Finding Straight Line Generators through the Approximate Synthesis of Symmetric Four-bar Coupler Curves

Aravind Baskar, Mark Plecnik, and Jonathan Hauenstein

Abstract In this paper, equations for the approximate synthesis of symmetric four-bar coupler curves are formulated. Our approach specifies a number of desired trace points, and finds a number of four-bar linkages with a coupler trace that approximately passes through these points. The computed linkages correspond to all the minima of the posed objective. The objective posed simultaneously enforces kinematic accuracy, loop closure, and leads to polynomial first order necessary conditions with a monomial structure that remains the same for any number of specified desired trace points. This last characteristic makes our result more general. To simplify computations, ground pivot locations are set as chosen parameters, and a root count analysis is conducted that shows our objective has a maximum of 73 critical points. The theoretical work is applied to the computational design of straight line coupler paths. To perform this exercise, the choice of ground pivots was varied, and a parameter homotopy for each choice (504 in total) was executed. These computations found the expected linkages (Watt, Evans, Roberts, Chebyshev) and other linkages resembling them but with sizable variations on their dimensions. The t-SNE algorithm was employed to organize the computed straight line generators into a visual atlas.

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1 Introduction

The synthesis of a point path by a four-bar linkage has been addressed in [1] for the exact case, and in [2] for the approximate case. Here we address a subcase, that is the synthesis of symmetric coupler curves. We are motivated to study symmetric curves as we note that many of the special straight line generators found over time produce symmetric curves, e.g. the Watt linkage, the Evans linkage, the Roberts linkage, the Chebyshev linkage, and the Chebyshev lambda linkage [3]. In search of more such interesting geometries, symmetry constraints are installed. This reduces the well known nine dimensional design space of four-bar linkages down to seven dimensions. In addition, to aid in computational tractability, the positions of ground pivots were set, reducing the design space to three dimensions. The relevant kinematic constraints were formulated into an optimization problem which was solved completely for all minima using polynomial homotopy continuation. The result is used to search for straight line generators by systematically varying ground pivot locations and computing several parameter homotopies. Our computational search found the well known straight line generators as well as several variants of their geometries. The resulting linkage designs are organized into an atlas using the t-SNE unsupervised machine learning algorithm.

2 Mathematical Formulation of Four-bars

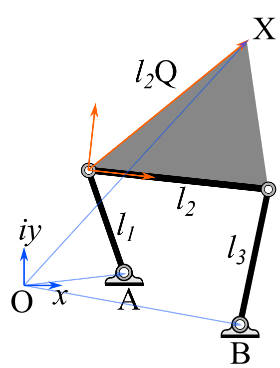


Fig. 1: Schematic of a four-bar linkage in the complex plane

Consider a planar four-bar linkage as shown in Fig. 1 in the complex plane. Let A and B represent the two fixed pivots, respectively. For representing vector variables such as the fixed pivots, isotropic coordinates [4] are used here. Hence, additional variables A^* and B^* denoting the conjugate variables of A and B, respectively, are introduced. This is an alternative approach to the Cartesian framework in order to gain certain advantages [4] during the mathematical formulation stage as well as in the implementation of numerical continuation solution technique that follows. Let l_1, l_2, and l_3 denote the lengths of the three moving links as shown. The coupler trace point (normalized by the coupler base length l_2) is represented in the local frame of the coupler as Q and its conjugate counterpart Q^*. Thus, the design variables of the four-bar linkage are summarized as $\boldsymbol{d} = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$. If X and its conjugate X^* denote the locus of the trace point of interest in the global frame, then it satisfies the equation $f(\boldsymbol{d}, X, X^*) = 0$ given by:

$$\begin{vmatrix} Q^*(A-X) & g(X,X^*) & l_2 Q(A^*-X^*) & 0 \\ 0 & l_2 Q^*(A-X) & g(X,X^*) & Q(A^*-X^*) \\ (-1+Q^*)(B-X) & h(X,X^*) & l_2(-1+Q)(B^*-X^*) & 0 \\ 0 & l_2(-1+Q^*)(B-X) & h(X,X^*) & (-1+Q)(B^*-X^*) \end{vmatrix} = 0, \quad (1)$$

where

$$g(X,X^*) = -l_1^2 + l_2^2 QQ^* + (A-X)(A^*-X^*) \qquad \text{and}$$
$$h(X,X^*) = -l_3^2 + l_2^2(-1+Q)(-1+Q^*) + (B-X)(B^*-X^*).$$

As is well known for four-bar linkages, Eq. (1) is a sextic equation with circularity 3. It comprises of 16 distinct monomial terms in X, X^*, namely,

$$\left\{ \begin{matrix} X^3X^{*3}, X^3X^{*2}, X^3X^*, X^3, X^2X^{*3}, X^2X^{*2}, \\ X^2X^*, X^2, XX^{*3}, XX^{*2}, XX^*, X, X^{*3}, X^{*2}, X^*, 1 \end{matrix} \right\}$$

in which the coefficient of the leading term X^3X^{*3} is equal to 1. Four-bar linkages that share an identical coupler locus occur as *Roberts cognate triplets* in the four-bar design space (see pp. 168-176 of [5]). For a design $\boldsymbol{d}_1 = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$, its other two cognates can be expressed as:

$$\begin{aligned} \boldsymbol{d}_2 = \Big\{ & B, B^*, A + Q(B-A), A^* + Q^*(B^*-A^*), l_2\sqrt{(1-Q)(1-Q^*)}\,, \\ & l_3\sqrt{(1-Q)(1-Q^*)}, l_1\sqrt{(1-Q)(1-Q^*)}, \frac{1}{1-Q}, \frac{1}{1-Q^*} \Big\}, \\ \boldsymbol{d}_3 = \Big\{ & A + Q(B-A), A^* + Q^*(B^*-A^*), A, A^*, l_3\sqrt{QQ^*}, \\ & l_1\sqrt{QQ^*}, l_2\sqrt{QQ^*}, \frac{Q-1}{Q}, \frac{Q^*-1}{Q^*} \Big\}. \end{aligned} \quad (2)$$

In our experiment, we restrict the model to four-bars that generate symmetric coupler curves. We do this for two reasons. First, much of the straight line linkages reported in the literature [3] such as Watt, Evans, Roberts, and Chebyshev linkages generate symmetric coupler curves about some axis of symmetry in the plane. And second, the inclusion of additional conditions on the design variables to this effect simplifies the model significantly and enables faster computations.

2.1 Symmetric coupler curves

The following derives the necessary and sufficient conditions for a four-bar linkage to generate symmetric coupler curves. While some of these conditions can be found in the literature, we present a direct proof here via analytical geometry and subsequent algebraic analysis.

Following the isotropic coordinates convention, points (P, P^*) on a generic line in the complex plane satisfy:

$$L(P, P^*) = K^* P + K P^* + c = 0, \tag{3}$$

where K, K^* $(\neq 0)$ are isotropic parameters and c is a real parameter. If (X, X^*) is any point in the plane, then its symmetric reflection about the axis given by Eq. (3) is

$$(X_m, X_m^*) = \left(-\frac{c + K X^*}{K^*}, -\frac{c + K^* X}{K}\right). \tag{4}$$

It follows that, for a four-bar coupler curve to be symmetric about an axis $L(P, P^*) = 0$, (X_m, X_m^*) given by Eq. (4) must also satisfy Eq. (1), that is, $f(\boldsymbol{d}, X_m, X_m^*) = 0$. Since the equation and its reflection must be identical, the coefficients of the 16 monomial terms in X, X^* can be equated element-wise to arrive at 15 conditions (disregarding the unit leading term of the monomial $X^3 X^{*3}$) on the design variables $\boldsymbol{d}$ and the axis parameters K, K^*, c. As the symmetric behavior is unaffected by scaling, rotation, and translation, the fixed pivots can be plugged in as $A = A^* = 0$ and $B = B^* = 1$ which further simplifies the conditions. Note that this choice of fixed pivots is made only for enabling the derivation of the conditions of symmetry and is not a global choice for the latter sections. The conditions corresponding to the monomials $X^3 X^{*2}, X^3 X^*, X^3$ are, respectively, as follows:

$$3c + K + K^* + K^* Q + K Q^* = 0, \tag{5}$$

$$-3c^2 - 2cK^* - 2cK^* Q - K^{*2} Q + K^2 Q^* = 0, \tag{6}$$

$$c(c + K)(c + K Q^*) = 0. \tag{7}$$

The conjugate of these conditions also occur for the monomials $X^2 X^{*3}, X X^{*3}, X^{*3}$.

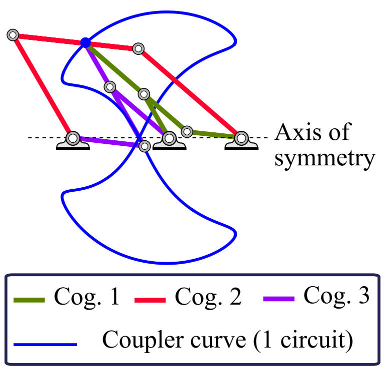
Eq. (7) shows that either $c = 0$, $c = -K$, or $c = -KQ^*$. Each of these three conditions can be analyzed separately in conjunction with Eqs. (5) and (6), and then with the other 12 coefficient conditions (not all independent). The algebra is not included for brevity and we present only the results:

Four-bar linkages with design variables $\boldsymbol{d} = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$ that generate coupler curves symmetric about an axis $K^* P + K P^* + c = 0$ can be of the following two classes:

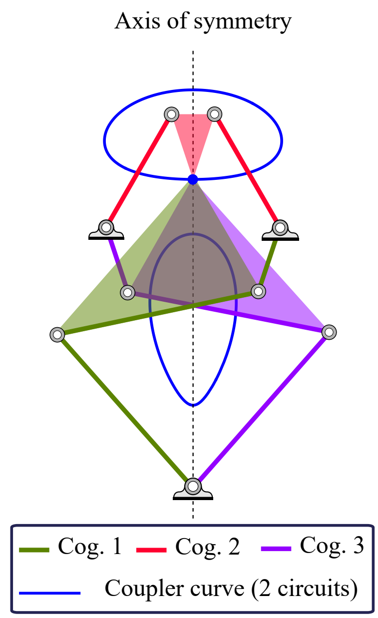
Class A $c = 0, K = -K^*, Q = Q^*$. These correspond to four-bars whose trace point lies along the line that connects the two floating pivots. For these, the reflection of the linkage about its ground link in any given configuration is also part of its configuration space, thus enabling the occurrence of symmetric coupler curves. The cognates of such four-bars also meet these conditions with the ground-pivots of all three cognates lying along the axis of symmetry.

Class B This class can be split into three types which themselves form a Roberts cognate triplet.

1. $c = 0, Q = -\frac{K}{K^*} = \frac{1}{Q^*}, l_1 = l_2$
2. $c = -K = -K^*, Q + Q^* = 1, l_1 = l_3$
3. $c = -KQ^* = -K^*Q, Q^* = \frac{Q}{Q-1}, l_2 = l_3$



(a) Class A



(b) Class B

Fig. 2: Two classes of four-bar linkages which generate symmetric coupler curves

In Fig. 2, examples of the two classes of four-bars that generate symmetric coupler curves are shown as cognate triplets. Arguably, four-bars of Class B are more interesting because, unlike Class A, the symmetric curves generated by them are not simply reflections about the ground link. The two classes of four-bars overlap in the design space in some cases, notably the Chebyshev and Watt straight line linkages. Another well-known symmetric straight line linkage, the Roberts linkage, is of Class B. The axis of symmetry in the four-bars of Class B is the perpendicular bisector of the fixed link corresponding to the cognate #2, while passing through the ground pivot shared between the cognates #1 and #3. For more geometric description of the four-bars of Class B and their conditions, refer to [6].

In this work, we limit the following design procedure to the four-bars of Class B based on the reasoning above. In particular, we solve for cognate #2 of Class B and compute cognates #1 and #3 based on the transformations presented in Eq. (2).

3 Optimization Model For Approximating Straight Lines

For four-bar linkages of Class B and cognate #2, a generic design is represented by $\boldsymbol{d} = \{A, A^*, B, B^*, l, l_2, l, Q, 1-Q\}$. Note that $l_1 = l_3 = l$ and $Q^* = 1-Q$ based on the conditions derived earlier. This simplifies the coupler equation in terms of $\{A, A^*, B, B^*, l, l_2, Q\}$. As the variables l and l_2 occur only in the form of squares, $l_{2s} = l_2^2$ and $l_s = l^2 - l_2^2 Q(1-Q)$ are introduced to simplify the equation further and to reduce the total degree. At this stage, a decision is made to treat A, A^*, B, B^* as specified design parameters instead of treating them as variables. This brings down the number of variables to 3, namely, l_s, l_{2s} and Q, as opposed to being 7 which would be a much harder problem outside the scope of this work.

As mentioned earlier, the coupler curve of a four-bar linkage is degree six. Hence, if the exact synthesis approach is taken, a maximum of only six design positions along a straight line can be specified. Approximate synthesis process allows for as many design specifications as desired. The optimization problem is one of minimizing the error residue of the coupler equation over all the design positions. We chose the $\boldsymbol{L}^2$-norm to retain the polynomial nature of the objective function, thus allowing the use of a numerical continuation approach to solve any resulting polynomial system.

The objective of the optimization problem is a sum of squares of the residue of the coupler equation over all the design positions, $j = 1, 2, ..., N$:

$$\frac{1}{2}\sum_{j=1}^{N} \eta_j^2, \tag{8}$$

where $\eta_j = f(A, A^*, B, B^*, l_s, l_{2s}, Q, X_j, X_j^*)$. The design variables are l_s, l_{2s}, Q, while A, A^*, B, B^* are the design parameters and X_j, X_j^* are the design positions. The first-order necessary conditions of optimality are then derived symbolically as:

$$\sum_{j=1}^{N} \eta_j \begin{pmatrix} \frac{\partial \eta_j}{\partial l_s} \\ \frac{\partial \eta_j}{\partial l_{2s}} \\ \frac{\partial \eta_j}{\partial Q} \end{pmatrix} = \mathbf{0}. \tag{9}$$

This system of 3 equations in 3 unknowns has a monomial structure that is invariant to the number of design positions N. This allows us to specify more design positions without increasing the complexity of the system. In particular, the total degree of this polynomial system is 648, which forms a trivial upper bound of the number of critical points of the objective function. One can computer tighter bounds such as a 2-homogeneous Bézout bound [4] of 186 and the BKK bound [4] of 73. This is confirmed by explicitly solving a randomly chosen *ab initio* system using the numerical continuation solver `Bertini` [7, 8] via a 2-homogeneous homotopy of 186 startpoints. Solving this *ab initio* system yielded 73 solutions matching the BKK bound. Thus, one can use a parameter homotopy [8] and track 73 solution paths to solve any other system with the same monomial structure.

4 Design of Experiments

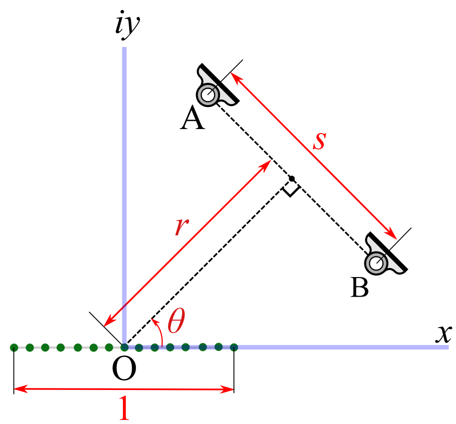


Fig. 3: Design specification for approximate straight line generating four-bar linkages

Parameter homotopy runs are carried out for the design of approximate straight line generating four-bar linkages with cognate #2 symmetric four-bars of Class B being the primary focus. The design specification is chosen to be discrete points of equal step-length along the x-axis in the range $[-0.5, 0.5]$. Specifying a high number of design positions ($N = 100$ in this work) reduces the possibility of undesirable coupler curve behavior between the desired positions. The ground link is described by four parameters, two for each fixed pivot. We add a constraint that restricts the ground link such that the axis of symmetry passing through the mid-point of the design specification as shown in Fig. 3 resulting in three parameters r, θ, and s as illustrated. The parameter θ can be restricted to be within $[0°, 90°]$ as the other values are topologically equivalent. We sample the space by employing a discretization scheme as follows:

$$r \in \{0.25i\}_{i=0}^{8} \qquad \theta \in \{15° j\}_{j=0}^{6} \qquad s \in \{0.25k\}_{k=1}^{8}, \tag{10}$$

which yields a total of $9 \cdot 7 \cdot 8 = 504$ distinct problems. The computation time required for solving a single parameter homotopy run of 73 paths is about 15s and only the solutions that correspond to physical linkages are investigated. Moreover, since this computation yields all critical points, only the local minima for each computation are retained while all saddle points are rejected. This yields 2461 linkages which are then further refined based on an allowable structural error tolerance of $\frac{1}{100}$ of unity in the y direction of the desired segment and a maximum link length constraint of 2. This results in 59 linkages of which cognates #1 and #3 are computed based on Eq. (2).

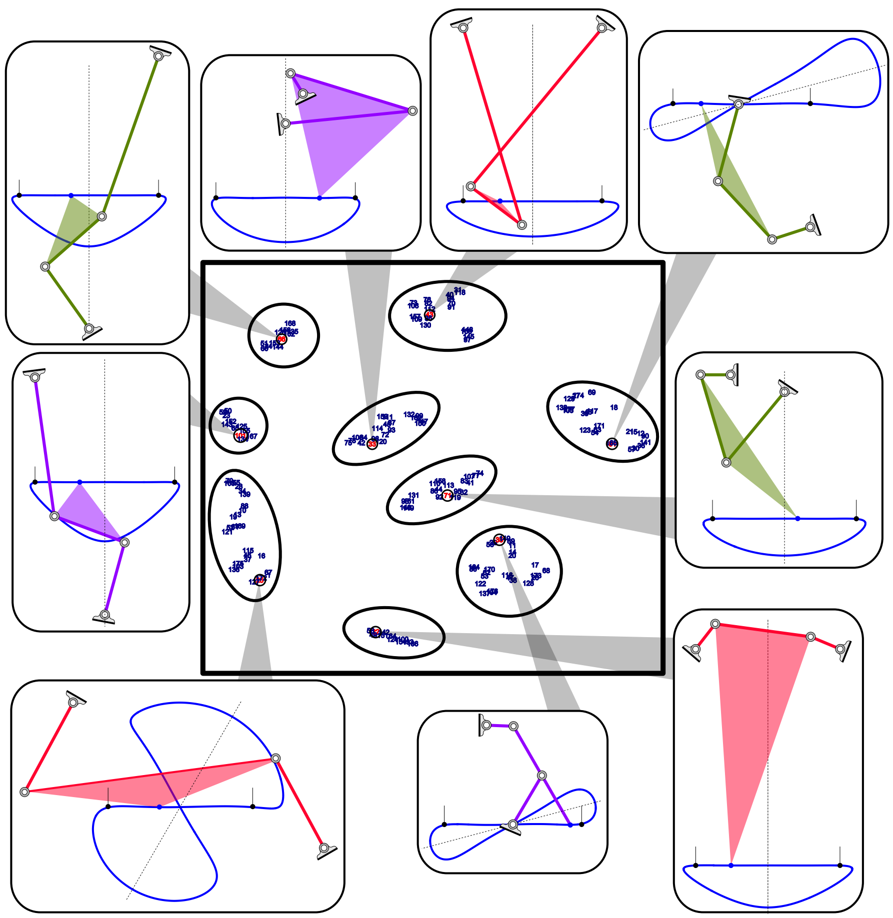


Fig. 4: An atlas of four-bar linkages that generate an approximate straight line segment visualized using t-SNE.

For exhibiting these $59 \cdot 3 = 177$ linkages, we used the machine learning technique t-SNE [9], a nonlinear dimensional reduction tool to allow us to visualize data in 2D. Using the link dimensions to represent each four-bar linkage and setting the hyper-parameter of t-SNE, namely, *perplexity*, at 5, Fig. 4 is produced. It shows bunches of linkages that qualitatively resemble classical straight line generators as observed in the representative set of nine four-bar linkages displayed to scale. This computational approach produced many that serve as a useful atlas for designers.

5 Conclusion

In this paper, the synthesis equations were formulated, characterized, and solved for a four-bar linkage with ground pivots specified to produce a desired symmetric coupler curve. The solution is applied to search for four-bar approximate straight line generators. The validity of our approach is affirmed by rediscovering the classical approximate straight line generators. In addition, we found more approximate straight line generators, each of which seems to be a variant of the classical linkages, but with substantially different dimensions. Using the t-SNE algorithm, our results are organized into a 2D atlas, which could be a useful reference for mechanical designers in need of more straight line options. A future direction related to this paper would be to more thoroughly investigate what we term as Class A linkages.

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