Numerically Testing Generically Reduced Projective Schemes for the Arithmetic Gorenstein Property

Noah S. Daleo[1] and Jonathan D. Hauenstein[2]

[1] Department of Mathematics, North Carolina State University,
Raleigh, NC 27695 (nsdaleo@ncsu.edu, www.math.ncsu.edu/~nsdaleo).
[2] Department of Applied and Computational Mathematics and Statistics, University of Notre Dame,
Notre Dame, IN 46556 (hauenstein@nd.edu, www.nd.edu/~jhauenst).

Abstract. Let $X \subset \mathbb{P}^n$ be a generically reduced projective scheme. A fundamental goal in computational algebraic geometry is to compute information about X even when defining equations for X are not known. We use numerical algebraic geometry to develop a test for deciding if X is arithmetically Gorenstein and apply it to three secant varieties.
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1 Introduction

When the defining ideal of a generically reduced projective scheme $X \subset \mathbb{P}^n$ is unknown, numerical methods based on sample points may be used to determine properties of X. In [4], numerical algebraic geometry was used to decide if X is arithmetically Cohen-Macaulay based on the Hilbert functions of subschemes of X. In our present work, we expand this to decide if X is arithmetically Gorenstein. Our method relies on numerically interpolating points approximately lying on a general curve section of X as well as a *witness point set* for X, which is defined in Section 2.4. This test does not assume that one has access to polynomials vanishing on X, e.g., X may be the image of an algebraic set under a polynomial map. In such cases, our method is an example of *numerical elimination theory* (see [2, Ch. 16] and [3]).

Much of the literature regarding arithmetically Gorenstein schemes focuses on the case in which the codimension is at most three (see, e.g., [6,8,10]), but less is known for larger codimensions. Our test is applicable to schemes of any codimension. For example, Sections 4.2 and 4.3 consider schemes of codimension 6.

The rest of this article is organized as follows. In Section 2, we provide prerequisite background material. In Section 3, we describe a numerical test for whether or not a scheme is arithmetically Gorenstein. In Section 4, we demonstrate this test on three examples.

2 Background

2.1 Arithmetically Cohen-Macaulay and arithmetically Gorenstein

If $X \subset \mathbb{P}^n$ is a projective scheme with ideal sheaf $\mathcal{I}_X$, then X is said to be *arithmetically Cohen-Macaulay (aCM)* if

$$H^i_*(\mathcal{I}_X) = 0 \quad \text{for} \quad 1 \leq i \leq \dim X$$

where $H^i_*(\mathcal{I}_X)$ is the i^{th} cohomology module of $\mathcal{I}_X$. In particular, all zero-dimensional schemes are aCM and every aCM scheme is pure-dimensional. If X is aCM, then its *Cohen-Macaulay* *type* is the rank of the last free module in a minimal free resolution of $\mathcal{I}_X$. An aCM scheme X is said to be *arithmetically Gorenstein (aG)* if X has Cohen-Macaulay type 1.

We will make use of the following fact about Cohen-Macaulay type [11, Cor. 1.3.8].

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an aCM scheme with $\dim X \geq 1$ and $H \subset \mathbb{P}^n$ be a general hypersurface of degree $d \geq 1$. Then $X \cap H$ is aCM and has the same Cohen-Macaulay type as X.*

2.2 Hilbert functions

Suppose that $X \subset \mathbb{P}^n$ is a nonempty scheme and consider the corresponding homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$. Let $\mathbb{C}[x_0, \ldots, x_n]_t$ denote the vector space of homogeneous polynomials of degree t, which has dimension $\binom{n+t}{t}$, and $I_t = I \cap \mathbb{C}[x_0, \ldots, x_n]_t$. Then, the *Hilbert function of X* is the function $HF_X : \mathbb{Z} \to \mathbb{Z}$ defined by

$$HF_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \binom{n+t}{t} - \dim I_t & \text{otherwise.} \end{cases}$$

The *Hilbert series of X*, denoted HS_X, is the generating function of HF_X, namely,

$$HS_X(t) = \sum_{j=0}^{\infty} HF_X(j) \cdot t^j.$$

There is a polynomial $P(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_r t^r$ with $\deg X = P(1)$ such that

$$HS_X(t) = \frac{P(t)}{(1-t)^{\dim X + 1}}.$$

The vector of coefficients $[c_0\ c_1\ c_2\ \cdots\ c_r]$ is called the *h-vector* of X. If X is aG, i.e., aCM of Cohen-Macaulay type 1, then the h-vector of X is symmetric: $c_i = c_{r-i}$ [13, Thm. 4.1]. Therefore, two necessary conditions on X to be aG are pure-dimensionality and a symmetric h-vector. These conditions can be used to identify schemes which are not aG, e.g., see Section 4.2.

2.3 Cayley-Bacharach property

Let $Z \subset \mathbb{P}^n$ be a nonempty reduced zero-dimensional scheme with h-vector $[c_0\ c_1\ c_2\ \cdots\ c_r]$. The scheme Z is said to have the *Cayley-Bacharach (C-B) property* if, for every subset $Y \subset Z$ with $|Y| = |Z| - 1$, $HF_Y(r-1) = HF_Z(r-1)$. The following, which is [5, Thm. 5], relates the C-B property to aG schemes.

Theorem 2. *If $Z \subset \mathbb{P}^n$ is a nonempty reduced zero-dimensional scheme, Z is arithmetically Gorenstein if and only if Z has the Cayley-Bacharach property and its h-vector is symmetric.*

2.4 Witness point sets

For a pure-dimensional generically reduced scheme $X \subset \mathbb{P}^n$, let $\mathcal{L} \subset \mathbb{P}^n$ be a general linear space with $\dim \mathcal{L} = \operatorname{codim} X$. The set $W = X \cap \mathcal{L}$ is called a *witness point set* for X.

3 Method

For a pure-dimensional generically reduced scheme $X \subset \mathbb{P}^n$, one can determine that X is arithmetically Gorenstein by combining Theorems 1 and 2. We describe the zero-dimensional and positive-dimensional cases below. A generalization of this approach, using Macaulay dual spaces, for pure-dimensional schemes that are not generically reduced is currently being written by the authors and will be presented elsewhere.

3.1 Reduced zero-dimensional schemes

If $\dim X = 0$, we can simply apply Theorem 2 to determine if X is aG. That is, given a numerical approximation of each point in X, we use the numerical interpolation approach described in [7] to compute the Hilbert function of X. In particular, there is an integer $\rho_X \geq 0$, which is called the *index of regularity* of X, such that

$$0 = HF_X(-1) < 1 = HF_X(0) < \cdots < HF_X(\rho_X - 1) < HF_X(\rho_X) = HF_X(\rho_X + 1) = \cdots = |X|.$$

The h-vector for X is $[c_0\ c_1\ \cdots\ c_{\rho_X}]$ where $c_t = HF_X(t) - HF_X(t-1)$. Thus, we can now test for symmetry of the h-vector, i.e., $c_i = c_{\rho_X - i}$.

If the h-vector is symmetric, we then test for the Cayley-Bacharach property. That is, for each $Y \subset X$ with $|Y| = |X| - 1$, we use [7] to compute $HF_Y(\rho_X - 1)$. If $HF_Y(\rho_X - 1) = HF_X(\rho_X - 1)$ for every such subset Y, then X has the C-B property.

Hence, if the h-vector is symmetric and X has the C-B property, then X is aG.

Example 1. Consider $X = \{[0,1,1],[0,1,2],[0,1,3],[1,1,-1]\} \subset \mathbb{P}^2$. It is easy to verify that $\rho_X = 2$ and the h-vector for X is [1 2 1], which is symmetric. However, X does not have the C-B property and thus is not aG, since $HF_Y(1) = 2 \neq 3 = HF_X(1)$ for $Y = \{[0,1,1],[0,1,2],[0,1,3]\}$.

3.2 Generically reduced positive-dimensional schemes

If $\dim X \geq 1$, Theorems 1 and 2 show that X is aG if and only if X is aCM and a witness point set for X is aG, i.e., has a symmetric h-vector and has the C-B property. We start with the witness point set condition and then summarize the aCM test presented in [4].

Let $W = X \cap \mathcal{L}$ be witness point set for X defined by the general linear slice $\mathcal{L}$. We apply the strategy of Section 3.1 to W with one simplification for deciding that W has the C-B property. This simplification arises from the fact that witness point sets for an irreducible scheme has the so-called *uniform position property.* That is, if X is irreducible, then W has the C-B property if and only if $HF_Y(\rho_W - 1) = HF_W(\rho_W - 1)$ for *any* $Y \subset W$ with $|Y| = |W| - 1$. In general, if X has k irreducible components, say $X_1, \ldots, X_k$ with $W_i = X_i \cap \mathcal{L}$, then W has the C-B property if and only if, for $i = 1, \ldots, k$, $HF_{Z_i}(\rho_W - 1) = HF_W(\rho_W - 1)$ where $Z_i = \bigcup_{j \neq i} W_j \cup Y_i$ for *any* $Y_i \subset W_i$ with $|Y_i| = |W_i| - 1$.

If W is aG, then X is aG if and only if X is aCM. The arithmetically Cohen-Macaulayness of X is decided using the approach of [4] by comparing the Hilbert function of W and the Hilbert function of a general curve section of X as follows. Let $\mathcal{M} \subset \mathbb{P}^n$ be a general linear space with $\dim \mathcal{M} = \operatorname{codim} X + 1$ and $C = X \cap \mathcal{M}$, i.e., $\dim C = 1$. By numerically sampling points approximately lying on C, we compute $HF_C(t)$ via [7] for $t = 1, \ldots, \rho_W + 1$. The following is a version of [4, Cor. 3.3] that decides the arithmetically Cohen-Macaulayness of X via HF_W and HF_C.

Theorem 3. *With the setup described above, X is arithmetically Cohen-Macaulay if and only if $HF_W(t) = HF_C(t) - HF_C(t-1)$ for $t = 1, \ldots, \rho_W + 1$.*

4 Examples

It has been speculated that the homogeneous coordinate ring of any secant variety of any Segre product of projective spaces is Cohen-Macaulay [12], but some examples of such secant varieties are known to not be arithmetically Gorenstein [9]. We demonstrate our test on two such secant varieties in Sections 4.1 and 4.2. Section 4.3 considers a secant variety of a Veronese variety.

4.1 $\sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$

Let $X = \sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{15}$, which is the third secant variety to the Segre product of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $\dim X = 13$. We computed a witness point set W for X using `Bertini` [1] and found that $\deg X = 16$. Using [7], we compute

$$\rho_W = 6, \quad HF_W = 1, 3, 6, 10, 13, 15, 16, 16, \quad \text{and} \quad h = [1\ 2\ 3\ 4\ 3\ 2\ 1].$$

Clearly, the h-vector for W is symmetric. Since X is irreducible, we selected one subset $Y \subset W$ consisting of 15 points. The witness point set W has the Cayley-Bacharach property since $HF_Y(5) = 15 = HF_W(5)$ and thus we conclude W is arithmetically Gorenstein by Theorem 2.

Next, we consider the arithmetically Cohen-Macaulayness of X. Let $\mathcal{M} \subset \mathbb{P}^{15}$ be a general linear space with $\dim \mathcal{M} = 3$ and $C = X \cap \mathcal{M}$. Via sampling C, we find that

$$HF_C = 1, 4, 10, 20, 33, 48, 64, 80.$$

Therefore, by Theorem 3, X is arithmetically Cohen-Macaulay and, hence, we can conclude it is arithmetically Gorenstein by Theorem 1. In fact, since X is aCM, we can observe from HF_W that two polynomials of degree 4 must vanish on X. We found that these two polynomials generate the ideal of X meaning that X is actually a complete intersection.

4.2 $\sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2)$

We next consider $X = \sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2) \subset \mathbb{P}^{23}$ where $\dim X = 17$. We computed a witness point set W for X using `Bertini` and found that $\deg X = 316$. Using [7], we compute

$$\rho_W = 6, \quad HF_W = 1, 7, 28, 84, 171, 261, 316, 316, \quad \text{and} \quad h = [1\ 6\ 21\ 56\ 87\ 90\ 55].$$

Since h is not symmetric, we conclude that W and, hence, X are not arithmetically Gorenstein.

Remark 1. Although the lack of symmetry in h is sufficient to show that W is not aG, we note that W satisfies the Cayley-Bacharach property and X is aCM. Since X is aCM, we can observe from HF_W that 39 polynomials of degree 4 must vanish on X which generate the ideal of X.

4.3 $\sigma_3(\nu_4(\mathbb{P}^2))$

Let ν_4 be the degree 4 Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^{14}$ and $X = \sigma_3(\nu_4(\mathbb{P}^2)) \subset \mathbb{P}^{14}$ where $\dim X = 8$. We computed a witness point set W for X using `Bertini` and found that $\deg X = 112$. Using [7], we compute

$$\rho_W = 6, \quad HF_W = 1, 7, 28, 84, 105, 111, 112, 112, \quad \text{and} \quad h = [1\ 6\ 21\ 56\ 21\ 6\ 1].$$

Clearly, the h-vector for W is symmetric. Since X is irreducible, we selected one subset $Y \subset W$ consisting of 111 points. The witness point set W has the Cayley-Bacharach property since $HF_Y(5) = 111 = HF_W(5)$ and thus we conclude W is arithmetically Gorenstein by Theorem 2.

Next, we consider the arithmetically Cohen-Macaulayness of X. Let $\mathcal{M} \subset \mathbb{P}^{14}$ be a general linear space with $\dim \mathcal{M} = 7$ and $C = X \cap \mathcal{M}$. Via sampling C, we find that

$$HF_C = 1, 8, 36, 120, 225, 336, 448, 560.$$

Therefore, by Theorem 3, X is arithmetically Cohen-Macaulay and, hence, we can conclude it is arithmetically Gorenstein by Theorem 1. In fact, since X is aCM, we can observe from HF_W that 105 polynomials of degree 4 must vanish on X and they generate the ideal of X.

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