Equivariant Witness Sets

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Many varieties are naturally invariant under a finite group action. For example,

$$f = q_{400}q_{040}q_{004} - q_{400}q_{022}^2 - q_{040}q_{202}^2 - q_{004}q_{220}^2 - 2q_{220}q_{202}q_{022} \tag{1}$$

from [4] defines a hypersurface $\mathcal{H} \subset \mathbb{P}^5$ that is invariant under the group $G = \langle\sigma\rangle$ where

$$\sigma(q_{ijk}) = q_{kij}.$$

This hypersurface $\mathcal{H}$ is related to the Luroth hypersurface [5] which also invariant under the same group G. Other examples include secant varieties of Segre varieties.

Problem 1. *Suppose that V is an irreducible variety and G is a finite group that acts invariantly on V, i.e., $g(v) \in V$ for every $v \in V$ and $g \in G$. Can one utilize G to create an "equivariant" witness set for V. Such a witness set should be able to be used, e.g., for testing membership in V by tracking less paths than a standard witness set for V.*

Some special cases were used in [1, 2, 3].

Returning to f as in (1) with $[q_{400}, q_{040}, q_{004}, q_{220}, q_{022}, q_{202}] \in \mathbb{P}^5$, the system

$$\begin{bmatrix} f \\ (q_{400}+q_{040}+q_{004}) + 3(q_{220}+q_{202}+q_{022}) \\ 2q_{400}+3q_{040}-q_{004}-2q_{220}+5q_{022}-3q_{202} \\ 2q_{040}+3q_{004}-q_{400}-2q_{022}+5q_{202}-3q_{220} \\ 2q_{004}+3q_{400}-q_{040}-2q_{202}+5q_{220}-3q_{022} \end{bmatrix} = 0$$

has 3 solutions which are invariant under G, approximately

$$\begin{array}{rrrrrr} [1.0000, & -0.0964, & -0.9036, & -0.2268, & -0.2985 & 0.5253], \\ [-0.9036, & 1.0000, & -0.0964, & 0.5253, & -0.2268 & -0.2985], \\ [-0.0964, & -0.9036, & 1.0000, & -0.2985, & 0.5253 & -0.2268]. \end{array}$$

Hence, due to this invariance, only one point is needed rather than $3 = \deg f$.

Problem 1 aims to generalize this construction. Remark 4.1 of [2] shows that some caution is needed in this generalization.

References

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