : 2

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x, y) \end{bmatrix} : \begin{array}{l} \deg_x g_1 = 2, \\ \deg_x g_2 = \deg_y g_2 = 1 \end{array} \right\}$$

Number of paths = number of isolated solutions for g : 2

Bertini

```
input variable_group x;
      variable_group y;
function f1,f2;
f1 = x^2 + 2*x - 8;
f2 = x*y + 2*x + 4*y - 3;
```

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Polyhedral (BKK, Huber-Sturmfels (1995)):

$$\mathcal{F} = \left\{ \begin{bmatrix} a_1 x^2 + a_2 x + a_3 \\ a_4 xy + a_5 x + a_6 y + a_7 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ y - 1 \end{bmatrix} \quad H = (1 - t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g : 2

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- ▶ Extra structure in the coefficients of f .

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g : 1

Example

Isolated Solutions

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Since \mathcal{F} is no longer linear, use a parameter homotopy:

$$H = p(x, y; a(t))$$

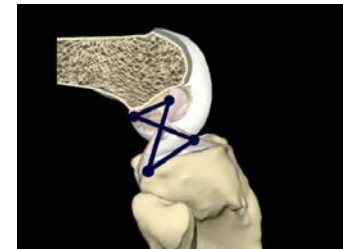
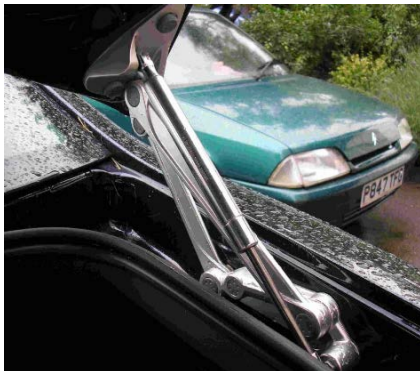
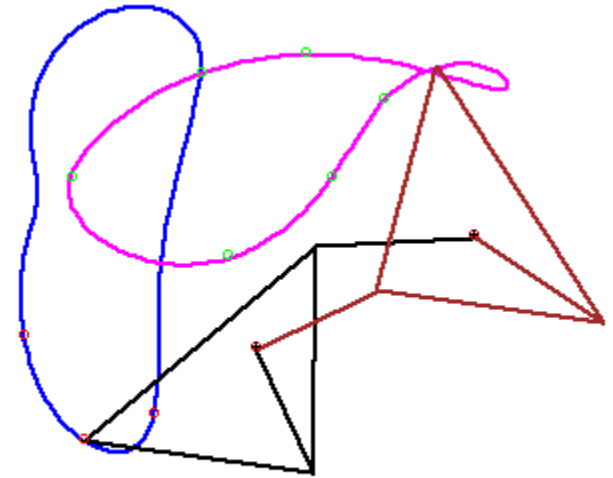
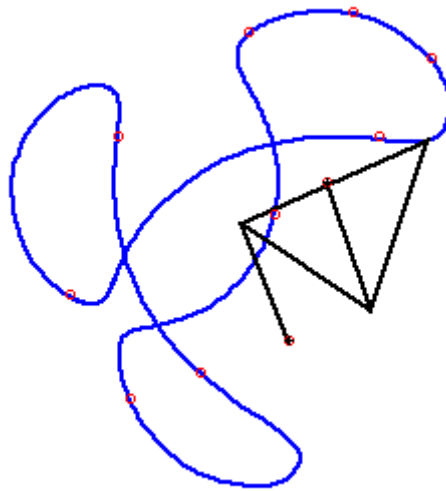
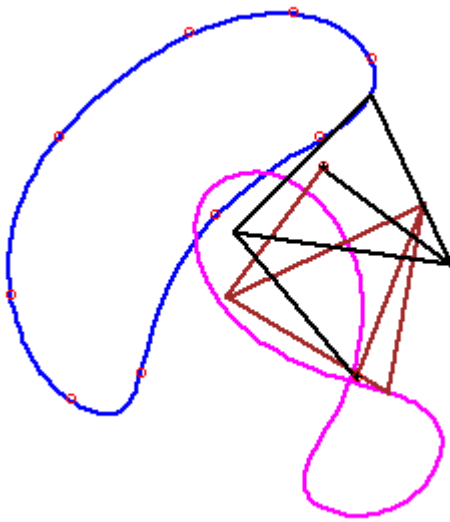
where $a(t) = (1 - \tau(t))(-4, 2, 2, -3) + \tau(t)(1, -1, 0, -1)$

$$\tau(t) = \frac{\gamma t}{1 - t + \gamma t}$$

Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.



Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

- ▶ $8652 = 6 \cdot 1442$ (Wampler-Morgan-Sommese (1992))

Their polynomial system: 4 quadratics and 8 quartics

Bézout	1,048,576	$= 2^4 \cdot 4^8$
M-hom Bézout	286,720	$= 2^{12} \cdot \binom{8}{4}$
Polyhedral	79,135	
Product decomp.	18,700	
Actual	8,652	

Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

- ▶ $8652 = 6 \cdot 1442$ (Wampler-Morgan-Sommese (1992))

Their polynomial system: 4 quadratics and 8 quartics

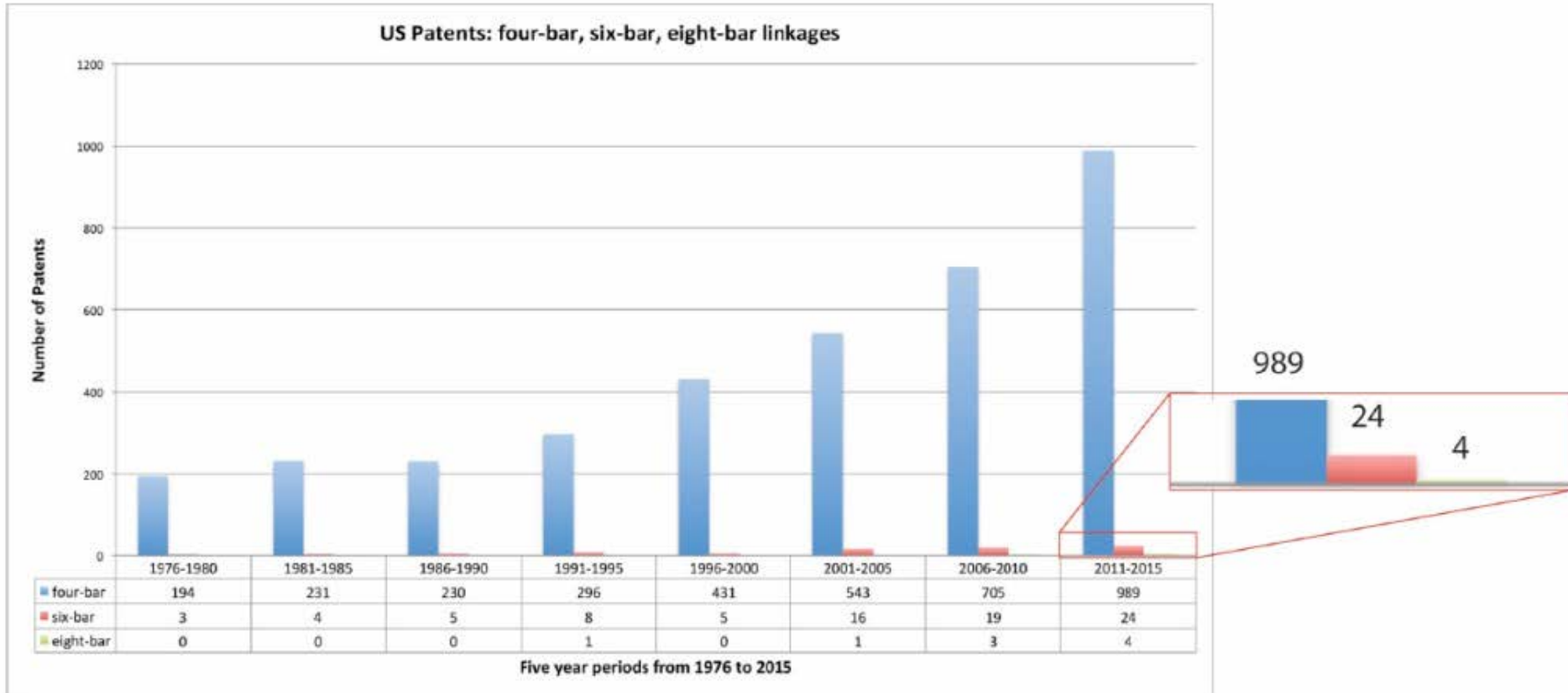
Bézout	1,048,576	$= 2^4 \cdot 4^8$
M-hom Bézout	286,720	$= 2^{12} \cdot \binom{8}{4}$
Polyhedral	79,135	
Product decomp.	18,700	
Actual	8,652	

To date: only verification via numerical algebraic geometry

- ▶ What structure can be exploited to prove 8,652 is correct?

Isolated Solutions

Mechanical Design 101 | MECHANICAL DESIGN EDUCATIONAL RESOURCE



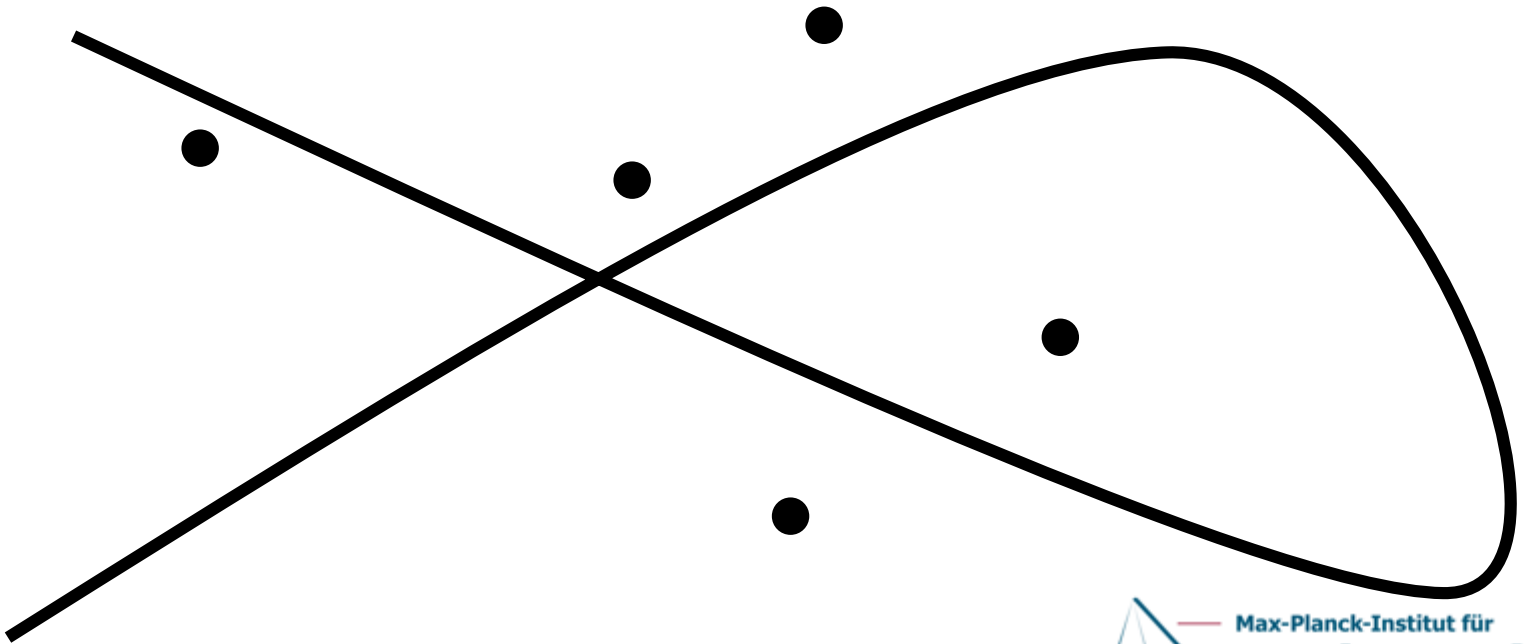
Early 21st Century



Witness Set

Describe all solutions of

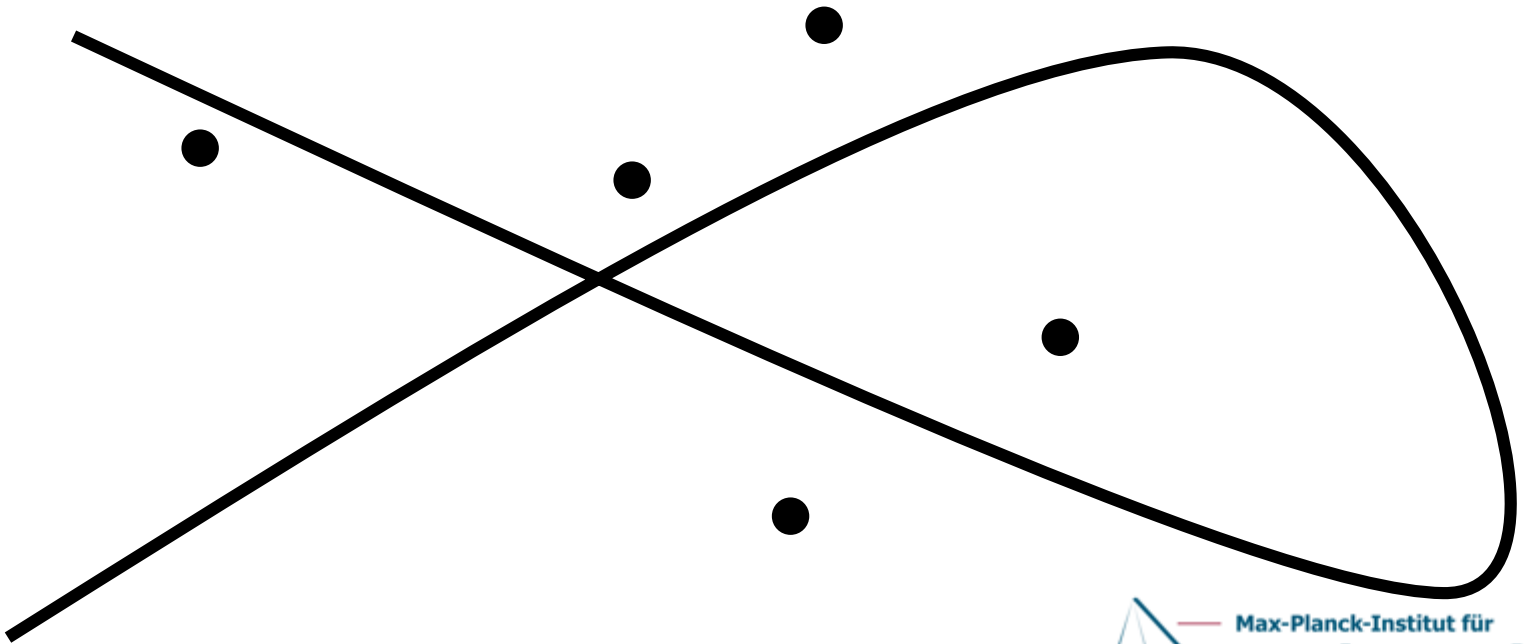
$$f(x) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_k(x_1, \dots, x_n) \end{bmatrix} = 0$$



Witness Set

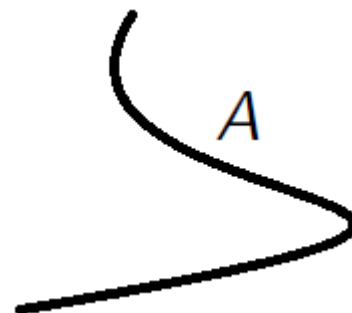
Numerical irreducible decomposition:

- ▶ decompose into irreducible components
- ▶ provide a numerical description of each irreducible component



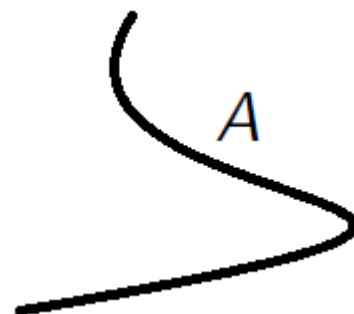
Witness Set

How to represent an irreducible algebraic variety A on a computer?



Witness Set

How to represent an irreducible algebraic variety A on a computer?



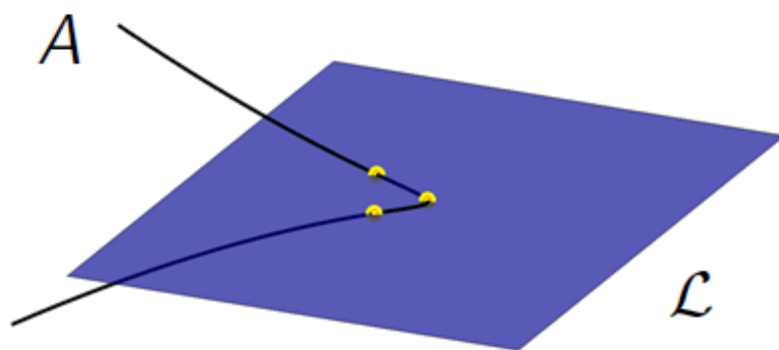
- ▶ algebraic: prime ideal $I(A) = \{g \mid g(a) = 0 \text{ for all } a \in A\}$
 - ▶ Hilbert Basis Theorem (1890): there exists f_1, \dots, f_k such that

$$I(A) = \langle f_1, \dots, f_k \rangle$$

Witness Set

How to represent an irreducible algebraic variety A on a computer?

- ▶ geometric: witness set $\{f, \mathcal{L}, W\}$ where
 - ▶ f is polynomial system where A is an irred. component of $\mathcal{V}(f)$
 - ▶ \mathcal{L} is a linear space with $\text{codim } \mathcal{L} = \dim A$
 - ▶ $W = \mathcal{L} \cap A$ where $\#W = \deg A$

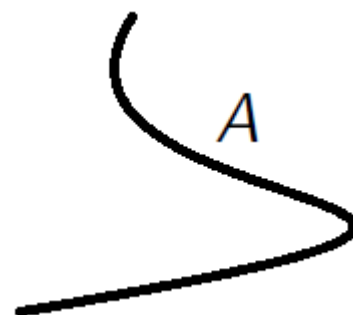


Witness Set

Example

$A = \{[s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

► $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$



Witness Set

Example

$A = \{[s^3, s^2 t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

▶ $I(A) = \langle x_1^2 - x_0 x_2, x_1 x_2 - x_0 x_3, x_2^2 - x_1 x_3 \rangle$

▶ $\{f, \mathcal{L}, W\}$ where

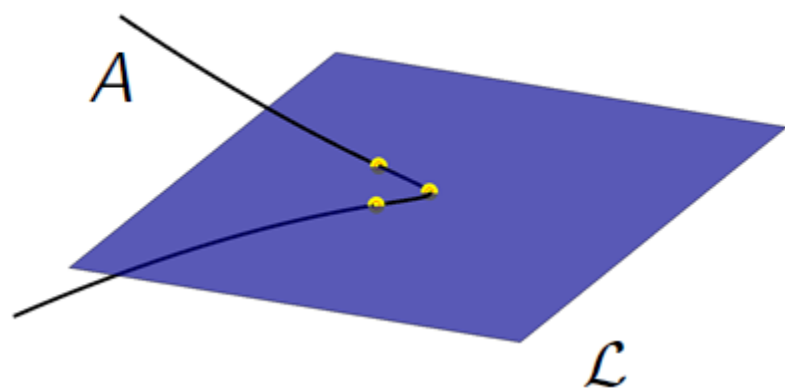
▶ $f = \begin{bmatrix} x_1^2 - x_0 x_2 \\ x_1 x_2 - x_0 x_3 \end{bmatrix}$

▶ $\mathcal{L} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0\} \subset \mathbb{P}^3$

▶ $\text{codim } \mathcal{L} = \dim A = 1$

▶ $W = \left\{ \begin{array}{l} [1, 3.2731, 10.7130, 35.0644], \\ [1, 0.8596, 0.7389, 0.6351], \\ [1, -2.1326, 4.5481, -9.6995] \end{array} \right\}$

▶ $\deg A = 3$



Witness Set

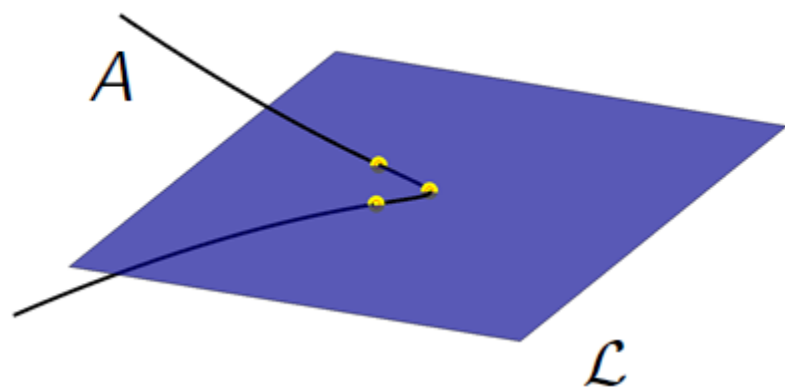
Example

$A = \{[s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$ – twisted cubic curve

▶ $I(A) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3 \rangle$

▶ $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$

$\mathcal{V}(f) = A \cup \{x_0 = x_1 = 0\}$



- ▶ Witness sets “localize” computations to A effectively ignoring the other irreducible components.
- ▶ Sample points from A by moving the linear slice \mathcal{L} .

Example

$$\blacktriangleright f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$$

Witness Set

Bertini
input

```
CONFIG
```

```
TrackType: 1;
```

```
END;
```

```
INPUT
```

```
hom_variable_group x0,x1,x2,x3;
```

```
function f1,f2;
```

```
f1 = x1^2 - x0*x2;
```

```
f2 = x1*x2 - x0*x3;
```

```
END;
```

Dimension 1: 2 classified components

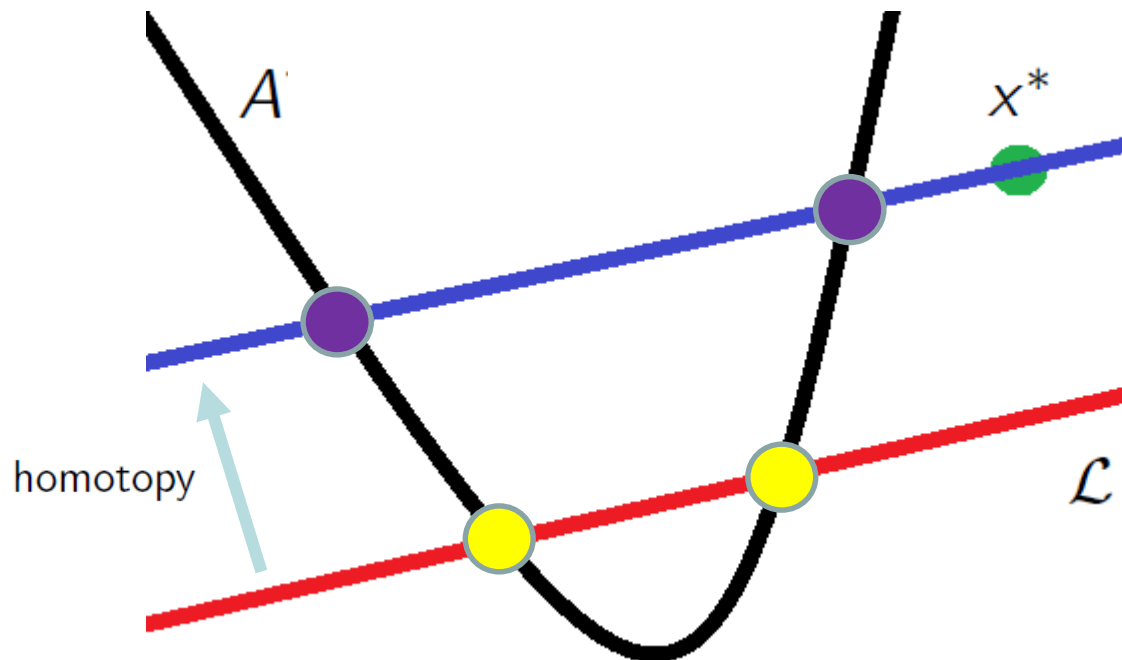
degree 1: 1 component

degree 3: 1 component

Witness Set

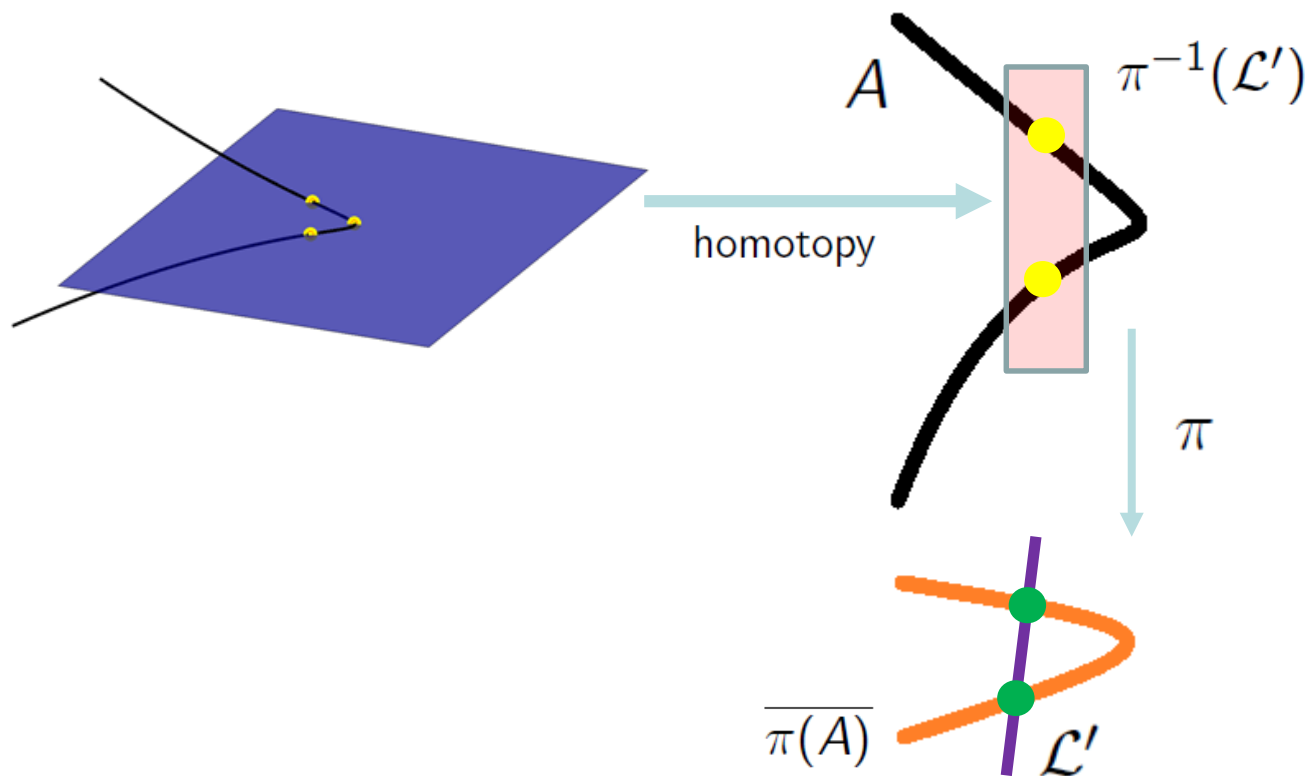
Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- ▶ membership testing: is $x^* \in A$?
- ▶ decide if $g(x^*) = 0$ for every $g \in I(A)$ without knowing $I(A)$



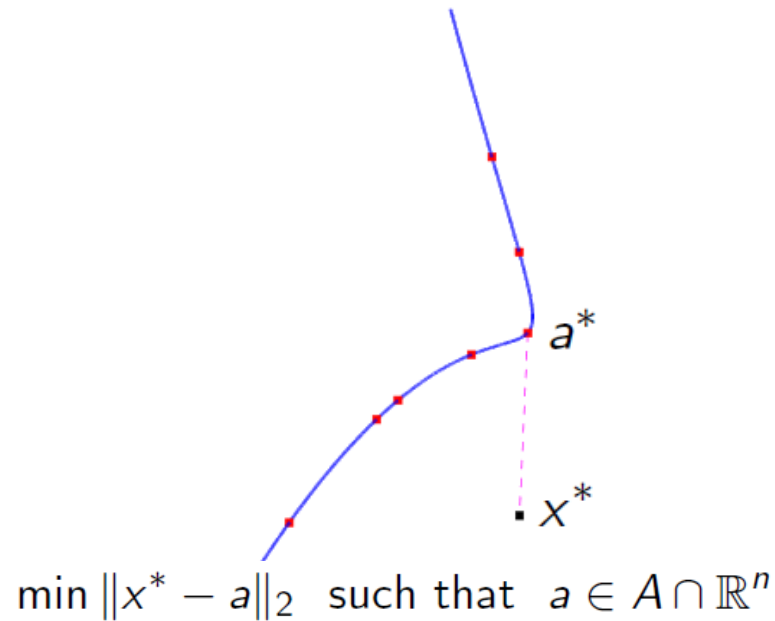
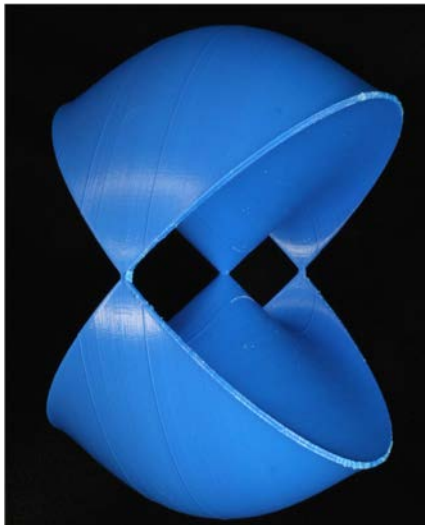
Witness Set

- ▶ projection: $\overline{\pi(A)}$
- ▶ perform computations on $\overline{\pi(A)}$ without knowing *any* polynomials that vanish on $\overline{\pi(A)}$



Witness Set

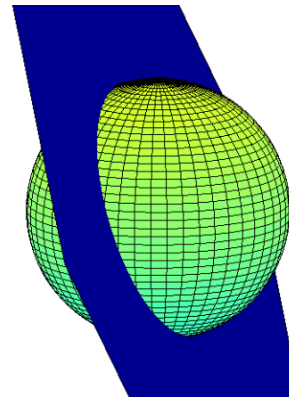
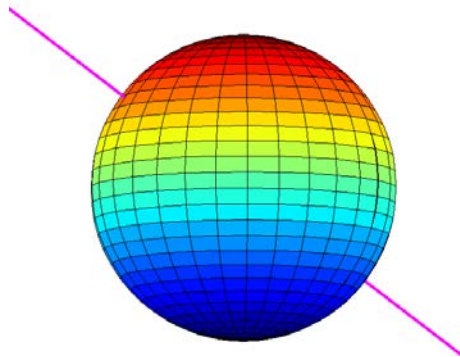
- ▶ intersection: $A \cap B$
 - ▶ special case is *regeneration*
 - ▶ $\mathcal{V}(f_1, \dots, f_k, f_{k+1}) = \mathcal{V}(f_1, \dots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
 - ▶ compute A_{sing}
 - ▶ compute critical points of optimization problem



Witness Set

Test other algebraic properties of A

- ▶ is A arithmetically Cohen Macaulay?
- ▶ is A arithmetically Gorenstein?
- ▶ is A a complete intersection?



Witness Set

Example

$$A = \sigma_4(\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^4) \subset \mathbb{P}^{35}$$

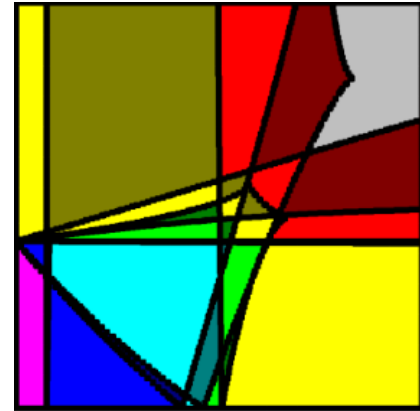
- ▶ $\dim A = 31$
- ▶ $\deg A = 345$

- ▶ $I(A)$ contains 10 poly. of degree 6 and 20 poly. of degree 9
 - ▶ Bates-Oeding (2011), Friedland-Gross (2012)

- ▶ used sampling to show that A was aCM and that these polynomials generate $I(A)$

Future?

- ▶ Specialized/structured homotopies
- ▶ Real solutions especially over parameter spaces
- ▶ Certification for singular and positive-dimensional sets
- ▶ Many applications in math, stats, science, and engineering
 - ▶ Local methods (too many solutions to find all of them?)



Summary

Numerical algebraic geometry provides a toolbox for solving polynomial systems.

- ▶ “If a problem was easy, someone else would have solved it.”
 - ▶ Gröbner basis computation probably did not terminate
- ▶ think carefully about what information you want/need
- ▶ art in building efficient homotopies that incorporate structure
- ▶ preconditioning is important
 - ▶ transform problem into form suitable for num. computations

Summary

England-Errami-Grigoriev-Radulescu-Sturm-Weber (2017)

$$\begin{aligned} & 71063292573000k_{29}x_5^3 - 299076127852260x_5^2x_6^3 - 106615407090630x_5^3x_6^2 \\ & - 3505609439955600x_5^2x_6^2 + 3557586742819200k_{29}x_6^3 + 91244417457024k_{29}x_6^4 \\ & - 3557586742819200x_5x_6^3 - 479383905861000x_5^3x_6 - 91244417457024x_5x_6^4 \\ & - 3796549898085x_5^4x_6 - 71063292573000x_5^4 + 3505609439955600k_{29}x_5x_6^2 \\ & + 479383905861000k_{29}x_5^2x_6 + 299076127852260k_{29}x_5x_6^3 + 3796549898085k_{29}x_5^3x_6 \\ & - 185019487578700k_{30}x_5x_6^3 - 598701732300k_{30}x_5^3x_6 + 106615407090630k_{29}x_5^2x_6^2 \\ & - 83232870778950k_{30}x_5^2x_6^2 \end{aligned}$$

$$\begin{aligned} & 71063292573000k_{28}x_5^3 - 106615407090630x_5^2x_6^3 - 3796549898085x_5^3x_6^2 \\ & - 479383905861000x_5^2x_6^2 + 3557586742819200k_{28}x_6^3 + 91244417457024k_{28}x_6^4 \\ & - 91244417457024k_{30}x_6^4 - 3505609439955600x_5x_6^3 - 71063292573000x_5^3x_6 \\ & - 299076127852260x_5x_6^4 - 3557586742819200x_6^4 - 91244417457024x_6^5 \\ & + 3505609439955600k_{28}x_5x_6^2 + 479383905861000k_{28}x_5^2x_6 + 299076127852260k_{28}x_5x_6^3 \\ & + 3796549898085k_{28}x_5^3x_6 - 114056640273560k_{30}x_5x_6^3 - 3197848165785k_{30}x_5^3x_6 \\ & + 106615407090630k_{28}x_5^2x_6^2 - 23382536311680k_{30}x_5^2x_6^2 \end{aligned}$$



Summary

England-Errami-Grigoriev-Radulescu-Sturm-Weber (2017)

$$\begin{aligned}
 & 71063292573000k_{29}x_5^3 - 299076127852260x_5^2x_6^3 - 106615407090630x_5^3x_6^2 \\
 & - 3505609439955600x_5^2x_6^2 + 3557586742819200k_{29}x_6^3 + 91244417457024k_{29}x_6^4 \\
 & - 3557586742819200x_5x_6^3 - 479383905861000x_5^3x_6 - 91244417457024x_5x_6^4 \\
 & - 3796549898085x_5^4x_6 - 71063292573000x_5^4 + 3505609439955600k_{29}x_5x_6^2 \\
 & + 479383905861000k_{29}x_5^2x_6 + 299076127852260k_{29}x_5x_6^3 + 3796549898085k_{29}x_5^3x_6 \\
 & - 185019487578700k_{30}x_5x_6^3 - 598701732300k_{30}x_5^3x_6 + 106615407090630k_{29}x_5^2x_6^2 \\
 & - 83232870778950k_{30}x_5^2x_6^2
 \end{aligned}$$

$$\begin{aligned}
 & 71063292573000k_{28}x_5^3 - 106615407090630x_5^2x_6^3 - 3796549898085x_5^3x_6^2 \\
 & - 479383905861000x_5^2x_6^2 + 3557586742819200k_{28}x_6^3 + 91244417457024k_{28}x_6^4 \\
 & - 91244417457024k_{30}x_6^4 - 3505609439955600x_5x_6^3 - 71063292573000x_5^3x_6 \\
 & - 299076127852260x_5x_6^4 - 3557586742819200x_6^4 - 91244417457024x_6^5 \\
 & + 3505609439955600k_{28}x_5x_6^2 + 479383905861000k_{28}x_5^2x_6 + 299076127852260k_{28}x_5x_6^3 \\
 & + 3796549898085k_{28}x_5^3x_6 - 114056640273560k_{30}x_5x_6^3 - 3197848165785k_{30}x_5^3x_6 \\
 & + 106615407090630k_{28}x_5^2x_6^2 - 23382536311680k_{30}x_5^2x_6^2
 \end{aligned}$$



$$\begin{aligned}
 & 0.071063292573k_{29}x_5^3 - 0.29907612785226x_5^2x_6^3 - 0.10661540709063x_5^3x_6^2 \\
 & - 3.5056094399556x_5^2x_6^2 + 3.5575867428192k_{29}x_6^3 + 0.091244417457024k_{29}x_6^4 \\
 & - 3.5575867428192x_5x_6^3 - 0.479383905861x_5^3x_6 - 0.091244417457024x_5x_6^4 \\
 & - 0.003796549898085x_5^4x_6 - 0.071063292573x_5^4 + 3.5056094399556k_{29}x_5x_6^2 \\
 & + 0.479383905861k_{29}x_5^2x_6 + 0.29907612785226k_{29}x_5x_6^3 + 0.003796549898085k_{29}x_5^3x_6 \\
 & - 0.1850194875787k_{30}x_5x_6^3 - 0.0005987017323k_{30}x_5^3x_6 + 0.10661540709063k_{29}x_5^2x_6^2 \\
 & - 0.08323287077895k_{30}x_5^2x_6^2
 \end{aligned}$$

$$\begin{aligned}
 & 0.071063292573k_{28}x_5^3 - 0.10661540709063x_5^2x_6^3 - 0.003796549898085x_5^3x_6^2 \\
 & - 0.479383905861x_5^2x_6^2 + 3.5575867428192k_{28}x_6^3 + 0.091244417457024k_{28}x_6^4 \\
 & - 0.091244417457024k_{30}x_6^4 - 3.5056094399556x_5x_6^3 - 0.071063292573x_5^3x_6 \\
 & - 0.29907612785226x_5x_6^4 - 3.5575867428192x_6^4 - 0.091244417457024x_6^5 \\
 & + 3.5056094399556k_{28}x_5x_6^2 + 0.479383905861k_{28}x_5^2x_6 + 0.29907612785226k_{28}x_5x_6^3 \\
 & + 0.003796549898085k_{28}x_5^3x_6 - 0.11405664027356k_{30}x_5x_6^3 - 0.003197848165785k_{30}x_5^3x_6 \\
 & + 0.10661540709063k_{28}x_5^2x_6^2 - 0.02338253631168k_{30}x_5^2x_6^2
 \end{aligned}$$



Summary

Bertini

- ▶ Written in C
- ▶ Stable (Bertini 1.0: April 20, 2008, Bertini 1.6: May 22, 2018)
- ▶ Bertini_real, Bertini.m2, BertiniLab, Paramotopy, ...

Bertini2

- ▶ Written in C++ and Python (interactive session)
- ▶ In development
- ▶ Backwards compatible with Bertini

Danke schön

