

# COMPUTATIONS AND EQUATIONS FOR SEGRE-GRASSMANN HYPERSURFACES

NOAH S. DALEO, JONATHAN D. HAUENSTEIN, AND LUKE OEDING

ABSTRACT. In 2013, Abo and Wan studied Waring’s problem for systems of skew-symmetric forms and identified several defective systems. The cases of particular interest occur when a certain secant variety of a Segre-Grassmann variety is expected to fill the natural ambient space, but is actually a hypersurface. In these cases, one aims to obtain both a defining polynomial for these hypersurfaces along with a representation theoretic description of the defectivity. In this note, we combine numerical algebraic geometry with representation theory to accomplish this task. In particular, numerical algebraic geometric algorithms implemented in `Bertini` [BHSW06] are used to determine the degrees of several hypersurfaces with representation theory using this data as input to understand the hypersurface.

This approach allows us to answer [AW13, Problem 6.5] and show that each member of an infinite family of hypersurfaces is minimally defined by a (known) determinantal equation. While led by numerical evidence, we provide non-numerical proofs for all of our results.

## 1. INTRODUCTION

Secant varieties, while a classical topic in algebraic geometry, have received much attention over the past several years largely due to the vast number of applications to many fields such as Geometric Complexity Theory and Signal Processing (e.g., see [Lan14a] and [SC14]).

Suppose  $X$  is an algebraic variety in  $\mathbb{P}^N$ , and for simplicity, assume that  $X$  is not contained in any linear subspace. The  $X$ -rank of a point  $[p] \in \mathbb{P}^N$  is the minimum number  $r$  such that  $p = \sum_{i=1}^r x_i$  with  $[x_i] \in X$ . The Zariski closure of the points of  $X$ -rank  $r$  is the  $r$ -secant variety to  $X$ , denoted  $\sigma_r(X)$ .<sup>1</sup> We say that the points of  $\sigma_r(X)$  have  $X$ -border rank  $r$ . For tensors and related algebraic varieties,  $X$ -rank and  $X$ -border rank provide a useful perspective; see [BL13]. The reader may find the recent lecture notes [CGOar] to be useful for general background on secant varieties, as well as an extensive list of references contained therein.

The first question one asks about  $X$ -rank for  $X \subset \mathbb{P}^N$  is which  $X$ -border rank fills the ambient space  $\mathbb{P}^N$ . Indeed, the famous Alexander-Hirschowitz Theorem [AH92] answered this question when  $X$  is the Veronese embedding of projective space (see also [BO08, Pos12] for modern accounts). The analogous question for the Segre embedding of the Cartesian product of projective spaces into the projectivization of a tensor product of vector spaces was addressed in [AOP09]. Many cases were settled, but this problem is not yet completely solved (see [COV14] for recent progress). The skew-symmetric version of this question was addressed in [AOP12, BDDG07], again with some cases solved and some cases remaining.

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<sup>1</sup>Note that taking the Zariski closure often causes a failure of upper semi-continuity of  $X$ -rank, for instance in the case of tensors of order 3 or more.

Another question one may ask regarding  $X$ -border rank is to describe the defining equations of  $\sigma_r(X)$ . From such equations, one can easily decide the  $X$ -border rank of any given point in  $\mathbb{P}^N$ . Versions of this test are extremely important, for instance, in algebraic complexity theory [Lan08, HIL13].

The purpose of this paper is twofold. The first objective is to find equations for secant varieties of certain Segre-Grassmann varieties. We focus on two cases where the secant variety in question is a hypersurface. One of these cases solves a problem left open in [AW13], while the other case, which is actually an entire family of hypersurfaces, confirms a guess in Abo and Wan’s work that an Ottaviani-type construction gives the requisite equations. The second objective is to demonstrate the power and use of combining tools from numerical algebraic geometry and representation theory, which we hope will be used to address many other problems in the future. While partially skew-symmetric tensors are certainly less studied than the fully symmetric and non-symmetric cases, it is often the case that methods for finding equations for border rank in one symmetry class inform techniques for another. For instance, Ottaviani’s approach to Aronhold’s invariant for symmetric tensors as a Pfaffian led to a new construction of Strassen’s invariant for non-symmetric tensors [Ott09, LO11a].

Here is an outline of the rest of this paper. In Section 2, we provide notation along with more background information. Section 3 summarizes the construction of the hypersurface whose defining equation is left open in [AW13, Problem 6.5]. Sections 4 and 5 describe the algorithms used from numerical algebraic geometry and representation theory, respectively, with Theorem 5.1 answering [AW13, Problem 6.5]. In Section 6 we consider an infinite family of hypersurfaces, and show that the known determinantal equations define them (Theorem 6.3). In Section 7 we study the irreducibility of a determinant of the tensor product of two skew symmetric matrices, which we use in the proof of Theorem 6.3.

## 2. NOTATION AND PRELIMINARIES

Let  $\bigwedge^{k+1}\mathbb{C}^{n+1}$  denote the vector space of alternating  $k+1$  forms on an  $n+1$  dimensional (complex) vector space, whose natural basis is given by the pure wedge products  $e_{j_1} \wedge \cdots \wedge e_{j_{k+1}}$ , with  $1 \leq j_1 < \cdots < j_{k+1} \leq n+1$  and  $\{e_j\}$  a basis of  $\mathbb{C}^{n+1}$ .

Now consider the following space of partially skew-symmetric tensors  $\mathbb{C}^{m+1} \otimes \bigwedge^{k+1}\mathbb{C}^{n+1}$ . We will write  $\{x_{i,j_1,\dots,j_k}\}$  for coordinates on  $\mathbb{C}^{m+1} \otimes \bigwedge^{k+1}\mathbb{C}^{n+1}$ , where  $1 \leq i \leq m+1$  and  $1 \leq j_1 < \cdots < j_k \leq n+1$ . By slicing in the first tensor mode, a point in this space may be thought of as an  $m+1$ -dimensional system of alternating  $k+1$  forms on  $n+1$  variables. It is natural to consider the points of rank 1 to be those points which are “pure tensors” or “indecomposable tensors” with the required symmetry.

Let  $X = \text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$  be a Segre-Grassmann variety, which is the Segre product of a projective  $m$ -plane and the Grassmann variety of  $k$ -dimensional projective subspaces of an  $n$  dimensional projective space. The natural embedding of  $X$  is by a Segre-Plücker embedding into  $\mathbb{P}(\mathbb{C}^{m+1} \otimes \bigwedge^{k+1}\mathbb{C}^{n+1})$ . A general point on  $\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$  is (a pure tensor) of the form

$$[v \otimes (w_0 \wedge \cdots \wedge w_k)],$$

where  $[v] \in \mathbb{P}^m$ , and  $w_0, \dots, w_k$  form a basis of a  $k$ -dimensional (projective) linear subspace of  $\mathbb{P}^n$ . Let  $\sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)))$  denote the  $s$ -th secant variety of the Segre-Grassmann

variety. A general point on this variety is of the form

$$(2.1) \quad \left[ \sum_{i=1}^s v^i \otimes (w_0^i \wedge \cdots \wedge w_k^i) \right],$$

where the superscripts are just formal placeholders, and the other terms have the same interpretation as before. Thus, the points of  $X$ -rank  $s$  in  $\mathbb{C}^{m+1} \otimes \wedge^{k+1} \mathbb{C}^{n+1}$  may be thought of as those points which have the interpretation as a formal linear combination of  $s$  terms, each term being an  $(m+1)$ -dimensional system of  $k$ -planes in  $\mathbb{P}^n$ .

Here is a straightforward way to use this description to obtain coordinates for the points (and hence a parametrization of the variety). Let  $v = (v_0, \dots, v_m)$ , and let  $E = (e_{i,j})$  be a  $(k+1) \times (n+1)$  matrix. One obtains an  $(m+1) \times \binom{n+1}{k+1}$  vector for a point on  $\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$  as

$$(v_i \cdot \Delta_I(E))_{i,I},$$

where  $\Delta_I$  is the maximal minor of  $E$  described by the columns of  $I = (i_1, \dots, i_{k+1})$ . Moreover, one may generate pseudo-random points on  $\sigma_s(\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)))$  by letting  $v$  and  $E$  be (respectively) a random vector and a random matrix, and summing  $s$  pseudo-random points of  $\text{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$ .

The main tool for determining the dimension of a secant variety is the well-known Terracini lemma. For an algebraic variety  $X \subset \mathbb{P}^N$ , let  $\widehat{X}$  denote the cone over  $X$  in  $\mathbb{C}^{N+1}$ , and if  $[x] \in X$  is a smooth point, let  $\widehat{T}_x X$  denote the cone over the tangent space of  $X$  at  $[x]$ .

**Lemma 2.1** (Terracini). *Let  $X \subset \mathbb{P}^N$  be an algebraic variety, and let  $[x_1], \dots, [x_k]$  be general points of  $X$ , set  $p = \sum_{i=1}^k x_i$  and suppose that  $[p] \in \sigma_k(X)$  is a general point. Then the tangent space of the secant variety is the sum of tangent spaces to the original variety:*

$$\widehat{T}_p \sigma_k(X) = \widehat{T}_{x_1} X + \cdots + \widehat{T}_{x_k} X.$$

**2.1. Symmetry.** Let  $V \cong \mathbb{C}^{m+1}$  and  $W \cong \mathbb{C}^{n+1}$ . Notice that all of our definitions have the feature that they display the natural symmetry. The Segre-Grassmann variety  $\text{Seg}(\mathbb{P}V \times \mathbb{G}(k, \mathbb{P}W))$  is left invariant under the action of  $\text{GL}(V) \times \text{GL}(W)$ . Its secant variety inherits the same symmetry. Moreover, the graded coordinate ring

$$\mathbb{C}[V \otimes \wedge^{k+1} W] = \bigoplus_{d \geq 0} S^d(V \otimes \wedge^{k+1} W)^*$$

also inherits this symmetry. A consequence of Schur-Weyl duality is that each degree  $d$  piece decomposes as

$$(2.2) \quad S^d(V \otimes \wedge^{k+1} W)^* = \bigoplus_{\lambda \vdash d, \pi \vdash (k+1)d} S_\lambda V^* \otimes S_\pi W^* \otimes \mathbb{C}^{[\lambda, \pi]},$$

where  $S_\lambda V^*$  and  $S_\pi W^*$  are Schur modules and  $\mathbb{C}^{[\lambda, \pi]}$  is the multiplicity space associated to the partitions  $\lambda, \pi$ .

This decomposition may be obtained via a character computation. This computation is conveniently carried out in the program LiE [vLCL92] (see Section 5 for an example). An explicit basis of  $\mathbb{C}^{[\lambda, \pi]}$  may be obtained by a careful application of Young symmetrizers. We will explain this construction in the example in Section 5.

### 3. THE ABO-WAN HYPERSURFACE $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$

Abo and Wan [AW13] classified many cases of defective Segre-Grassmann varieties. One of the sporadic cases of defectively was  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$ , which is a hypersurface in  $\mathbb{P}^{59}$ , even though the naive dimension count implies that one expects this variety to fill the ambient space. Here is a summary of their method applied to this specific case.

In order to show that  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$  is a hypersurface, Abo and Wan bounded the dimension of  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$  from above by using a geometric argument and from below using a so-called “randomized algorithm.” Since both the upper and lower bounds were 58, they determined  $\dim \sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))) = 58$ .

More specifically, to prove that  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$  has dimension at most 58, Abo and Wan first showed the existence of a rational normal curve  $C$  of degree 8 in  $\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$  that passes through five generic points  $p_1, \dots, p_5$  of  $\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$ . For each  $i \in \{1, \dots, 5\}$ , the affine cone  $\widehat{T}_{p_i}C$  is a subspace of the affine cone  $\widehat{T}_{p_i}\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$ , so we may choose a complement, denoted  $L_i$ , forming a direct sum. Then it follows from Terracini’s lemma that the affine cone over the projectivized tangent space of  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$  at the generic point  $q$  of the linear span of  $p_1, \dots, p_5$  is

$$\begin{aligned} \sum_{i=1}^5 \widehat{T}_{p_i}\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)) &= \sum_{i=1}^5 (L_i \oplus \widehat{T}_{p_i}C) \\ &= \sum_{i=1}^5 L_i + \sum_{i=1}^5 \widehat{T}_{p_i}C. \end{aligned}$$

Since each  $\widehat{T}_{p_i}C$  is contained in the linear space  $\langle C \rangle$  of  $C$ , we obtain

$$\begin{aligned} \dim T_q \sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))) &= \sum_{i=1}^5 \dim L_i + \sum_{i=1}^5 \dim \widehat{T}_{p_i}C - 1 \\ &\leq \sum_{i=1}^5 \dim L_i + \sum_{i=1}^5 \dim \langle C \rangle \\ &= 5 (\dim \text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)) - \dim C) + \deg C \\ &\leq 5(2 + 9 - 1) + 8 \\ &= 58. \end{aligned}$$

To prove that the dimension of  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$  is at least 58, Abo and Wan chose five random points of  $\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$ , and then they determined that the dimension of the linear subspace spanned by the projectivized tangent spaces of  $\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$  at these five points is 58. By semi-continuity, the dimension of the linear subspace spanned by the projectivized tangent spaces of  $\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))$  at five generic points is therefore greater than or equal to 58. Thus, an application of Terracini’s lemma shows the inequality  $\dim \sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))) \geq 58$ .

The follow section uses numerical algebraic geometric algorithms to determine the degree of this hypersurface and several other related ones. These degrees are used as input to determine an equation defining each hypersurface, using representation theory in Section 5 and careful multi-linear algebra in Sections 6,7.

FIGURE 1. The trace test

Let  $\mathcal{H}$  be an irreducible hypersurface and  $\mathcal{L}$  be a line so that  $\deg \mathcal{H} = |\mathcal{H} \cap \mathcal{L}|$ .

- (1) Generate a point  $x \in \mathcal{H} \cap \mathcal{L}$ . Initialize  $\mathcal{W} := \{x\}$ .
- (2) Perform a random monodromy loop starting at the points in  $\mathcal{W}$ :
  - (a) Pick a random loop  $\mathcal{M}(t)$  in the space of lines so that  $\mathcal{M}(0) = \mathcal{M}(1) = \mathcal{L}$ .
  - (b) Trace the curves  $\mathcal{H} \cap \mathcal{M}(t)$  starting at the points in  $\mathcal{W}$  at  $t = 0$  to compute the endpoints  $\mathcal{E}$  at  $t = 1$ . (Hence,  $\mathcal{E} \subset \mathcal{H} \cap \mathcal{L}$ ).
  - (c) Update  $\mathcal{W} := \mathcal{W} \cup \mathcal{E}$ .
- (3) Repeat (2) until the trace test verifies that  $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$ .

#### 4. USING Bertini TO DETERMINE THE DEGREE OF A PARAMETRIZED HYPERSURFACE

Computing the degree and defining equation for a parametrized hypersurface is a classical problem in elimination theory (e.g., see [CLO07, Chap. 3]). Since classical techniques have thus far failed to produce information regarding the Abo-Wan hypersurfaces, we turn to numerical algebraic geometry, namely techniques in *numerical elimination theory* [HS10, HS13] summarized in [BHSW13, Chap. 16]. For the applications here, we will use such numerical techniques to compute the degree of each hypersurface under consideration. Once the degree is known, we then use representation theory and linear algebra, described in the next sections, to compute a defining equation for each hypersurface.

Before describing in detail the computation involving  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$ , we first summarize the procedure from a geometric point of view. Suppose that  $\mathcal{H} \subset \mathbb{P}^n$  is an irreducible hypersurface. Since  $\deg \mathcal{H} = |\mathcal{H} \cap \mathcal{L}|$  for a general line  $\mathcal{L} \in \mathbb{G}(1, n)$ , one simply needs to compute the finite set of points  $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$ , called a *witness point set* for  $\mathcal{H}$ . More details regarding witness sets are provided in [SW05, Chap. 13].

To compute  $\mathcal{W}$ , one needs to first generate a point in  $\mathcal{W}$ . In the cases of interest here, we have a parametrization of  $\mathcal{H}$  so it is trivial to compute a smooth point  $x \in \mathcal{H}$ . One may then pick  $\mathcal{L}$  to be a general line passing through  $x$  where  $x \in \mathcal{W} = \mathcal{H} \cap \mathcal{L}$ .

Starting from one point in  $\mathcal{W}$ , we then use random *monodromy loops* [SVW02] to attempt to generate additional points in  $\mathcal{W}$ . We first select a random path  $\mathcal{M} : [0, 1] \rightarrow \mathbb{G}(1, n)$  with  $\mathcal{M}(0) = \mathcal{M}(1) = \mathcal{L}$ . Then, for each  $w \in \mathcal{W}$ , we trace the path  $p_w(t) \in \mathcal{H} \cap \mathcal{M}(t)$  with  $p_w(0) = w$  to compute the point  $p_w(1) \in \mathcal{W}$ .

As stated, such random monodromy loops allow one to potentially generate additional points in  $\mathcal{W}$  without a definitive criterion for when we have computed all points in  $\mathcal{W}$ . A heuristic criterion is when several of such loops fail to generate new points. The definitive criterion we will use is the *trace test* [SVW02]. Let  $\mathcal{P} : \mathbb{R} \rightarrow \mathbb{G}(1, n)$  define a family of parallel lines with  $\mathcal{P}(0) = \mathcal{L}$  and  $\mathcal{W}' \subset \mathcal{W}$ . Then,  $\mathcal{W}' = \mathcal{W}$  if and only if

$$\text{every coordinate of } \sum_{w \in \mathcal{W}'} p_w(t) \text{ is linear in } t,$$

where  $p_w(t) \in \mathcal{H} \cap \mathcal{P}(t)$  with  $p_w(0) = w$ . In practice, this linearity condition is verified by testing at three values of  $t$ , typically  $-1$ ,  $0$ , and  $1$ . The trace test procedure is summarized Figure 1.

For the problems at hand, we need to modify this procedure for irreducible hypersurfaces that arise as the closure of the image of an irreducible algebraic set, say  $\mathcal{H} = \overline{\pi(Y)}$ . This

results in a problem in numerical elimination theory in which computations are performed on  $Y$  and witness sets are simply replaced by *pseudowitness sets* [HS10].

This approach facilitated by path tracking using *Bertini* [BHSW06] yielded the following.

**Theorem 4.1.** *The following hold.*

- (1) *The hypersurface  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5))) \subset \mathbb{P}^{59}$  has degree 6.*
- (2) *The hypersurface  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 6))) \subset \mathbb{P}^{62}$  has degree 21.*
- (3) *The hypersurface  $\sigma_8(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 10))) \subset \mathbb{P}^{164}$  has degree 33.*
- (4) *The hypersurface  $\sigma_{11}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 14))) \subset \mathbb{P}^{314}$  has degree 45.*

In our execution of the procedure for the hypersurface  $\mathcal{H} = \sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$ , it took 6 random monodromy loops to compute the six points in  $\mathcal{H} \cap \mathcal{L}$ . The total procedure lasted 50 seconds using a single 2.3 GHz core of an AMD Opteron 6376 processor.

The last 3 hypersurfaces come from [AW13] and are part of an infinite family that will be considered in Section 6. In our execution for these hypersurfaces, it took 13, 12, and 13 random monodromy loops to yield the degree many points for each case, respectively. Using a total of sixteen 2.3 GHz cores, the total procedure lasted 2.5 minutes, 32 minutes, and 5.5 hours, respectively.

*Remark 4.2.* All 4 cases of Theorem 4.1 have numerical proofs via the method presented in this section. One may object that the results hold only up to the numerical precision of our calculations. However, we have used these computations as strong evidence and motivation to search for, and eventually find, non-numerical proofs of these results as well as generalizations. These proofs are provided in Sections 5 and 6.

## 5. USING YOUNG SYMMETRIZERS TO CONSTRUCT POLYNOMIAL INVARIANTS

By Theorem 4.1(1), we know that we are looking for a degree 6 equation for  $\sigma_5(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(2, 5)))$ . Moreover, by the symmetry of the variety, we know that we are looking for a degree 6 polynomial invariant for  $\text{SL}(3) \times \text{SL}(6)$  acting on  $\mathbb{C}^3 \otimes \wedge^3 \mathbb{C}^6$ . Using [vLCL92], we computed the entire isotopic decomposition of the degree 6 part of the coordinate ring  $\mathbb{C}[\mathbb{C}^3 \otimes \wedge^3 \mathbb{C}^6]$  in (2.2) above via the LiE command `sym_tensor(6, [1, 0] ^ [0, 0, 1, 0, 0], A2A5)` (which does the appropriate character computation necessary to determine the dimensions of the multiplicity spaces).

The output is a long polynomial, but the occurrence of `1X[0, 0, 0, 0, 0, 0]` tells us, in particular, that the trivial representation occurs with multiplicity one. Now that we know that there is only one non-trivial degree 6 invariant (up to trivial rescaling), we can apply a Young symmetrizer construction to produce the invariant as follows. We will describe the entire process with the degree 6 Abo-Wan example. The algorithm we present here is a modification of the Landsberg-Manivel algorithm [LM04], and uses ideas from [FH91, GW98, Ott13a] and [Lan12]. See [BO11] for an example using this algorithm for 3-tensors.

First, we start with the partitions  $(2, 2, 2)$  and  $(3, 3, 3, 3, 3, 3)$  associated (respectively) to the trivial representations of  $\text{GL}(3)$  and  $\text{GL}(6)$  in degrees 6 and 18, respectively. Then, we must find fillings of the associated tableaux so that the associated Young symmetrizer produces a non-zero image.

After an exhaustive search, we found that the following pair of fillings will produce a non-zero image.

$$\begin{array}{|c|c|} \hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array},$$

where, in the second filling, we use each letter three times indicating that we are parametrizing an invariant of degree 6 on  $\bigwedge^3(W) \subset W^{\otimes 3}$ . We will use this filling to show how to construct the associated Young symmetrizer and compute its image.

Using this filling we construct a generic polynomial in terms of auxiliary variables associated to the letters in the fillings by constructing matrices associated to the columns. For

$$\begin{array}{|c|c|} \hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array},$$

we associate the product of determinants

$$p_V = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \begin{vmatrix} c_1 & c_2 & c_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

Similarly, for the filling

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array},$$

we associate the product of determinants  $p_W =$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{vmatrix} \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \end{vmatrix}.$$

The next step is to extract the terms of the polynomial  $p_V p_W$  one at a time and replace parts of the monomials with our target variables  $x_{i,j,k,l}$ , where  $1 \leq i \leq 3$  and  $1 \leq j < k < l \leq 6$ .

Let the symbol  $\lrcorner$  denote the contraction performed by “taking the coefficient.” For example, if we have a polynomial

$$p = a_1 b_2 d_3 c_1 e_3 f_3 a_{11} a_{22} a_{33} b_{34} c_{25} c_{36} \cdot q,$$

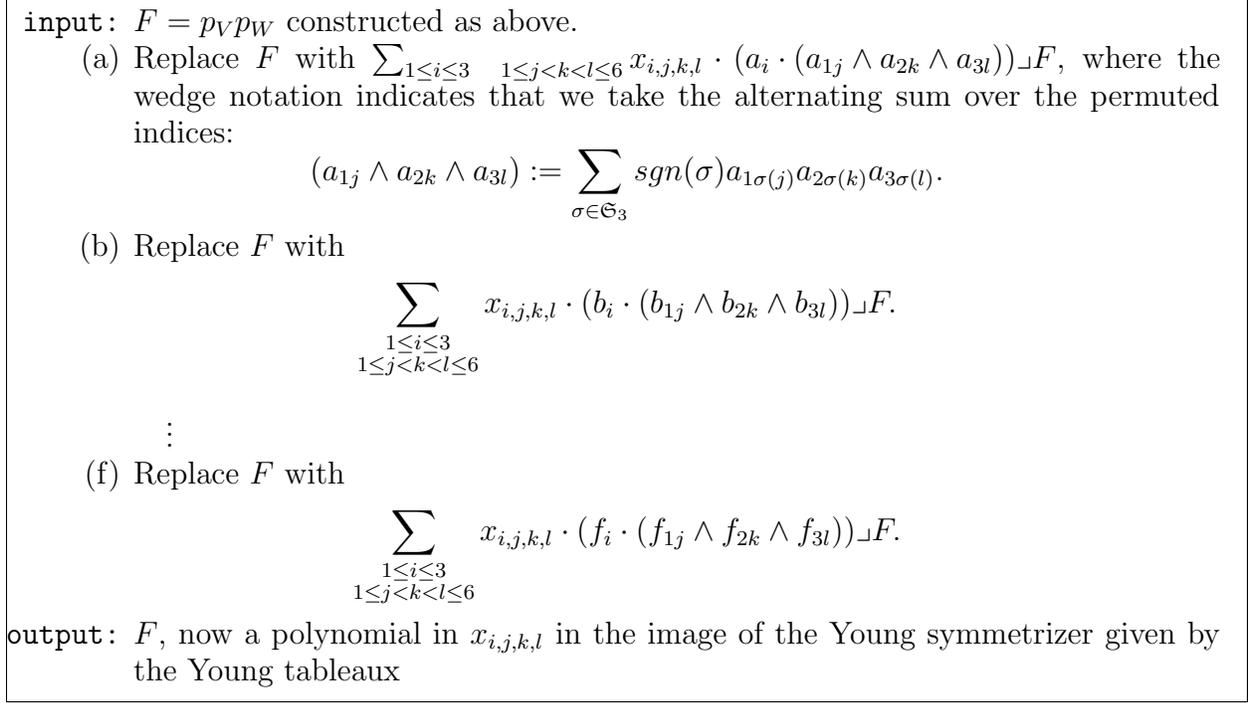
where  $q$  does not depend on the variables  $a$ , then we can contract:

$$(a_1 a_{11} a_{22} a_{33}) \lrcorner p = b_2 d_3 c_1 e_3 f_3 b_{34} c_{25} c_{36} \cdot q.$$

We perform contractions to produce a polynomial in the  $x_{i,j,k,l}$  that is in the image of the Young Symmetrizer associated to our initial fillings the algorithm in Figure 2.

To test whether this algorithm will produce a non-zero result, it is crucial to recognize that the procedure has a built-in evaluation option. That is, at each step (a-f) in the algorithm in Figure 2, one may evaluate the partial result at a fixed pre-determined point. The intermediate steps will become much less memory consuming and the evaluation will

FIGURE 2. Evaluating Young symmetrizers



happen much more quickly than producing the polynomial and then evaluating it. We used this method to find a filling that would produce a non-zero result and then, knowing that the filling we found would produce a non-zero polynomial, we applied the full algorithm to the filling we have recorded above. We then check that the polynomial we produced is both non-zero (because it evaluates non-zero at at least one point of the ambient space) and vanishes on  $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(2, 5))$  (because it vanishes on all parametrized points, i.e. on a Zariski open set).

**Theorem 5.1.** *The prime ideal of the hypersurface  $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(2, 5))$  is generated by the single degree 6 polynomial (up to scale) constructed via the image of the Young symmetrizer associated to the filling*

$$\begin{array}{|c|c|} \hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array}.$$

*Proof.* Let  $F$  denote the polynomial resulting from the recipe given in the statement above. In particular,  $F$  has precisely 10080 monomials, 5040 of which have coefficient  $+1$  and 5040 of which have coefficient  $-1$ . It can be downloaded from the ancillary files associated to the arXiv version of this paper. One can check that  $F$  vanishes on the irreducible Abo-Wan hypersurface  $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(2, 5))$ . The proof is complete if we can show that  $F$  is irreducible.

We know that  $F$  is non-zero, has degree 6, and is invariant under the  $\text{SL}(3) \times \text{SL}(6)$  action. It is easy to check, in LiE for instance, that there are no non-trivial invariants of degree less than 6, and there is only one (up to scale) invariant in degree 6. If  $F$  were to factor into factors of positive degree, the individual factors would define invariant hypersurfaces of lower degree. Since this can't happen,  $F$  is irreducible. Note this solves [AW13, Problem 6.5].  $\square$

*Remark 5.2.* We suppose that this equation may have an expression as a root of a determinant of a special matrix, similar to Ottaviani's degree 15 equation in [Ott09], however our initial attempts at finding such an expression were unsuccessful. A natural guess is to start with  $T \in V \otimes \wedge^3 W$  and use it to produce the  $18 \times 36$  matrix  $A_T: W \otimes W \rightarrow (V \otimes W)^*$ , which has rank 3 when  $T$  has rank 1 and rank  $\leq 3r$  when  $T$  has rank  $r$ . However, this map actually factors through a map  $\wedge^2 W \rightarrow (V \otimes W)^*$  but this matrix is  $18 \times 15$  with maximum rank of 15. This means that this construction cannot distinguish rank 5 tensors from rank 6 tensors.

## 6. THE ABO-WAN HYPERSURFACES $\sigma_{3\ell+2}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2)))$

Abo and Wan also studied the following family of secant varieties that are hypersurfaces

$$(6.1) \quad \sigma_{3\ell+2}(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \subset \mathbb{P}(V \otimes \wedge^2 W) = \mathbb{P}^3 \binom{4\ell+3}{2} - 1$$

for  $\ell \geq 1$ , [AW13, Sec. 4]. For these secant varieties, an Ottaviani-type flattening construction produces an equation that vanishes on them and shows that they are defective. This approach, which was adapted from a construction by Ottaviani [Ott09], was used by [AW13] to prove that this secant variety (and an entire class of varieties similar to it) is defective. Namely, the dimension of each of these secant varieties is less than expected (it is expected to fill the ambient space) because of the existence of a non-trivial polynomial in the ideal. In particular, for the  $3 \cdot (4\ell + 3) \times 3 \cdot (4\ell + 3)$  “flattening” matrix  $\varphi_T$  associated to a general  $T \in \mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^{3 \cdot (4\ell+3)}$ , they showed that  $\det \varphi_T$  is both nontrivial and vanishes on (6.1).

It remains, however, an open problem to show that such polynomials are irreducible. This is the missing ingredient to describing the generator of the corresponding prime ideal.

*Remark 6.1.* This flattening construction and its variants have also been used successfully to find equations for other secant varieties in a wide array of cases in [LO11a], and led to new results in complexity [Lan14b, LO11b]. An analogous construction was used for partially symmetric tensors in [CEO12], and for arbitrary tensors for the so-called “salmon problem” in [Fri13, BO11, FG12].

We consider the construction of this equation in the case when  $\ell = 1$ . Here,  $V = \mathbb{C}^3$ , (so  $\wedge^2 V \cong V^*$ ) and  $W = \mathbb{C}^7$ . For a tensor  $T \in V \otimes \wedge^2 W$  we can view  $T$  as an element in  $\wedge^2 V^* \otimes \wedge^2 W$ , and associate to  $T$  the natural linear map it induces:

$$\varphi_T: V \otimes W^* \rightarrow V^* \otimes W,$$

which is skew-symmetric in  $W$  and (separately) skew-symmetric in  $V$ . The following provides an explicit construction of  $\varphi_T$  in coordinates.

Choose a basis  $a, b, c$  of  $V$ , and a basis  $e_{i,j}$  of  $\wedge^2 W$ . Then  $\varphi_T$  is constructed from the  $21 \times 21$  Kronecker product of two matrices:

$$\varphi_T = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} \\ -e_{12} & 0 & e_{23} & e_{24} & e_{25} & e_{26} & e_{27} \\ -e_{13} & -e_{23} & 0 & e_{34} & e_{35} & e_{36} & e_{37} \\ -e_{14} & -e_{24} & -e_{34} & 0 & e_{45} & e_{46} & e_{47} \\ -e_{15} & -e_{25} & -e_{35} & -e_{45} & 0 & e_{56} & e_{57} \\ -e_{16} & -e_{26} & -e_{36} & -e_{46} & -e_{56} & 0 & e_{67} \\ -e_{17} & -e_{27} & -e_{37} & -e_{47} & -e_{57} & -e_{67} & 0 \end{pmatrix}.$$

By replacing  $a \otimes e_{jk}$  with  $ajk$  (similarly for  $b \otimes e_{jk}$  and  $c \otimes e_{jk}$ ), we obtain the (symmetric) matrix  $\varphi_T =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & 0 & -b_{12} & -b_{13} & -b_{14} & -b_{15} & -b_{16} & -b_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & b_{12} & 0 & -b_{23} & -b_{24} & -b_{25} & -b_{26} & -b_{27} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & a_{37} & b_{13} & b_{23} & 0 & -b_{34} & -b_{35} & -b_{36} & -b_{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & a_{47} & b_{14} & b_{24} & b_{34} & 0 & -b_{45} & -b_{46} & -b_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & a_{57} & b_{15} & b_{25} & b_{35} & b_{45} & 0 & -b_{56} & -b_{57} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & a_{67} & b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & 0 & -b_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{17} & -a_{27} & -a_{37} & -a_{47} & -a_{57} & -a_{67} & 0 & b_{17} & b_{27} & b_{37} & b_{47} & b_{57} & b_{67} & 0 \\ 0 & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16} & -a_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} \\ a_{12} & 0 & -a_{23} & -a_{24} & -a_{25} & -a_{26} & -a_{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{12} & 0 & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} \\ a_{13} & a_{23} & 0 & -a_{34} & -a_{35} & -a_{36} & -a_{37} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{13} & -c_{23} & 0 & c_{34} & c_{35} & c_{36} & c_{37} \\ a_{14} & a_{24} & a_{34} & 0 & -a_{45} & -a_{46} & -a_{47} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{14} & -c_{24} & -c_{34} & 0 & c_{45} & c_{46} & c_{47} \\ a_{15} & a_{25} & a_{35} & a_{45} & 0 & -a_{56} & -a_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{15} & -c_{25} & -c_{35} & -c_{45} & 0 & c_{56} & c_{57} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & 0 & -a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{16} & -c_{26} & -c_{36} & -c_{46} & -c_{56} & 0 & c_{67} \\ a_{17} & a_{27} & a_{37} & a_{47} & a_{57} & a_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_{17} & -c_{27} & -c_{37} & -c_{47} & -c_{57} & -c_{67} & 0 \\ 0 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & 0 & -c_{12} & -c_{13} & -c_{14} & -c_{15} & -c_{16} & -c_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{12} & 0 & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & c_{12} & 0 & -c_{23} & -c_{24} & -c_{25} & -c_{26} & -c_{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{13} & -b_{23} & 0 & b_{34} & b_{35} & b_{36} & b_{37} & c_{13} & c_{23} & 0 & -c_{34} & -c_{35} & -c_{36} & -c_{37} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{14} & -b_{24} & -b_{34} & 0 & b_{45} & b_{46} & b_{47} & c_{14} & c_{24} & c_{34} & 0 & -c_{45} & -c_{46} & -c_{47} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 & b_{56} & b_{57} & c_{15} & c_{25} & c_{35} & c_{45} & 0 & -c_{56} & -c_{57} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{16} & -b_{26} & -b_{36} & -b_{46} & -b_{56} & 0 & b_{67} & c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & 0 & -c_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{17} & -b_{27} & -b_{37} & -b_{47} & -b_{57} & -b_{67} & 0 & c_{17} & c_{27} & c_{37} & c_{47} & c_{57} & c_{67} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If  $T$  has rank 1 as a tensor (up to the action of  $\mathrm{GL}(3) \times \mathrm{GL}(7)$ ), we may assume that  $T_{112} = 1$  and all other coordinates are zero. In this case,  $\varphi_T$  has rank 4. The construction is linear in  $T$ , so if  $T$  has rank  $r$  then  $\varphi_T$  has rank  $\leq 4r$  (because matrix rank is sub-additive). In particular if  $T$  has rank 5, then  $\varphi_T$  has rank  $\leq 20$ , so the determinant of  $\varphi_T$  must vanish.

One checks that for random  $T$ ,  $\varphi_T$  has rank 21 so the  $21 \times 21$  determinant of  $\varphi_T$  is non-trivial and produces the equation of  $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 6)))$ . We verified these computations using Macaulay2 [GS13].

The Bertini computation described above that is summarized in Theorem 4.1 indicates that (with high probability) this polynomial is irreducible. Indeed, since the degree found by this computation and the degree of determinant of  $\varphi_T$  are both 21, we know that the determinant of  $\varphi_T$  is irreducible. A similar argument holds for the cases  $\ell = 2, 3$  as well. For the first three non-trivial cases,  $\ell = 1, 2, 3$ , our numerical computation implies that the following results hold with high probability.

**Theorem 6.2.** *With high probability the prime ideal of each hypersurface*

- (1)  $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 6))) \subset \mathbb{P}(V \otimes \Lambda^2 W) = \mathbb{P}^{62}$ , for  $W = \mathbb{C}^7$ ,
- (2)  $\sigma_8(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 10))) \subset \mathbb{P}(V \otimes \Lambda^2 W) = \mathbb{P}^{164}$ , for  $W = \mathbb{C}^{11}$ ,
- (3)  $\sigma_{11}(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 14))) \subset \mathbb{P}(V \otimes \Lambda^2 W) = \mathbb{P}^{314}$ , for  $W = \mathbb{C}^{15}$ ,

is minimally generated by the determinant of the matrix

$$\varphi_T: V \otimes W^* \rightarrow V^* \otimes W,$$

which has size  $21 \times 21$  when  $W = \mathbb{C}^7$ ,  $33 \times 33$  when  $W = \mathbb{C}^{11}$ , and  $45 \times 45$  when  $W = \mathbb{C}^{15}$ .

Motivated by these results, we prove a more general statement without the “with high probability” qualifier.

**Theorem 6.3.** *For each  $\ell \geq 1$  the prime ideal of the irreducible hypersurface*

$$\sigma_{3\ell+2}(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell + 2))) \subset \mathbb{P}(V \otimes \Lambda^2 W) = \mathbb{P}^{3\binom{4\ell+3}{2}-1}$$

is generated by the determinant of the  $3(4\ell + 3) \times 3(4\ell + 3)$  matrix  $\varphi_T: V \otimes W^* \rightarrow V^* \otimes W$ .

*Proof.* We first explain how to construct the matrix  $\varphi_T$  in general. To that end, choose a basis  $v_1, v_2, v_3$  of  $V$ , and a basis  $e_{i,j}$  of  $\Lambda^2 W$  and write  $E = (e_{i,j}) \in \Lambda^2 W$  which is a  $(4\ell + 3) \times (4\ell + 3)$  skew symmetric matrix, i.e.,  $E = (e_{i,j}) = -E^t$ . Then,  $\varphi_T$  is the  $3(4\ell + 3) \times 3(4\ell + 3)$  matrix

constructed via a  $\boxtimes$  product (see Section 7). Namely, we take the usual Kronecker product of matrices

$$\begin{pmatrix} 0 & v_1 & -v_2 \\ -v_1 & 0 & v_3 \\ v_2 & -v_3 & 0 \end{pmatrix} \otimes E,$$

and replace each  $v_i e_{j,k}$  with the variable  $x_{ijk}$  (as explained in Section 7). The resulting matrix  $\varphi_T$  represents a point  $T \in V \otimes \bigwedge^2 W \cong \bigwedge^2 V^* \otimes \bigwedge^2 W$ . Note, this variable replacement is crucial, because the identity (7.1) implies that before our replacement of  $v_i e_{j,k}$  with  $x_{ijk}$ , the determinant of the matrix we construct is zero. On the other hand, Lemma 4.1 of [AW13] constructs a tensor  $T$  for which  $\varphi_T$  has full rank. In particular,  $\det(\varphi_T) \neq 0$ . Abo and Wan also explained why  $\det \varphi_T$  vanishes on the appropriate secant variety, which is a consequence of the flattening construction.

We will prove that the ideal generated by  $\det \varphi_T$  is prime by showing that  $\det \varphi_T$  is irreducible, which will be a consequence of Theorem 7.1 below.  $\square$

*Remark 6.4.* The case  $\ell = 0$  is the well-known  $3 \times 3$  determinantal hypersurface since

$$\sigma_2(\text{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 2))) \cong \sigma_2(\text{Seg}(\mathbb{P}^2 \times (\mathbb{P}^2)^*)).$$

## 7. TENSOR PRODUCTS OF MATRICES

Let  $P = (p_{i,j}) \in A^* \otimes B$  and  $Q = (q_{k,l}) \in C^* \otimes D$  be 1-generic matrices and consider their tensor product

$$P \boxtimes Q \in A^* \otimes B \otimes C^* \otimes D,$$

which we view as a 4-dimensional tensor. One flattening is to view  $P \boxtimes Q$  in  $(A \otimes C)^* \otimes (B \otimes D)$ . In this flattening we see  $P \boxtimes Q$  as a matrix with rows indexed by the double index  $(i, k)$  and columns indexed by the double index  $(j, l)$ , and the entry in position  $((i, k), (j, l))$  is the tensor product of variables  $p_{i,j} \otimes q_{k,l}$ .

Note the usual Kronecker product of matrices (unfortunately denoted by the tensor product symbol  $\otimes$ ) would put the symmetric product  $p_{i,j} q_{k,l}$  in that position, so we use the symbol  $\boxtimes$  to make this distinction.

The usual Kronecker product satisfies the well-known property that if  $P$  and  $Q$  are square matrices of size  $m$  and  $n$  respectively, then

$$(7.1) \quad \det(P \otimes Q) = \det(P)^n \det(Q)^m.$$

The question we are led to by our study of the equations of Abo-Wan hypersurfaces is whether the determinant of the  $\boxtimes$  product is irreducible or not.

As we will see, introducing the non-commutative feature in this tensor product causes interesting behavior of determinants.

For our purposes, we are interested in the case that  $P$  and  $Q$  are skew-symmetric, and  $P$  is a  $3 \times 3$  matrix. In this case, the usual tensor product will always have vanishing determinant because of (7.1). On the other hand, the initial cases of the  $\boxtimes$  product behave as follows.

**Theorem 7.1.** *Let  $P$  and  $Q$  be respectively  $3 \times 3$  and  $s \times s$  skew-symmetric 1-generic matrices.*

- (1) *If  $s = 1$  or  $s = 2$ , then  $\det(P \boxtimes Q) = 0$ .*
- (2) *If  $s = 3$ , then  $\det(P \boxtimes Q)$  factors as the cube of a cubic polynomial.*
- (3) *If  $s = 4$ , then  $\det(P \boxtimes Q)$  factors as the square of a sextic polynomial.*
- (4) *If  $s \geq 5$ , then  $\det(P \boxtimes Q)$  is irreducible.*

*Proof.* The case  $s = 1$  is trivial because the determinant is just the determinant of  $P$  in renamed variables. The cases  $s = 2, 3, 4$  are easy to verify in `Macaulay2` directly by constructing the usual tensor product matrix, substituting new variables for each product  $p_{i,j}q_{k,l}$ , and using the `factor` command. As  $s$  grows, this computation becomes much more difficult.

For  $s = 5, 6, \dots, 15$  we specialized the variables in the matrix  $P \boxtimes Q$  to a random line, computed the determinant, and checked that the resulting homogeneous polynomial in 2 variables had the same degree and did not factor over  $\mathbb{Q}$ . This provides a certificate that the original polynomial is irreducible. (Note, if the specialized polynomial factors, this test is inconclusive but gives evidence that the original polynomial probably factors.)

For  $s \geq 8$  we proceed by induction from the case  $s - 3$  to the case  $s$ . Our proof for each case is the same building on the base cases  $s = 3, \dots, 7$  which were computed directly as described above.

**Summary of proof:** Our induction step is somewhat lengthy, but the idea is straightforward. Specifically, we will show that if  $\det(P \boxtimes Q)$  has a factorization as a product of non-trivial invariants this will force a non-trivial factorization of a  $P \boxtimes Q$  of smaller size, which can't happen by induction. The rest of the proof is a careful study of why this phenomenon occurs.

We partially compute  $\det(P \boxtimes Q)$  via Laplace expansion. Let  $Q'$  be the first  $3 \times 3$  principal submatrix of  $Q$  and let  $Q'^c$  denote the principal minor of  $Q$  with complementary indices, which is necessarily the last  $(s - 3) \times (s - 3)$  principal minor of  $Q$ .

Now  $P \boxtimes Q'$  and  $P \boxtimes Q'^c$  are complementary principal minors of  $P \boxtimes Q$ , which are respectively of size  $9 \times 9$  and  $3(s - 3) \times 3(s - 3)$ . Note that the term

$$H := \det(P \boxtimes Q') \cdot \det(P \boxtimes Q'^c)$$

occurs in  $\det(P \boxtimes Q)$ . We know that  $H$  is non-zero by cases 3 (a base case) and  $s - 3$  (by the induction hypothesis).

Recall [Stu08, Ch.4] that we can assign a **weight** or **multi-degree** to a polynomial in

$$S^d(\wedge^2(\mathbb{C}^3)^* \otimes \wedge^2 \mathbb{C}^s) \cong S^d(\mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^s)$$

as follows. Let  $e_1, e_2, e_3$  be the standard basis of  $\mathbb{Z}^3$  and let  $f_1, \dots, f_s$  denote the standard basis on  $\mathbb{Z}^s$ . To the variable  $p_{i,j} \otimes q_{k,l}$  we assign the weight  $e_h + f_k + f_l$ , where  $h$  is such that  $\{i, j, h\} = \{1, 2, 3\}$ , and write this as  $wt(p_{i,j} \otimes q_{k,l}) = e_h + f_k + f_l$ . We assign weights to monomials by declaring that the weight function  $wt()$  is additive over products of variables. A polynomial in  $S^d(\mathbb{C}^3 \otimes \wedge^2 \mathbb{C}^s)$  is called **isobaric** if every term has weight

$$a(e_1 + e_2 + e_3) + b(f_1 + \dots + f_s)$$

and must satisfy the condition that  $a = d/3$  and  $b = 2d/s$  are integers. Note that all  $\mathrm{SL}(3) \times \mathrm{SL}(s)$ -invariant polynomials must be isobaric.

Now we compute

$$wt(\det(P \boxtimes Q')) = 3(e_1 + e_2 + e_3) + 6(f_1 + f_2 + f_3),$$

and

$$wt(\det(P \boxtimes Q'^c)) = (s - 3)(e_1 + e_2 + e_3) + 6(f_4 + \dots + f_s).$$

The weights reflect the fact that  $H$  is a bi-homogeneous polynomial in two disjoint sets of variables (those appearing in  $P \boxtimes Q'^c$  and those in  $P \boxtimes Q'$ ). Since  $H$  contains the monomials of highest possible degree among the variables appearing in each matrix,  $H$  cannot be canceled by other terms in the block Laplace expansion of  $\det \varphi_T$ . A trivial, but necessary, remark is

that  $\det(P \boxtimes Q)$  is not equal to  $H$ . One way to see this is that  $\det(P \boxtimes Q)$  is  $\mathrm{SL}(3) \times \mathrm{SL}(s)$ -invariant, but  $H$  is not.

Let  $M$  denote Newton polytope of  $H$ , which is the convex hull of all the exponent vectors of  $H$ . By the above discussion,  $M$  consists of all exponent vectors of monomials arising as a product of monomials of weight  $3(e_1 + e_2 + e_3) + 6(f_1 + f_2 + f_3)$  with monomials of weight  $(s-3)(e_1 + e_2 + e_3) + 6(f_4 + \dots + f_s)$ .

Let  $N_{P \boxtimes Q'}$  (respectively  $N_{P \boxtimes Q'^c}$ ) denote the Newton polytope of  $\det(P \boxtimes Q')$  (respectively of  $\det(P \boxtimes Q'^c)$ ). Then  $M$  is the Minkowski sum  $M = N_{P \boxtimes Q'} \oplus N_{P \boxtimes Q'^c}$ , where the direct sum is due to the fact that  $\det(P \boxtimes Q')$  and  $\det(P \boxtimes Q'^c)$  use disjoint sets of variables.

Now we claim that  $H$  cannot have a non-trivial *isobaric* factorization. For contradiction, suppose  $\det(P \boxtimes Q) = f \cdot g$  with both  $f$  and  $g$  isobaric, which means that the weights of  $f$  and  $g$  must be of the form:

$$a(e_1 + e_2 + e_3) + b(f_1 + \dots + f_s),$$

with  $a = d/3$  and  $b = 2d/s$  both integers, and  $d < 3s$ .

The possible  $(a, b)$  satisfying these conditions are (at most):

$$(s/6, 1), \quad (s/3, 2), \quad (s/2, 3), \quad (2s/3, 4), \quad (5s/6, 5),$$

and we don't need to consider the pairs  $(a, b) \notin \mathbb{Z}^2$ .

We have

$$(7.2) \quad f \cdot g = \det(P \boxtimes Q') \cdot \det(P \boxtimes Q'^c) + (\text{l. o. t.}),$$

where by "l. o. t." we mean terms using monomials not supported in  $M$ .

We can delete the lower order terms by setting to zero those variables that don't occur in  $P \boxtimes Q'$  or  $P \boxtimes Q'^c$ . Further, we know that  $\det(P \boxtimes Q')$  is non-zero, so we may partially evaluate  $P \boxtimes Q$  by assigning scalar values  $c_{i,j,k}$  to each  $x_{i,j,k}$  appearing in  $P \boxtimes Q'$  so that  $\det(P \boxtimes Q')$  evaluates to a non-zero scalar  $C$ . After this evaluation, (7.2) becomes

$$\tilde{f} \cdot \tilde{g} = \left( \sum_{(\alpha, \beta) \in N_{P \boxtimes Q'} \oplus N_{P \boxtimes Q'^c}} f_{\alpha, \beta} c^\alpha x^\beta \right) \cdot \left( \sum_{(\alpha, \beta) \in N_{P \boxtimes Q'} \oplus N_{P \boxtimes Q'^c}} g_{\alpha, \beta} c^\alpha x^\beta \right) = C \det(P \boxtimes Q'^c),$$

where  $\tilde{f}$  and  $\tilde{g}$  are respectively the images of  $f$  and  $g$  under this evaluation, and the equation is non-trivial and supported on  $N_{P \boxtimes Q'^c}$ . Finally, the degrees of  $f$  and  $g$  must be integers in

$$\{s/2, \quad s, \quad 3s/2, \quad 2s, \quad 5s/2\},$$

such that their sum is  $3s$  (if there are no such pair for our particular value of  $s$  then we could end the proof earlier). So the smallest possible degree of each of  $\tilde{f}$  or  $\tilde{g}$  is  $s/2 - 3 \geq 1$  for all  $s \geq 7$ , in particular, neither  $\tilde{f}$  nor  $\tilde{g}$  are constant.

Thus we have produced a non-trivial isobaric factorization of  $\det(P \boxtimes Q'^c)$ , which is impossible when the size of  $Q'^c$  is at least  $5 \times 5$ , i.e.  $s - 3 \geq 5$  which surely happens when  $s \geq 8$ . This provides the contradiction that ends the proof.  $\square$

*Remark 7.2.* If one reinterprets Theorem 7.1 in light of projective duality, it gives some hints to Ottaviani's open question #3 in [Ott13b], which is the skew-symmetric version of a problem on hyperdeterminants considered in [Oed12].

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC  
*E-mail address:* [nsdaleo@ncsu.edu](mailto:nsdaleo@ncsu.edu)  
*URL:* [www.math.ncsu.edu/~nsdaleo](http://www.math.ncsu.edu/~nsdaleo)

DEPARTMENT OF APPLIED AND COMPUTATIONAL MATHEMATICS AND STATISTICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN  
*E-mail address:* [hauenstein@nd.edu](mailto:hauenstein@nd.edu)  
*URL:* [www.nd.edu/~jhauenst](http://www.nd.edu/~jhauenst)

DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AUBURN, AL  
*E-mail address:* [oeding@auburn.edu](mailto:oeding@auburn.edu)  
*URL:* [www.auburn.edu/~oeding](http://www.auburn.edu/~oeding)