Problem Session for Numerical Algebraic Geometry

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Problem 1 (Cubic surfaces). *Consider a cubic surface in $\mathbb{C}^3$ defined by*

$$f(x) = a_{000} + a_{100} \cdot x + a_{010} \cdot y + a_{001} \cdot z + \cdots + a_{003} \cdot z^3 = 0.$$

a. *Setup a parameter homotopy, where the parameters are the 20 coefficients $a_{000}, \ldots, a_{003}$, that computes the 27 lines on the corresponding cubic surface.*

b. *Use the parameter homotopy to verify that all 27 lines on the Clebsch cubic are real:*

$$2\sqrt{2}y^3 + 2x^2z - 8y^2z - 2x^2 + 8y^2 + 3\sqrt{2}yz^2 - 10\sqrt{2}yz - z^3 + 3\sqrt{2}y + 3z^2 - 3z + 1 = 0.$$

c. *Use the solution to (b) to compute all Eckardt points [points on the cubic surface where 3 of the 27 lines meet] for the Clebsch cubic.*

d. *What is the behavior of the endpoints of the parameter homotopy when applied to Cayley's nodal cubic:*

$$xyz + xy + xz + yz = 0?$$

e. *Repeat (d) with Whitney's umbrella:*

$$y^2z - x^2 = 0?$$

f. *Compute the degree of the hypersurface $\mathcal{H}$ of singular cubics by computing a (pseudo)witness set for $\mathcal{H}$.*

g. *Use (f) to verify that Cayley's nodal cubic and Whitney's umbrella are contained in $\mathcal{H}$ and that the Clebsch cubic is not contained in $\mathcal{H}$.*

Problem 2 (Bitangents and tritangents)**.** *A general genus* 3 *curve is canonically represented as a quartic plane curve and has* 28 *bitangent lines [lines that are simultaneously tangent at two points on the curve]. A general genus* 4 *curve is canonically represented as a space sextic which is the complete intersection of quadric and cubic hypersurfaces and has* 120 *tritangent planes [planes that are simultaneously tangent at three points on the curve].*

a. *Setup a parameter homotopy for computing bitangents of quartic plane curve where the parameters are the coefficients of the quartic.*

b. *Use the parameter homotopy to verify that all 28 bitagents of the Trott curve are real:*

$$1.44(x^4 + y^4) + 3.5x^2y^2 - 2.25(x^2 + y^2) + 0.81 = 0.$$

c. *Setup a parameter homotopy for computing tritangents of a complete intersection of quadric and cubic hypersurfaces in space.*
   - *If having difficulty solving, the solution to an example is available at* `www.nd.edu/~jhauenst/Leipzig2018`.

d. *Use the parameter homotopy to compute the number of real tritangents [tritangent plane defined by real coefficients] and totally real tritangents [real tritangent plane that is tangent at 3 real points] for the following examples in* $\mathbb{P}^3$*:*

$$xw - yz = 0.25x^3 - 0.24x^2y - 0.14y^3 - 0.89x^2z - 0.55xyz - 0.31y^2z + 0.86yzw + 0.74z^2w - 0.45zw^2 - 0.62w^3 = 0$$

$$xw - yz = 0.89x^3 - 0.41x^2y - 0.87xy^2 - 0.25y^2z - 0.26xz^2 + 0.56yz^2 + 0.87z^3 + 0.42y^2w - 0.67zw^2 - 0.42w^3 = 0$$

(Open) e. *Is there a way to "easily" observe the generic number of solutions are 28 and 120, respectively, directly from the polynomial system formulation and then "easily" construct a start system with 28 and 120 solutions, respectively?*

Problem 3 (Plane conics)**.** *A classical enumerative geometry problem is to count the number of plane conics in $\mathbb{C}^3$ that pass through k points and intersect $8-2k$ lines in general position. The following table lists the degrees based on k:*

k	*number of plane conics*
3	1
2	4
1	18
0	92

a. *Setup a parameter homotopy for each of these problems.*

b. *Verify that all 92 plane conics that intersect the lines $\mathcal{L}_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$ are real:*

$$\begin{array}{ll}
p_1 = (0.46978, -3.988, -2.3527) & v_1 = (2.9137, 1.546, -0.27448) \\
p_2 = (3.19, 0.5752, 3.0953) & v_2 = (0.56569, 1.108, 4.3629) \\
p_3 = (0.40308, 0.78659, 0.9053) & v_3 = (-3.0656, -1.4638, 1.4096) \\
p_4 = (-4.3743, 4.0046, -1.0243) & v_4 = (-0.9163, 3.6495, -2.6528) \\
p_5 = (1.5198, -0.86125, -4.5963) & v_5 = (-3.8418, 3.9541, 2.5494) \\
p_6 = (0.46801, -4.0308, -2.4411) & v_6 = (1.0225, 1.6422, 1.5925) \\
p_7 = (-3.3382, 3.8432, 1.693) & v_7 = (-4.4657, 1.9618, 1.6865) \\
p_8 = (1.3536, 3.6311, 0.42864) & v_8 = (-3.1442, -2.4915, -0.63586).
\end{array}$$

c. *For $k = 1$, take the point to be the origin (without loss of generality). Experiment with different choices of 6 real lines to count the possible number of real solutions.*

(Open) d. *Taking the point to be the origin, is it possible to find 6 real lines for which there are no real solutions? If this is impossible, what structure in the system requires there to always be a real solution when the parameters (which define the real lines) are real?*

(Open) e. *Is there a way to "easily" observe the generic number of solutions are 4, 18, and 92 respectively, directly from the polynomial system formulation and then "easily" construct a start system with 4, 18, and 92 solutions, respectively?*

Problem 4 (Kuramoto/power flow)**.** *For n oscillators, fix $s_n = 0$ and $c_n = 1$, parameters $\alpha \in \mathbb{C}^{n-1}$ and symmetric matrix $B \in \mathbb{C}^{n\times n}$, and consider the polynomial system*

$$F(s,c;\alpha,B) = \begin{bmatrix} \alpha_i - \sum_{j=1}^{n} B_{ij}(s_i c_j - s_j c_i) & i = 1,\ldots,n-1 \\ s_i^2 + c_i^2 - 1 & i = 1,\ldots,n-1 \end{bmatrix} = 0$$

which consists of $2(n-1)$ equations in $2(n-1)$ variables.

a. *Setup a parameter homotopy for $n = 3$ and $n = 4$ when α and $B = B^T$ are general.*

b. *For $n = 3$ and $n = 4$, construct a parameter homotopy on the subparameter space for generic α and $B = vv^T$ where v is generic (rank 1 coupling case).*

c. *Experiment with the parameters to show that all solutions can be real in both the general and rank 1 cases when $n = 3$.*

d. *For $n = 4$, show that $\alpha = (0.5, 0.5, -0.5, -0.5)$ and $v = (1,1,1,1)$ with $B = vv^T$ has 10 real solutions. What happened to the other 4 solutions? What happens when one slightly perturbs α?*

e. *For $n = 4$, show that $\alpha = 0$ and* $B = \begin{bmatrix} 0 & -3.9524 & 0.3177 & 4.3192 \\ -3.9524 & 0 & 6.3855 & -7.9773 \\ 0.3177 & 6.3855 & 0 & -7.4044 \\ 4.3192 & -7.9773 & -7.4044 & 0 \end{bmatrix}$ *(data adapted from Zachary Charles) has 18 real solutions.*

(Open) f. *Is 10 the maximum number of real solutions for $n = 4$ with rank 1 coupling?*

(Open) g. *Is it possible to have all 20 solutions real for $n = 4$ with arbitrary coupling?*

(Open) h. *Determine the generic number of solutions as a function of $r = \operatorname{rank} B$ and n.*

Problem 5 (Special orthogonal and special Euclidean). *Let*

$$SO(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I, \det(A) = 1\}$$

be the set of special orthogonal matrices and

$$SE(n) = \{(A, x, y, r) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid A \in SO(n), y + Ax = 2r + x^T x = 0\}$$

be a representation of the special Euclidean group. Let $\mathcal{SO}_n$ *and* $\mathcal{SE}_n$ *be the Zariski closure of* $SO(n)$ *and* $SE(n)$*, respectively.*

a. *Compute* $\deg \mathcal{SE}_2$*. (This corresponds with the generic number of assembly configurations for planar pentads.)*

b. *Experiment to find the possible number of real witness points for* $\mathcal{SE}_2$*.*

c. *Compute* $\deg \mathcal{SO}_3$ *(This corresponds with the generic number of assembly configurations for spherical pentads.)*

d. *Experiment to find the possible number of real witness points for* $\mathcal{SO}_3$*.*

e. *Compute* $\deg \mathcal{SE}_3$*. (This corresponds with the generic number of assembly configurations for Stewart-Gough platforms.)*

f. *Verify that all witness points for* $\mathcal{SE}_3$ *with respect to the linear system*

$$\ell_i = r + b_i^T x + p_i^T y + p_i^T M b_i - (b_i^T b_i + p_i^T p_i - d_i^2)/2 = 0, \quad i = 1, \ldots, 6,$$

are real for the following data from Dietmaier (1998):

$$B = \begin{bmatrix} 0 & 1.107915 & 0.549094 & 0.735077 & 0.514188 & 0.590473 \\ 0 & 0 & 0.756063 & -0.223935 & -0.526063 & 0.094733 \\ 0 & 0 & 0 & 0.525991 & -0.368418 & -0.205018 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.542805 & 0.956919 & 0.665885 & 0.478359 & -0.137087 \\ 0 & 0 & -0.528915 & -0.353482 & 1.158742 & -0.235121 \\ 0 & 0 & 0 & 1.402538 & 0.107672 & 0.353913 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 & 0.645275 & 1.086284 & 1.503439 & 1.281933 & 0.771071 \end{bmatrix}$$

where b_i *and* p_i *is the* i^{th} *column of* B *and* P*, respectively. (This computation verifies that every assembly configuration for a Stewart-Gough platform can be real.)*

(Open) g. *Determine the maximum number of real witness points for* $\mathcal{SO}_N$ *and* $\mathcal{SE}_N$*. Can they all be real?*

Problem 6 (Equivariant witness set)**.** *Many varieties are naturally invariant under a finite group action and the aim is to exploit this structure to compute simplify the computation of witness sets.*

a. *Verify that the hypersurface* $\mathcal{H} \subset \mathbb{P}^5$ *defined by*

$$f = q_{400}q_{040}q_{004} - q_{400}q_{022}^2 - q_{040}q_{202}^2 - q_{004}q_{220}^2 - 2q_{220}q_{202}q_{022} = 0$$

is invariant under the finite cyclic group $G = \langle \sigma \rangle$ *where* $\sigma(q_{ijk}) = q_{kij}$.

b. *Verify that*

$$\left[\begin{array}{c} (q_{400} + q_{040} + q_{004}) + 3(q_{220} + q_{202} + q_{022}) \\ 2q_{400} + 3q_{040} - q_{004} - 2q_{220} + 5q_{022} - 3q_{202} \\ 2q_{040} + 3q_{004} - q_{400} - 2q_{022} + 5q_{202} - 3q_{220} \\ 2q_{004} + 3q_{400} - q_{040} - 2q_{202} + 5q_{220} - 3q_{022} \end{array} \right] = 0$$

defines a line $\mathcal{L} \subset \mathbb{P}^5$ *that is invariant under* G.

c. *Compute* $W = \mathcal{H} \cap \mathcal{L}$ *and verify that* $\#W = \deg \mathcal{H} = \deg f = 3$*, i.e.,* $\mathcal{L}$ *intersects* $\mathcal{H}$ *transversely. Verify that the points in* W *are in the same* G*-orbit, i.e., we can write* $W = \{w_1, w_2, w_3\}$ *such that* $w_2 = \sigma(w_1)$ *and* $w_3 = \sigma(w_2) = \sigma(\sigma(w_1))$.

d. *The polynomial* f *arises from studying the Lüroth hypersurface* $\mathcal{A} \subset \mathrm{Sym}^4(\mathbb{C}^3) = \mathbb{P}^{14}$ *which is the closure of quartics in* $\mathbb{P}^2$ *of the form*

$$\ell_1 \cdot \ell_2 \cdot \ell_3 \cdot \ell_4 \cdot \ell_5 \cdot (\ell_1^{-1} + \ell_2^{-1} + \ell_3^{-1} + \ell_4^{-1} + \ell_5^{-1})$$

where ℓ_i *is a linear form on* $\mathbb{P}^2$*. Hence, each quartic of this form contains the* 10 *points of pairwise intersection of the five lines. Show that* $\deg \mathcal{A} = 54$.

e. *Use the (pseudo)witness set for* $\mathcal{A}$ *computed in (d) to verify that*

$$q_1 = x^3y + x^3z + 3x^2y^2 + 10x^2yz + 4x^2z^2 + 2xy^3 + 13xy^2z + 16xyz^2 + 3xz^3 + 2y^3z + 5y^2z^2 + 3yz^3$$

is contained in the Lüroth hypersurface while the following is not:

$$q_2 = x^3y + y^3z + z^3x.$$

f. *If we write* $\mathcal{A} \subset \mathbb{P}^{14}$ *using coordinates* $[q_{ijk} : i + j + k = 4$ *and* $i, j, k \geq 0]$ *which is the coefficient of* $x^iy^jz^k$ *in the quartic, verify that* $\mathcal{A}$ *is invariant under the group* G *as above.*

g. *Is there a line* $\mathcal{L} \subset \mathbb{P}^{14}$ *that is invariant under* G *and intersects* $\mathcal{A}$ *transversely? Compute the number of distinct* G*-orbits in* $\mathcal{A} \cap \mathcal{L}$.

(Open) h. *If an irreducible variety* V *is invariant under the action of a finite group* G*, can one create an "equivariant" witness set for* V*? When is this possible? One goal of such an "equivariant" witness set is to develop an "equivariant" membership test.*

Problem 7 (Monodromy groups and Alt-Burmester problems). *The monodromy group of a parameterized polynomial system $f(x;p) = 0$ for which $f(x;p^*) = 0$ has k solutions for generic p^* is a subgroup of the symmetric group S_k consisting of all permutations of the roots under monodromy loops in the parameter space with the branch locus removed.*

a. *Consider the parameterized polynomial*

$$f(x;p) = x^2 - x - p = 0.$$

Verify that $\{-1/4\}$ is the branch locus so we aim to create loops in $\mathbb{C} \setminus \{-1/4\}$. At $p = 0$, we have two solutions, say $x_1 = 0$ and $x_2 = 1$. Perform a monodromy loop that encircles the point $p = -1/4$, e.g., $p(\theta) = -1/4 + 1/4 \cdot e^{i\theta}$, and show that this loop generates a transposition of the roots. Hence, the monondromy group is the symmetric group S_2.

b. *Consider the parameterized polynomial*

$$f(x;p) = x^4 - 4x^2 + p = 0.$$

Since the solutions naturally arise in 2 groups of 2, the monodromy group cannot be the full symmetric group S_4. In fact, the monodromy group must be a subset of the wreath product $S_2 WrS_2 = D_4$, the dihedral group which consists of $2^2 \cdot 2! = 8$ elements. Starting from, say, $p = 3$, what is the element of the monodromy group generated by encircling $p = 0$? What about $p = 4$? From these two elements, show that the monodromy group is indeed $D_4 = S_2 WrS_2$ by showing that these two elements generate a group of size 8.

Burmester (1886) solved the motion generation (based on poses = position + orientation) and Alt (1923) formulated the path synthesis problem (based only on position) for four-bar linkages. The Alt-Burmester problems consist of a mix of pose constraints (M of them) and path point constraints (N of them).

Let $a_1, a_2, x_1, x_2, b_1, b_2, y_1, y_2$ be the variables that define a four-bar linkage. We write the constraints using isotropic coordinates based on

$$\begin{array}{llll} a = a_1 + a_2 i, & A = a_1 - a_2 i, & x = x_1 + x_2 i, & X = x_1 - x_2 i, \\ b = b_1 + b_2 i, & B = b_1 - b_2 i, & y = y_1 + y_2 i, & Y = y_1 - y_2 i. \end{array}$$

A pose constraints is described by the input data $(d_1, d_2, t_1, t_2) \in \mathbb{R}^4$ where $t_1^2 + t_2^2 = 1$. Hence, for isotropic coordinates $d = d_1 + d_2 i$, $D = d_1 - d_2 i$, $t = t_1 + t_2 i$, $T = t_1 - t_2 i$, we know $t \cdot T = 1$. The two polynomials to enforce the pose constraint are

$$(1) \qquad \begin{bmatrix} (1-t)Ax + (1-T)aX + tDx + TdX - Da - Ad + Dd \\ (1-t)By + (1-T)bY + tDy + TdY - Db - Bd + Dd \end{bmatrix} = 0$$

A path point constrain is described by the input data $(d_1, d_2) \in \mathbb{R}^2$ with isotropic coordinates $d = d_1 + d_2 i$ and $D = d_1 - d_2 i$ as above. Since the pose is not specified, we need to add two variables t_1, t_2 with isotropic coordinates $t = t_1 + t_2 i$ and $T = t_1 - t_2 i$. There are now three constraints: the two from (1) and $t \cdot T = 1$.

First, we trivially have $M \geq 1$ and ignore the first pose constraint as setting the frame of reference. To generically have finitely many solutions, we require $2M + N = 10$. This results in a square polynomial system of $8 + 2N$ variables with $2(M-1) + 3N = (2M+N) - 2 + 2N = 10 - 2 + 2N = 8 + 2N$ polynomials.

c. *For* $(M, N) = (5, 0)$*, verify that the* 16 *solutions arise naturally in* 4 *groups of size* 4*. This verifies Burmester's result from 1886 that there are* 4 *distinct mechanism to solve the motion generation problem of* 5 *poses for four-bar linkages.*

d. *Experiment to find the possible number of real solutions.*

e. *Experiment with random loops in the parameter space to observe that the monodromy group for the* $(M, N) = (5, 0)$ *problem is isomorphic to the symmetric group* S_4*.*

f. *For* $(M, N) = (4, 2)$*, verify that the* 60 *solutions arise naturally in* 30 *groups of* 2*. (In this case, the monodromy group is as large as possible, namely* $S_2 Wr S_{30}$*.*

g. *For* $(M, N) = (4, 2)$*, experiment to find the possible number of real solutions.*

(Open) h. *Is it possible for all solutions to be real? What about the other Alt-Burmester problems:* $(3, 4)$*,* $(2, 6)$*, and* $(1, 8)$*? [Note that* $(1, 8)$ *is equivalent to Alt's problem.]*