Maximum Number of Real Solutions for Four Oscillators

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Two related systems with four oscillators arise from the Kuramoto model and power flow equations. In both cases, the maximum number of real solutions is unknown.

Fix $s_4 = 0$ and $c_4 = 1$ and consider the general form as follows. The parameters of the system are $\alpha \in \mathbb{R}^3$ and symmetric matrix $B \in \mathbb{R}^{4\times 4}$. The system is

$$F(s,c;\alpha,B) = \begin{bmatrix} \alpha_i - \sum_{j=1}^4 B_{ij}(s_i c_j - s_j c_i) & i = 1,2,3 \\ s_i^2 + c_i^2 - 1 & i = 1,2,3 \end{bmatrix} = 0 \tag{1}$$

which consists of 6 equations in 6 variables.

1. Kuramoto system

This system corresponds with the classical Kuramoto system [2] when $B = \mathbb{1}$ (each entry is 1). In fact, we can generalize this by taking B to be a rank one matrix, i.e., $B = vv^T$ where $v \in \mathbb{R}^4$. In this case, the generic root count is 14.

Problem 1 (Conjecture 4.14 [1])**.** *For the Kuramoto system with rank one coupling, the maximum number of isolated real solutions is 10.*

Thus upper bound can be achieved [1, 4] and is the upper bound for special cases [1, Thm. 4.6].

2. Power flow system

The system corresponds to a power flow system of a lossless, four-bus system of PV buses connected by lines with arbitrary susceptances. If $\alpha = 0$, then the system is said to have zero power injections. In both cases, the generic root count is 20.

Problem 2. *Determine the maximum number of isolated real solutions.*

It was shown in [3] that 16 real solutions is possible and, for zero power injections, 16 is the upper bound when any of the parameters B_{ij} are at or near zero.

More recently, as part of work for [5], Zachary Charles has found an example with 18 real solutions that also has zero power injections. In light of [3], this is possible when all parameters are sufficiently far away from zero.

Thus, either 18 is indeed the upper bound or all 20 solutions can be real.

References

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