Alt's Problem

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Alt's problem [1] is to count the number of four-bar linkage coupler curves that pass through 9 general points in the plane as illustrated in Figure 1. The answer to this problem computed using homotopy continuation is 1442 [6] . Although this problem has been numerically solved many times using various formulations, e.g., [2, 3, 5], and validated using a numerical trace test [4], a complete understanding of why the answer is 1442 remains elusive. Moreover, such an understanding to Alt's problem for four-bar linkages would yield insight into how to solve the corresponding 15 point path synthesis problems for the various types of six-bar linkages.

Problem 1. *Prove that* 1442 *is the solution to Alt's problem.*

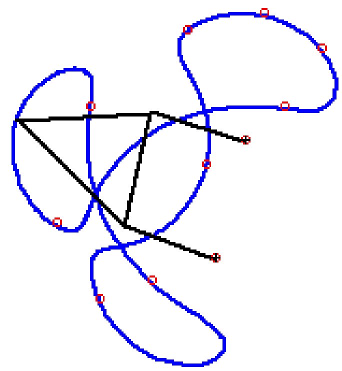

FIGURE 1. Four-bar linkage whose coupler curve passes through 9 given points

Alt's problem for 9 points is the most number of general points to still have a solution. The paper [2] considered both the 9 point problem (finitely many solutions) as well as the 3 point problem (codimension 2 set of solutions) which showed that a Gröbner basis approach on both failed to terminate in 24 hours. Thus, one approach to solve Problem 1 is to exploit structure to create a family of polynomial systems where the general root count is $8652 = 1442 \cdot 6$ – the factor of $6 = 2 \cdot 3$ arises from a trivial symmetry ($\times\ 2$) and Roberts' cognates ($\times\ 3$).

One can explore the n-point problems for $n = 2, \ldots, 9$ (the 1-point problem does not impose any conditions – merely sets the frame of reference). It is classically known that the $n = 2$ case defines a hypersurface of degree 7. The following table is derived from [3, Table 2] which shows the dimension and degrees for the n-point problems.

n	dimension	degree
2	7	7
3	6	43
4	5	234
5	4	1108
6	3	3832
7	2	8716
8	1	10858
9	0	8652

To get started on Problem 1, why does the $n = 3$ case have degree $43 < 7 \cdot 7 = 49$?

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