Real Solutions and Parameter Continuation

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April 27-29, 2012

Supported by NSF grant
DMS-1114336

Goal

For a parameterized system of polynomial equations

$$f(x,p) = 0,$$

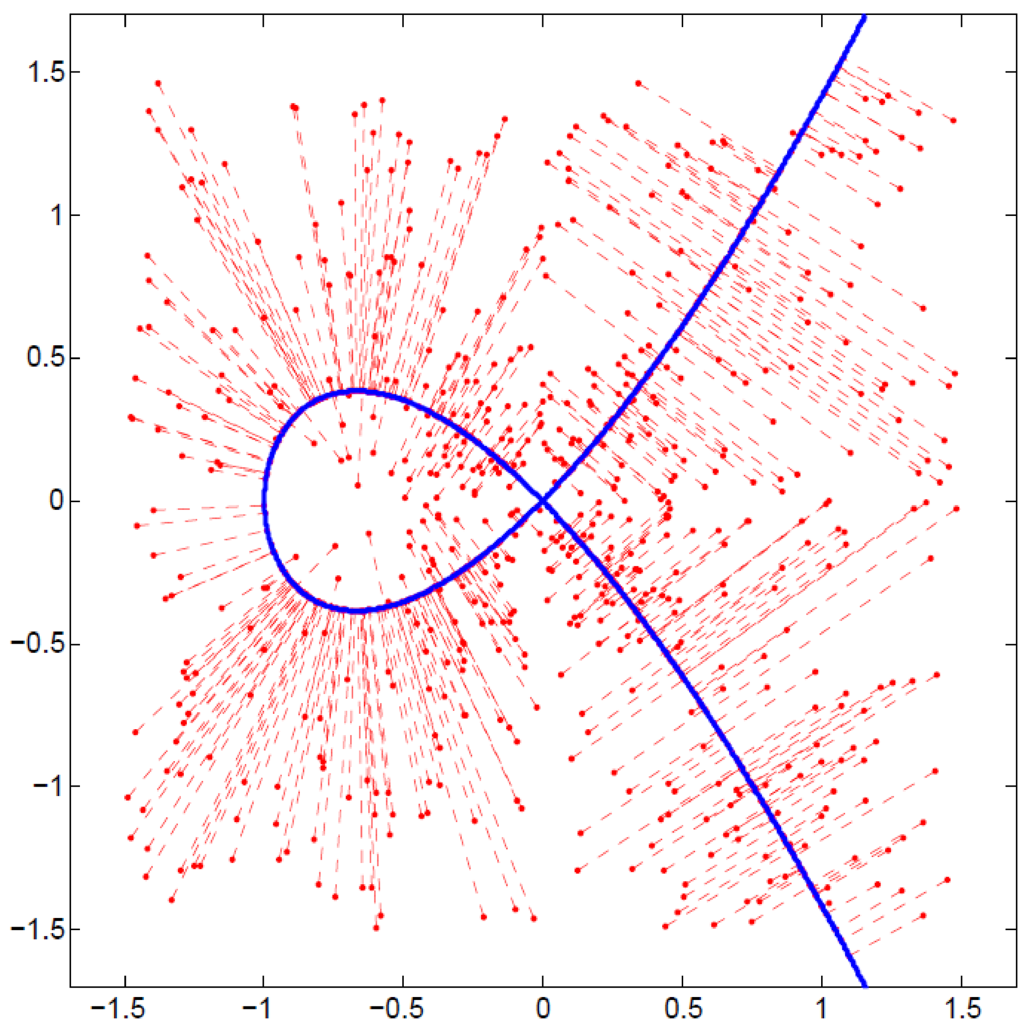
develop a homotopy-based approach for changing the parameter p to change the number of real solutions.

Approach: Gradient descent homotopies

For a real polynomial system g and a real point y,

$$H(x,\lambda,t) = \begin{bmatrix} g(x) - tg(y) \\ \lambda_0(x-y) + \lambda_1 \nabla g_1(x)^T + \cdots \lambda_n \nabla g_n(x)^T \end{bmatrix}$$

starting at $x = y$ and $\lambda = (1,0,0\ldots,0)$ when $t = 1$, is a *gradient descent homotopy* that aims to compute the solution of $g(x) = 0$ of minimal distance from y by computing a critical point of the distance function.



Plot using $g(x) = x_2^2 - x_1^2(x_1+1)$.

Application to discriminant locus

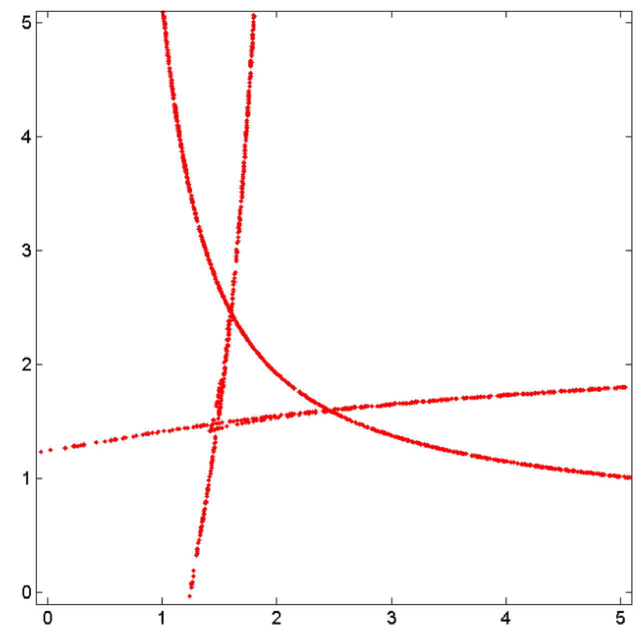
With minor modifications, gradient descent homotopies can compute points on the real discriminant locus for the parameterized polynomial system $f(x,p)$ by using

$$g(x,p) = \begin{bmatrix} f(x,p) \\ \det J_x f(x,p) \end{bmatrix}$$

starting with some real point (y,q).

We demonstrate using the system from [1], namely

$$f(x,z;p,r) = \begin{bmatrix} x^6 + pz^3 - z \\ z^6 + rx^3 - x \end{bmatrix}.$$



Plot of real discriminant locus computed using gradient descent homotopies.

Changing the number of real solutions

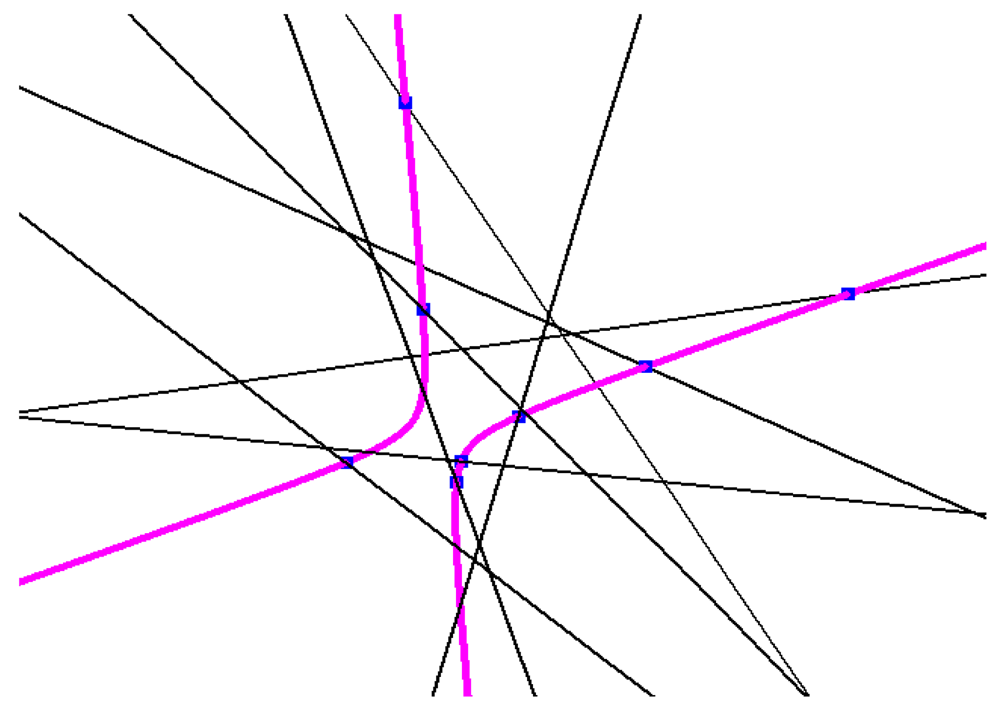
1. Use Dietmaier's local linearization approach [2] to move in well-conditioned areas of the parameter space towards the discriminant locus.
2. When near the discriminant locus, use our modified gradient descent homotopy to move to and through the discriminant locus.

Conics meeting 8 lines

We started with 8 randomly selected real lines in space having 82 real and 10 nonreal plane conics meeting the lines. Our approach then systematically increased the number of real conics up to 92.

Theorem. *There exists 8 lines in $\mathbb{R}^3$ such that all 92 plane conics meeting them are real.*

Proof. Smale's α-theoretic [4] certificate computed using exact rational arithmetic by `alphaCertified` [3].



Plot of a real conic meeting 8 given lines.

For more information:

http://people.tamu.edu/~zacgriffin21
http://math.tamu.edu/~jhauenst

References

[1] A. Dickenstein, J.M. Rojas, K. Rusek, and J. Shih. Extremal real algebraic geometry and $\mathcal{A}$-discriminants. *Mosc. Math. J.*, 7(3), 425–452, 2007.

[2] P. Dietmaier. The Stewart-Gough platform of general geometry can have 40 real postures. In *Advances in Robot Kinematics: Analysis and Control*, pp. 7–16, Kluwer, 1998.

[3] J.D. Hauenstein and F. Sottile. Algorithm 921: alphaCertified: Certifying solutions to polynomial systems. To appear in *ACM Trans. Math. Softw.*, 38(4), 2012.

[4] S. Smale. Newton's method estimates from data at one point. *The merging of disciplines: new directions in pure, applied, and computational mathematics*, pp. 185–196, Springer, New York, 1986.