Real solutions to systems of polynomial equations and parameter continuation

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May 7, 2013

Abstract

Given a parameterized family of polynomial equations, a fundamental question is to determine upper and lower bounds on the number of real solutions a member of this family can have and, if possible, compute where the bounds are sharp. A computational approach to this problem was developed by Dietmaier in 1998 which used a local linearization procedure to move in the parameter space to change the number of real solutions. He used this approach to show that there exists a Stewart-Gough platform that attains the maximum of forty real assembly modes. Due to the necessary ill-conditioning near the discriminant locus, we propose replacing the local linearization near the discriminant locus with a homotopy-based method derived from the method of gradient descent arising in optimization. This new hybrid approach is then used to develop a new result in real enumerative geometry.

Keywords. real solutions, parameter space, discriminant, numerical algebraic geometry, polynomial system, homotopy continuation, enumerative geometry

AMS Subject Classification. 65H10, 68W30, 14Q99

Introduction

Parameterized families of systems of real polynomial equations naturally arise in many areas including economics, engineering, enumerative geometry, and physics where the real solutions for each member are often the solutions of interest. A fundamental and difficult question is to compute upper and lower bounds on the number of real solutions and, if possible, compute parameter values where such bounds are attained. For systems arising from geometry, some bounds on the number of real solutions are known and

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summarized in [22]. Two notable examples are the real analog of Steiner's problem of conics [23] and the number of real assembly modes of Stewart-Gough platforms [8, 24].

Steiner's problem of conics is to count the number of plane conics tangent to five given plane conics in general position, with the answer being 3264. The solving of Steiner's problem lead to the development of intersection theory and ultimately to many techniques used in the field of numerical algebraic geometry (see [21] for a general introduction to numerical algebraic geometry). The real analog of Steiner's problem, proposed by Fulton [7], is to determine if there exists five real plane conics such that there are 3264 *real* plane conics tangent to the given five conics. This was answered in the affirmative later by Fulton, a result he did not publish, and independently in [16].

A general Stewart-Gough platform has 40 assembly modes. Dietmaier [6] considered, and answered in the affirmative, the existence of a Stewart-Gough platform with 40 real assembly modes. This platform was found using a computational approach that is summarized in Section 1.4. In short, the approach moves through the real parameter space based on solving a linear optimization problem obtained by constructing a local linearization of the system of equations. The shortcoming of this approach is the linearization near the discriminant locus is necessarily ill-conditioned.

This shortcoming of Dietmaier's method is addressed in Section 3 by introducing a homotopy-based approach for moving to and through the discriminant locus. This new approach uses homotopies that we call *gradient descent homotopies*, described in Section 2, based on their relationship to the method of gradient descent used for solving optimization problems. Gradient descent homotopies arise from the computation of critical points of the distance function between a given point and the set of real solutions of a system of polynomial equations. Such critical points have been used in several other algorithms in real algebraic geometry including [1, 11, 18, 19].

Section 4 demonstrates using this hybrid approach together with an *a posteriori* real certification technique developed in [13] arising from Smale's α-theory [4, 20] to prove the following real enumerative geometric theorem.

Theorem 1. *There exists eight lines in $\mathbb{R}^3$ met by 92 real plane conics.*

Since eight general lines in $\mathbb{C}^3$ are met by 92 plane conics, Theorem 1 shows that the real analog of this enumerative geometry problem is answered in the affirmative.

1 Background

1.1 Algebraic sets

For a polynomial system $g : \mathbb{C}^N \to \mathbb{C}^n$, let

$$\mathcal{V}(g) = \{x \in \mathbb{C}^N \mid g(x) = 0\} \text{ and } \mathcal{V}_\mathbb{R}(g) = \mathcal{V}(g) \cap \mathbb{R}^N = \{x \in \mathbb{R}^N \mid g(x) = 0\}$$

be the set of solutions and the set of real solutions of $g(x) = 0$, respectively. A set $A \subset \mathbb{C}^N$ is called an *algebraic set* if $A = \mathcal{V}(g)$ for some polynomial system g. An algebraic set $A \subset \mathbb{C}^N$ is *reducible* if there exists algebraic sets $B, C \subset \mathbb{C}^N$ such that $B, C \subsetneq A$ and $A = B \cup C$. If an algebraic set is not reducible, it is said to be *irreducible*.

Let $Jg(x)$ denote the $n \times N$ Jacobian matrix of g evaluated at x. When $N = n$, the system g is said to be a *square system*. In this case, a point $x \in \mathcal{V}(g)$ is *nonsingular* if $\det Jg(x) \neq 0$ and *singular* if $\det Jg(x) = 0$.

1.2 Discriminant locus

Let $\mathcal{P} \subset \mathbb{C}^P$ be an irreducible algebraic set and $\mathcal{P}_\mathbb{R} = \mathcal{P} \cap \mathbb{R}^P$. We will focus on parameterized polynomial systems $f : \mathbb{C}^N \times \mathcal{P} \to \mathbb{C}^N$ having *real* coefficients such that $f(x, p) = 0$ has finitely many isolated solutions in $\mathbb{C}^N$, all of which are nonsingular, for general parameter values $p \in \mathcal{P}$. Thus, there exists an integer $M_\mathcal{P} \geq 0$ and an algebraic set $\mathcal{Q} \subsetneq \mathcal{P}$ such that, for all $p \in \mathcal{P} \setminus \mathcal{Q}$,

$$f(x, p) = 0 \text{ has exactly } M_\mathcal{P} \text{ solutions, all of which are nonsingular.} \tag{1}$$

The *discriminant locus* of f, denoted $\Delta(f)$, is the set of $p \in \mathcal{P}$ such that (1) does not hold. The *real discriminant locus* of f is $\Delta_\mathbb{R}(f) = \Delta(f) \cap \mathbb{R}^P$.

If C is a connected component of $\mathcal{P}_\mathbb{R} \setminus \Delta_\mathbb{R}(f)$, the number of real solutions is constant on C. That is, there exist an integer $0 \leq M_C \leq M_\mathcal{P}$, with $M_\mathcal{P} - M_C$ even, such that $f(x, p) = 0$ has exactly M_C real solutions, all of which are nonsingular, for every $p \in C$.

Example 2. For $\mathcal{P} = \mathbb{C}^2$ and $f(x, p) = x^2 + p_1 x + p_2$, we have $M_\mathcal{P} = 2$ and $\Delta(f) = \mathcal{V}(p_1^2 - 4p_2)$. In particular, the equation $f(x, p) = 0$ has two distinct real solutions for $p \in \mathbb{R}^2$ such that $p_1^2 > 4p_2$, one real solution of multiplicity 2 when $p_1^2 = 4p_2$, and no real solutions when $p_1^2 < 4p_2$.

Example 3. Let $\mathcal{P} = \mathcal{V}(p_3^2 - p_1 p_3 + p_2) \subset \mathbb{C}^3$ and $f : \mathbb{C}^2 \times \mathcal{P} \to \mathbb{C}^2$ defined by

$$f(x, p) = \begin{bmatrix} x_1^2 + p_1 x_1 + p_2 \\ x_1 x_2 + p_3 x_2 - 1 \end{bmatrix}.$$

We have $M_\mathcal{P} = 1$ and $\Delta(f) = \mathcal{P} \cap \mathcal{V}(2p_3 - p_1)$. In particular, the system of equations $f(x, p) = 0$ has one real solution for $p \in \mathcal{P}_\mathbb{R}$ such that $2p_3 \neq p_1$ and no solutions in $\mathbb{C}^2$ when $p \in \Delta(f)$. We note that $M_{\mathbb{C}^3} = 2$.

1.3 Trackable paths

Homotopy methods rely upon the construction of solution paths which are *trackable*. The following, from [12], defines a trackable path starting at a nonsingular solution.

Definition 4. Let $H(x, t) : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ be polynomial in x and complex analytic in t. If $y \in \mathbb{C}^N$ is a nonsingular solution of $H(x, 1) = 0$, then y is said to be *trackable* for $t \in (0, 1]$ from $t = 1$ to $t = 0$ using $H(x, t)$ if there is a smooth map $\xi_y : (0, 1] \to \mathbb{C}^N$ such that $\xi_y(1) = y$ and, for $t \in (0, 1]$, $\xi_y(t)$ is a nonsingular solution of $H(x, t) = 0$.

The solution path starting at y is said to *converge* if $\lim_{t\to 0^+} \xi_y(t) \in \mathbb{C}^N$, where $\lim_{t\to 0^+} \xi_y(t)$ is called the *endpoint* (or *limit point*) of the path.

A trackable path may converge to a nonsingular solution, converge to a singular solution, or diverge. Numerical path tracking algorithms (see [21] for a general overview) use endgames and projective space to handle the singular and divergent cases.

1.4 Dietmaier's approach

The following is a summary of the approach used by Dietmaier [6] to compute a Stewart-Gough platform [8, 24] that achieves the maximum of 40 real assembly modes. In this section, we assume that $\mathcal{P} = \mathbb{C}^P$ and consider polynomial systems of the form $f : \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^N$ having real coefficients such that $f(x,p) = 0$ has $M_{\mathbb{C}^P} \geq 2$ solutions, all of which are nonsingular, for general parameter values $p \in \mathbb{C}^P$. The basic idea is to move in the real parameter space $\mathcal{P}_{\mathbb{R}} = \mathbb{R}^P$ so that two complex conjugate solutions first merge and then become two distinct real solutions. Clearly, the parameter value where the two solutions coincide is a point on the real discriminant locus $\Delta_{\mathbb{R}}(f)$.

Suppose that $p^* \in \mathbb{R}^P \setminus \Delta_{\mathbb{R}}(f)$ and $x^* \in \mathbb{C}^N \setminus \mathbb{R}^N$ such that $f(x^*, p^*) = 0$. If $\Delta x^* \in \mathbb{C}^N$ and $\Delta p^* \in \mathbb{R}^P$ so that $f(x^*+\Delta x^*, p^*+\Delta p^*) = 0$, linearizing at (x^*, p^*) yields

$$\Delta x^* \approx -J_x f(x^*, p^*)^{-1} J_p f(x^*, p^*) \Delta p^*$$

where $J_x f(x^*, p^*)$ and $J_p f(x^*, p^*)$ are the Jacobian matrices of f with respect to x and p evaluated at (x^*, p^*), respectively. Since $p^* \notin \Delta_{\mathbb{R}}(f)$, the matrix $J_x f(x^*, p^*)$ is indeed invertible. The value of Δp^* is chosen to minimize the distance, using some appropriate norm, between $x^* + \Delta x^*$ and $\mathbb{R}^N$ subject to the following conditions:

1. Since this linearization is only acceptable on a small neighborhood, Δp^* must be constrained to avoid too large of a step.

2. To maintain the same number of distinct real solutions, for every distinct pair $y_1, y_2 \in \mathbb{R}^N$ such that $f(y_i, p^*) = 0$, the distance between the approximations of the corresponding solutions of $f(x, p^* + \Delta p^*) = 0$, namely $y_i + \Delta y_i$, must remain larger than a given bound.

3. To maintain the same number of finite solutions, for every $z \in \mathbb{C}^N$ such that $f(z, p^*) = 0$, the norm of the approximation of the corresponding solution of $f(x, p^* + \Delta p^*) = 0$, namely $z + \Delta z$, must remain below a given bound.

For a properly selected Δp^*, the solutions to $f(x, p^* + \Delta p^*) = 0$ are computed with one natural approach being a parameter homotopy (see [2, 21]). If no acceptable Δp^* can be found, the process is restarted using a different nonreal solution of $f(x, p^*) = 0$. The process repeats until some complex conjugate pair merges (up to numerical tolerance). If no acceptable value of Δp^* can be found for all nonreal solutions, the process fails.

If a complex conjugate pair has successfully merged, the parameter value needs to be moved so that the merged pair becomes distinct real solutions. We note that [6] does not directly address how to move through the discriminant locus to initially yield two distinct real solutions, but one approach is to simply continue to move in the direction of the last Δp^* computed. Updating p^* to be the new parameter value with the newly created solutions $x_1, x_2 \in \mathbb{R}^N$ of $f(x, p^*) = 0$, the approach uses the same constraints above with the objective of maximizing the distance between $x_1 + \Delta x_1$ and $x_2 + \Delta x_2$. As before, once Δp^* is computed, the solutions to $f(x, p^* + \Delta p^*) = 0$ are computed and this process is repeated until the two new real solutions are sufficiently far apart.

2 Gradient descent homotopies

Suppose that $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system with real coefficients and $y \in \mathbb{R}^N$. Since we will be using deformations of f, we impose two assumptions on f. First, we assume that $N \geq n$ so that f is not overconstrained. Second, we assume that $Jf(x)$ has rank n for general $x \in \mathbb{C}^N$. That is, there exists an algebraic set $\mathcal{Q} \subsetneq \mathbb{C}^N$ such that rank $Jf(x) = n$ for all $x \in \mathbb{C}^N \setminus \mathcal{Q}$. We note that these assumptions can always be satisfied by replacing f with sums of squares.

For $x \in \mathbb{R}^N$, define $d_y(x) = \|x - y\|^2 = (x - y)^T (x - y)$ and consider the polynomial optimization problem

$$\text{(P)} \qquad \min \{d_y(x) \mid x \in \mathcal{V}_{\mathbb{R}}(f)\}.$$

The basic idea is to construct a homotopy and defines a solution path emanating from the given point y. The aim is to compute a *real critical point* for problem (P), which is a point $x \in \mathcal{V}_{\mathbb{R}}(f)$ such that

$$\operatorname{rank} \begin{bmatrix} x - y & \nabla f_1(x)^T & \cdots & \nabla f_n(x)^T \end{bmatrix} \leq n$$

where $\nabla f_i(x)$ is the gradient vector of f_i evaluated at x. In particular, the set of real crtical points for (P) is the set $\pi(\mathcal{V}(G)) \cap \mathbb{R}^N$ where $\pi(x, \lambda) = x$ and $G : \mathbb{C}^N \times \mathbb{P}^n \to \mathbb{C}^{N+n}$ is the polynomial system defined by

$$G(x, \lambda) = \begin{bmatrix} f(x) \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots \lambda_n \nabla f_n(x)^T \end{bmatrix}. \tag{2}$$

Consider the homotopy $H : \mathbb{C}^N \times \mathbb{P}^n \times \mathbb{C} \to \mathbb{C}^{N+n}$ defined by

$$H(x, \lambda, t) = \begin{bmatrix} f(x) - tf(y) \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots \lambda_n \nabla f_n(x)^T \end{bmatrix}. \tag{3}$$

Since we are interested in one solution path, namely the path starting at $(y, 1, 0, \ldots, 0)$ when $t = 1$, we will also consider an affine version of H which performs computations on an affine patch in $\mathbb{P}^n$. This homotopy $H^a : \mathbb{C}^N \times \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}^{N+n+1}$ is defined by

$$H^a(x, \lambda, t) = \begin{bmatrix} f(x) - tf(y) \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots \lambda_n \nabla f_n(x)^T \\ \lambda_0 + \alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n - \alpha_0 \end{bmatrix} \tag{4}$$

where $\alpha_i \in \mathbb{R} \setminus \{0\}$. For H^a, we will consider the start point $(y, \alpha_0, 0, \ldots, 0)$ at $t = 1$. Due to Proposition 6 below, we call H and H^a *gradient descent homotopies*.

Remark 5. We note that even though we have constructed homotopies H and H^a for a polynomial system f, one could consider these homotopies for a more general class of systems, e.g., analytic systems or twice differentiable systems.

2.1 Gradient descent

Consider the homotopy H^a defined by (4) when $f : \mathbb{C}^N \to \mathbb{C}$ is a polynomial with real coefficients, $y \in \mathbb{R}^N$ such that $f(y) \neq 0$ and $\nabla f(y) \neq 0$, and $\alpha_0, \alpha_1 \in \mathbb{R} \setminus \{0\}$. It is easy to see that the Jacobian matrix of H^a with respect to x and λ is

$$JH^a_{x,\lambda}(x, \lambda_0, \lambda_1, t) = \begin{bmatrix} \nabla f(x) & 0 & 0 \\ \lambda_0 I_N + \lambda_1 \mathcal{H}_f(x) & x - y & \nabla f(x)^T \\ 0 & 1 & \alpha_1 \end{bmatrix}$$

where I_N is the $N \times N$ identity matrix and $\mathcal{H}_f(x)$ is the Hessian of f. In particular,

$$JH^a_{x,\lambda}(y, \alpha_0, 0, 1) = \begin{bmatrix} \nabla f(y) & 0 & 0 \\ \alpha_0 I_N & 0 & \nabla f(y)^T \\ 0 & 1 & \alpha_1 \end{bmatrix}$$

is full rank since

$$|\det JH^a_{x,\lambda}(y, \alpha_0, 0, 1)| = |\alpha_0|^{N-1} \|\nabla f(y)\|^2 \neq 0. \tag{5}$$

The implicit function theorem yields that the solution path starting at $(y, \alpha_0, 0)$ at $t = 1$ exists, and is smooth, locally near $t = 1$. The following proposition shows that this path, as t decreases from 1, moves in the direction of gradient descent if $f(y) > 0$ and in the direction of gradient ascent if $f(y) < 0$.

Proposition 6. *Let $f : \mathbb{C}^N \to \mathbb{C}$ be a polynomial with real coefficients and $y \in \mathbb{R}^N$ such that $f(y) \neq 0$ and $\nabla f(y) \neq 0$. Let $\alpha_0, \alpha_1 \in \mathbb{R} \setminus \{0\}$ and H^a be the homotopy defined by (4). If $(x(t), \lambda(t))$ is the solution path starting with $(y, \alpha_0, 0)$ at $t = 1$, then*

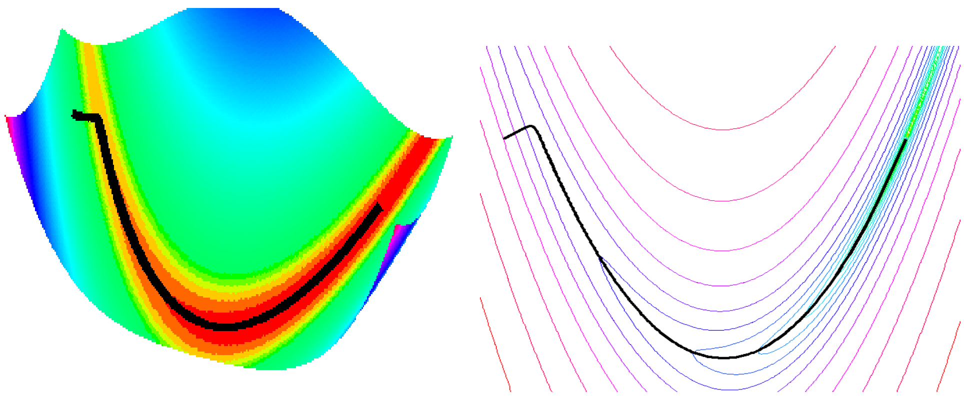
$$\left.\frac{dx}{dt}\right|_{t=1} = \frac{f(y)}{\|\nabla f(y)\|^2} \nabla f(y)^T.$$

Proof. Define $\gamma = f(y)/\|\nabla f(y)\|^2$. The result immediately follows from (5) and

$$JH^a_{x,\lambda}(y, \alpha_0, 0) \begin{bmatrix} \gamma \nabla f(y)^T \\ \gamma \alpha_0 \alpha_1 \\ -\gamma \alpha_0 \end{bmatrix} = \begin{bmatrix} f(y) \\ 0 \\ 0 \end{bmatrix} = -JH^a_t(y, \alpha_0, 0)$$

where $JH^a_t(y, \alpha_0, 0)$ is the vector corresponding to the Jacobian matrix of H^a with respect to t evaluated at $(y, \alpha_0, 0)$. $\square$

Figure 1: Plot of the x coordinates of the solution path for the Rosenbrock polynomial on (a) the graph and (b) a contour plot.



(a) (b)

Since the path need not move in the gradient descent or ascent directions for $t \neq 1$, this solution path avoids one of the drawbacks of the *method of gradient descent*, also known as the *method of steepest descent* (see [15, §3.2-3.3]), as shown in the following classical example.

Example 7. Let $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ be the Rosenbrock polynomial [17]. Clearly, $\mathcal{V}_{\mathbb{R}}(f) = \{(1,1)\}$ and $f > 0$ on $\mathbb{R}^2 \setminus \mathcal{V}_{\mathbb{R}}(f)$. For this polynomial, the iterative method of gradient descent has poor convergence due to this method using orthogonal steps to move through the "curved valley," which surrounds the parabola $x_2 = x_1^2$ and contains the unique real root. Following [17], we took $y = (-1.2, 1)$ so that the x coordinates of the solution path descend into the "curved valley" and then follow the valley around to the point $(1,1)$. Figure 1 plots the x coordinates of the solution path on the graph and on a contour plot.

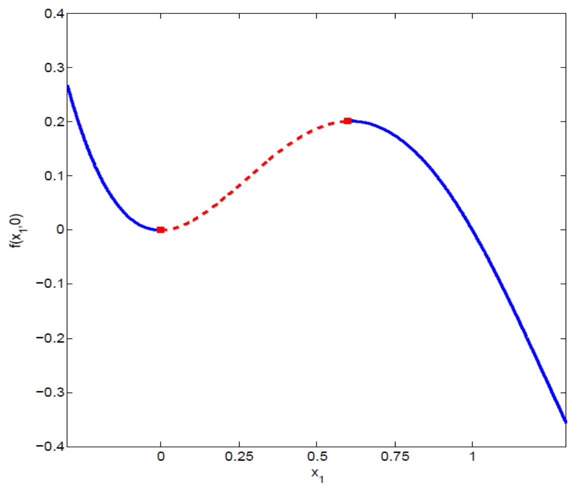
2.2 Theory

The following provides some theoretical results for gradient descent homotopies.

Proposition 8. *Let $N \geq n$, $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system with real coefficients, $y \in \mathbb{R}^N$ such that $f(y) \neq 0$, $\alpha \in (\mathbb{R} \setminus \{0\})^{n+1}$, and H^a be the homotopy defined by (4). If the solution path defined by H^a starting at $(y, \alpha_0, 0, \ldots, 0) \in \mathbb{R}^N \times \mathbb{R}^{n+1}$ is trackable on $(0,1]$ and converges as $t \to 0$ with endpoint (x^*, λ^*), then $x^* \in \mathcal{V}_{\mathbb{R}}(f)$ is a real critical point for (P).*

Proof. Since the homotopy H^a has real coefficients and the start point is real, trackability and convergence immediately imply that every point on the path for $t \in [0,1]$ is real.

Figure 2: Plot of the x_1 coordinate of the solution path on the graph of $f(x_1, 0)$.



Since $\alpha_0 \neq 0$ and $H^a(x^*, \lambda^*, 0) = 0$, we know $\lambda^* \neq 0$. Therefore, if we consider $\lambda^* \in \mathbb{P}^n$, this yields $G(x^*, \lambda^*) = 0$ where G is defined in (2). Thus, by definition, $x^* \in \mathcal{V}_{\mathbb{R}}(f)$ is a real critical point for (P). $\square$

Even though the endpoint is a real critical point for (P), the following example shows that it need not be the global minimizer of the distance measured from y.

Example 9. Consider the polynomial $f(x_1, x_2) = x_2^2 + x_1^2(x_1 - 1)(x_1 - 2)$ with $y = (0.6, 0)$. It is easy to verify that the Euclidean distance between y and $\mathcal{V}_{\mathbb{R}}(f)$ is 0.4 which is attained at the point $(1, 0) \in \mathcal{V}_{\mathbb{R}}(f)$. For the homotopy H^a defined in (4) with $\alpha = (4, 2)$, the endpoint of the path starting at $(0.6, 0, 4, 0)$ is $(0, 0, 0, 2)$. That is, this path ended at the real critical point $x^* = (0, 0)$ which is not the global minimizer of (P). Since every point on the path has an x_2 coordinate of zero, Figure 2 plots the x_1 coordinate of the solution path on the graph of $f(x_1, 0)$. The local maximum of $f(x_1, 0)$ between 0 and 1 occurs at $\beta = (9 - \sqrt{17})/8 \approx 0.6096$ which yields that the gradient descent path for $y = (z, 0)$ will yield $(0, 0)$ when $0 < z < \beta$ and $(1, 0)$ when $\beta < z < 1$.

Proposition 8 immediately yields the following.

Corollary 10. *Let $N \geq n$ and $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system with real coefficients such that $\mathcal{V}_{\mathbb{R}}(f) = \emptyset$. For any $y \in \mathbb{R}^N$ and any $\alpha \in \mathbb{R}^{n+1}$, the solution path defined by the homotopy H^a from* (4) *starting at* $(y, \alpha_0, 0, \ldots, 0)$ *is either not trackable on* $(0, 1]$ *or does not converge in* $\mathbb{C}^N \times \mathbb{C}^{n+1}$ *as* $t \to 0$.

The following examples demonstrate the two cases of Corollary 10.

Example 11. Consider the univariate polynomial $f(x) = x^2 + 1$ for which $\mathcal{V}_\mathbb{R}(f) = \emptyset$ with $y = 1$ and $\alpha = (3, 2)$. For the homotopy H^a defined by (4), namely

$$H^a(x, \lambda, t) = \begin{bmatrix} x^2 + 1 - 2t \\ \lambda_0(x - 1) + 2\lambda_1 x \\ \lambda_0 + 2\lambda_1 - 3 \end{bmatrix},$$

it is easy to verify that the solution path starting at $(1, 3, 0)$ is not trackable on $(0, 1]$ due to a singularity at $t = 1/2$.

Example 12. Consider the polynomial $f(x_1, x_2) = x_1^2 + (x_1x_2 - 1)^2$ from [10] for which $\mathcal{V}_\mathbb{R}(f) = \emptyset$ with $y = (1, 1)$ and $\alpha = (3, 2)$. Clearly, $f > 0$ on $\mathbb{R}^2$, but 0 is the infimum of f on $\mathbb{R}^2$ since $\lim_{s\to\infty} f(1/s, s) = 0$. For the homotopy H^a defined by (4), it is easy to verify that the solution path starting at $(1, 1, 3, 0)$ is trackable on $(0, 1]$, but does not converge in $\mathbb{C}^2 \times \mathbb{C}^2$ as $t \to 0$.

2.3 Application to discriminants

With only a few changes, the gradient descent homotopies can be applied to parameterized polynomial systems for computing points on the real discriminant locus. For simplicity, we assume that $r : \mathbb{C}^P \to \mathbb{C}^u$ is a polynomial system with real coefficients such that $\mathcal{P} = \mathcal{V}(r)$ is an irreducible algebraic set of dimension $P - u$. Let $f : \mathbb{C}^N \times \mathcal{P} \to \mathbb{C}^N$ be a polynomial system with real coefficients such that (1) holds generically on $\mathcal{P}$. That is, there is an algebraic set $\mathcal{A} \subsetneq \mathcal{P}$ such that (1) holds for all $p \in \mathcal{P} \setminus \mathcal{A}$. Given $(y, q) \in \mathbb{R}^N \times \mathbb{R}^P$ such that $q \in \mathbb{R}^P \setminus \Delta_\mathbb{R}(f)$, we will use gradient descent homotopies with the aim of computing a point on $\Delta_\mathbb{R}(f)$ of minimal distance to q. We note that q is not assumed to be in $\mathcal{P}_\mathbb{R}$.

Let $J_x f(x, p)$ be the Jacobian matrix of f with respect to x evaluated at (x, p). Consider the polynomial system $g : \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^{N+u+1}$ having real coefficients defined by

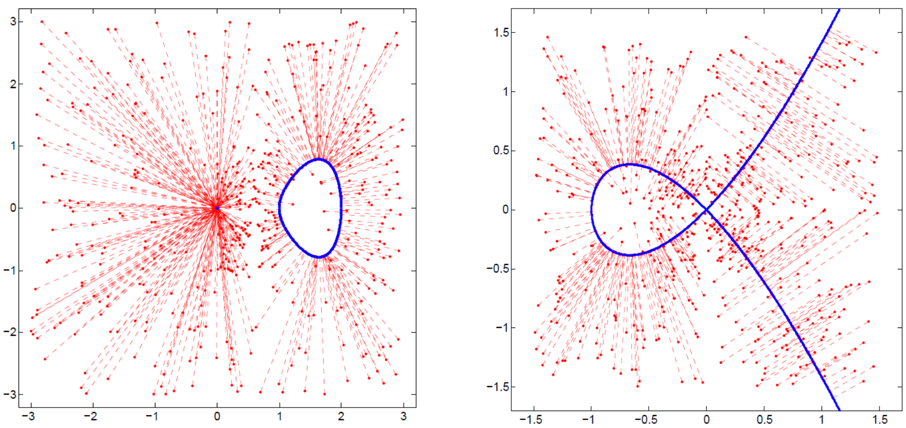
$$g(x, p) = \begin{bmatrix} f(x, p) \\ r(p) \\ \det J_x f(x, p) \end{bmatrix}. \tag{6}$$

One may attempt to use a gradient descent homotopy for g starting with (y, q). If successful, the limit point $(x^*, p^*) \in \mathcal{V}_\mathbb{R}(g)$ is a critical point of the distance function defined over $\mathbb{R}^N \times \mathbb{R}^P$ and x^* is a singular solution of $f(x, p^*) = 0$. However, since it is more natural to consider the critical points of the distance function defined over $\mathbb{R}^P$, i.e., remove the dependence on x, we simply replace

$$\lambda_0 \begin{bmatrix} x - y \\ p - q \end{bmatrix} \quad \text{with} \quad \lambda_0 \begin{bmatrix} 0 \\ p - q \end{bmatrix}$$

in the gradient descent homotopies.

Figure 3: Plot for (a) $f(x) = x_2^2 + x_1^2(x_1 - 1)(x_1 - 2)$ and (b) $g(x) = x_2^2 - x_1^2(x_1 + 1)$.



(a) (b)

Example 13. Consider $\mathcal{P} = \mathbb{C}^3$ with polynomial $f(x, p) = p_1 x^2 + p_2 x + p_3$, $y = 0.5$, and $q = (1, -3, 1)$. A gradient descent homotopy for g defined in (6), where $r(p)$ is simply removed since $\mathcal{P} = \mathbb{C}^3$, starting with (y, q) yields, to four decimal places, $x^* = 0.9172$ and $p^* = (1.4478, -2.6559, 1.2180)$. One can verify that (x^*, p^*) solves

$$\min\{\|(x, p) - (y, q)\|^2 \mid f(x, p) = 0, p \in \Delta_{\mathbb{R}}(f)\}$$

where $\|z\|^2 = z^T z$ for $z \in \mathbb{R}^4$.

After removing the dependency upon x, the gradient descent homotopy yields $x^* = 1$ and $p^* = (4/3, -8/3, 4/3)$. One can verify that p^* solves

$$\min\{\|p - q\|^2 \mid p \in \Delta_{\mathbb{R}}(f)\}$$

where $\|w\|^2 = w^T w$ for $w \in \mathbb{R}^3$.

2.4 Illustrative examples

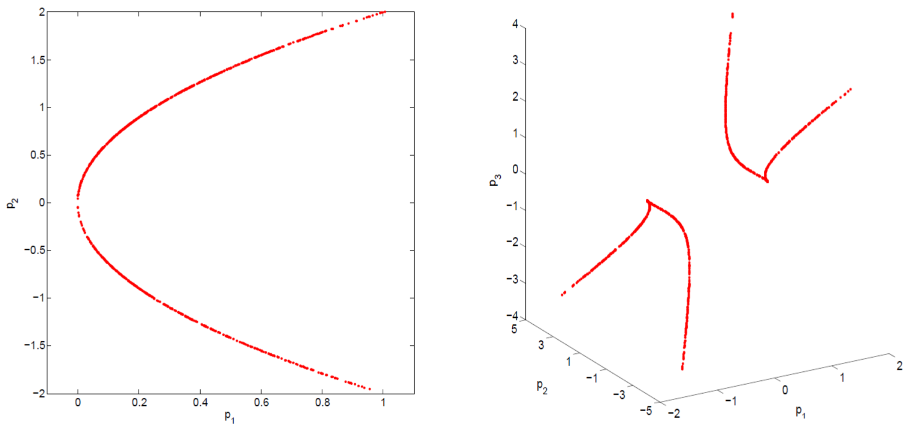
We conclude our discussion of gradient descent homotopies with illustrative examples.

Example 14. Let f be as in Example 9 and $g(x_1, x_2) = x_2^2 - x_1^2(x_1 + 1)$. Clearly, the curve $\mathcal{V}_{\mathbb{R}}(f)$ is compact while $\mathcal{V}_{\mathbb{R}}(g)$ is unbounded. Figure 3 displays $\mathcal{V}_{\mathbb{R}}(f)$ and $\mathcal{V}_{\mathbb{R}}(g)$ along with 500 random points and their corresponding endpoint on the real curves connected by a straight line computed using a gradient descent homotopy.

Example 15. For the Griewank-Osborne [9] polynomial system

$$f(x_1, x_2) = \begin{bmatrix} 29/16 x_1^3 - 2x_1 x_2 \\ x_2 - x_1^2 \end{bmatrix},$$

Figure 4: Plot of $\Delta_{\mathbb{R}}(f)$ for (a) $\mathcal{P} = \mathcal{V}_{\mathbb{C}}(p_3 - 1)$ and (b) $\mathcal{P} = \mathcal{V}(p_1^2 + p_2^2/2 - p_3^2 - 1)$.



(a) (b)

Newton's method diverges starting from every point in $\mathbb{C}^2 \setminus \{(0,0)\}$. Using a graident descent homotopy starting at the point $(1,1)$, the path converges to $(0,0)$. We note that the path starting with $(1,-1)$ is not trackable on $(0,1]$.

Example 16. Example 13 demonstrates computing points on the discriminant locus of a quadratic polynomial with $\mathcal{P} = \mathbb{C}^3$. We used gradient descent homotopies starting at 1000 random points to compute points on the real discriminant locus using the plane $\mathcal{P} = \mathcal{V}(p_3 - 1) \subset \mathbb{C}^3$ and the hyperboloid of one sheet $\mathcal{P} = \mathcal{V}(p_1^2 + p_2^2/2 - p_3^2 - 1)$. The points obtained are displayed in Figure 4.

Example 17. Let $\mathcal{P} = \mathbb{C}^2$ and consider the polynomial system $F : \mathbb{C}^2 \times \mathcal{P} \to \mathbb{C}^2$ from [5] defined by

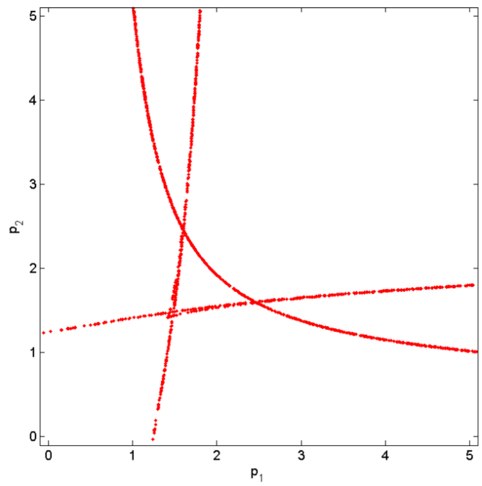
$$F(x,p) = \begin{bmatrix} x_1^6 + p_1 x_2^3 - x_2 \\ x_2^6 + p_2 x_1^3 - x_1 \end{bmatrix}.$$

The discriminant locus $\Delta(F)$ is an algebraic curve of degree 90, which was studied in [5] via $\mathcal{A}$-discriminants. Figure 5 plots the points on $\Delta_{\mathbb{R}}(F)$ computed by using gradient descent homotopies starting at 2300 random points.

3 Changing the number of real solutions

Dietmaier's approach [6], which is summarized in Section 1.4, uses a local linearization approach to move through a real parameter space changing the number of real solutions. Due to the necessary ill-conditioning near the discriminant locus, we propose incorporating a modified gradient descent homotopy into Dietmaier's approach. The modified

Figure 5: Plot of $\Delta_{\mathbb{R}}(F)$.



homotopies are presented in Section 3.1 and used in Sections 3.2 and 3.3 to increase and decrease, respectively, the number of real solutions.

Throughout this section, we assume that $\mathcal{P} = \mathbb{C}^P$ so that $\mathcal{P}_{\mathbb{R}} = \mathbb{R}^P$ and $f : \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^N$ is a polynomial system with real coefficients such that $f(x,p) = 0$ has $M_{\mathbb{C}^P} \geq 2$ solutions, all of which are nonsingular, for general parameter values $p \in \mathbb{C}^P$.

3.1 Modified gradient descent homotopies

Let $y \in \mathbb{C}^N \setminus \mathbb{R}^N$ and $q \in \mathbb{R}^P \setminus \Delta_{\mathbb{R}}(f)$ such that $f(y,q) = 0$. For the first modified gradient descent homotopy, we aim to move q to the real discriminant locus so that the solution corresponding to y becomes a real singular solution. Let $y = y_r + iy_i$ where $y_r, y_i \in \mathbb{R}^N$ and $i = \sqrt{-1}$. Consider $g, h : \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^P \to \mathbb{C}^N$ defined by

$$g(a,b,p) = \frac{f(a+ib,p) + f(a-ib,p)}{2} \quad \text{and} \quad h(a,b,p) = \frac{f(a+ib,p) - f(a-ib,p)}{2i}.$$

which are the real and imaginary parts of $f(a+ib,p)$, respectively, with

$$\begin{array}{ll} g(a,-b,p) = g(a,b,p), & g(a,0,p) = f(a,p), \\ h(a,-b,p) = -h(a,b,p), & h(a,0,p) = 0. \end{array}$$

Let $\alpha \in \mathbb{R}^{2N+1}$ and consider $H^c : \mathbb{C}^N \times \mathbb{C}^P \times \mathbb{C}^{2N+1} \times \mathbb{C} \to \mathbb{C}^{3N+P+1}$ defined by

$$H^c(x,p,\lambda,t) = \left[\begin{array}{c} g(x, t\cdot y_i, p) \\ h(x, t\cdot y_i, p) \\ \lambda_0 \left[\begin{array}{c} 0 \\ p-q \end{array} \right] + \sum_{j=1}^N \lambda_j \left[\begin{array}{c} \nabla_a g_j(x,t\cdot y_i,p)^T \\ \nabla_p g_j(x,t\cdot y_i,p)^T \end{array} \right] + \sum_{j=1}^N \lambda_{N+j} \left[\begin{array}{c} \nabla_a h_j(x,t\cdot y_i,p)^T \\ \nabla_p h_j(x,t\cdot y_i,p)^T \end{array} \right] \\ \lambda_0 + \alpha_1\lambda_1 + \cdots + \alpha_{2N}\lambda_{2N} - \alpha_0 \end{array} \right] \tag{7}$$

where $\nabla_a F(a,b,p)^T$ and $\nabla_p F(a,b,p)^T$ are the gradient vectors of a polynomial F with respect to a and p evaluated at (a,b,p), respectively, with start point $(y_r, q, \alpha_0, 0, \ldots, 0)$. Suppose that this solution path is trackable and converges to (x^*, p^*, λ^*). Since the start point and the coefficients of H^c are real, the endpoint is real, i.e., $x^* \in \mathbb{R}^N$ and $p^* \in \mathbb{R}^P$. Since x^* arises as the limit of complex solutions to a family of real polynomial systems, it immediately follows that x^* is a real singular solution of $f(x, p^*) = 0$.

Similarly, let $y_1, y_2 \in \mathbb{R}^N$ and $q \in \mathbb{R}^P \setminus \Delta_{\mathbb{R}}(f)$ such that $f(y_j, q) = 0$. For the second modified gradient descent homotopy, we aim to move q to the real discriminant locus so that the solutions corresponding to y_1 and y_2 coincide. Let $\alpha \in \mathbb{R}^{2N+1}$ and consider the homotopy $H^r : \mathbb{C}^N \times \mathbb{C}^P \times \mathbb{C}^{2N+1} \times \mathbb{C} \to \mathbb{C}^{3N+P+1}$ defined by

$$H^r(x,p,\lambda,t) = \left[\begin{array}{c} f(x,p) \\ f(x+t(y_2-y_1),p) \\ \lambda_0 \left[\begin{array}{c} 0 \\ p-q \end{array} \right] + \sum_{j=1}^N \lambda_j \left[\begin{array}{c} \nabla_x f_j(x,p)^T \\ \nabla_p f_j(x,p)^T \end{array} \right] + \sum_{j=1}^N \lambda_{N+j} \left[\begin{array}{c} \nabla_x f_j(x+t(y_2-y_1),p)^T \\ \nabla_p f_j(x+t(y_2-y_1),p)^T \end{array} \right] \\ \lambda_0 + \alpha_1\lambda_1 + \cdots + \alpha_{2N}\lambda_{2N} - \alpha_0 \end{array} \right] \tag{8}$$

with start point $(y_1, q, \alpha_0, 0, \ldots, 0)$. Suppose that this solution path is trackable and converges to (x^*, p^*, λ^*). As in the H^c case, since the start point and the coefficients of H^r are real, the limit point is real, i.e., $x^* \in \mathbb{R}^N$ and $p^* \in \mathbb{R}^P$. Since x^* arises as the limit of two real solutions to a family of real polynomial systems, it immediately follows that x^* is a real singular solution of $f(x, p^*) = 0$.

3.2 Increase the number of real solutions

Given $y \in \mathbb{C}^N \setminus \mathbb{R}^N$ and $q \in \mathbb{R}^P \setminus \Delta_{\mathbb{R}}(f)$ such that $f(y, q) = 0$, the first part of our two-part approach for attempting to increase the number of real solutions is outlined in the following summary.

1. Attempt to use H^c defined by (7) to yield a new parameter value such that the corresponding polynomial system has a real singular solution.
   (a) If successful, compute all solutions for the new parameter value.
   (b) If the real solutions have persisted, update the parameter values and solutions, and terminate the process.
2. If using H^c was not successful or some real solutions have disappeared, compute Δq using the approach of Dietmaier [6] attempting to move y closer to $\mathbb{R}^N$.
   (a) If Δq can not be computed, the process is terminated. Otherwise, compute all solutions for the new parameter value.
   (b) If the real solutions have persisted, update the parameter values and the solutions, and return to Item 1. Otherwise, return to Item 2 computing a shorter Δq.

If this process terminates, it has either computed a parameter $p^* \in \Delta_{\mathbb{R}}(f)$ and a point $x^* \in \mathbb{R}^N$ that is a singular solution of $f(x, p^*) = 0$, or has failed. If it has failed, the approach is repeated starting with a different nonreal solution to $f(x, q) = 0$. If all nonreal solutions of $f(x, q) = 0$ fail, the procedure has failed. One could then restart the procedure after picking another value of q and a nonreal solution of $f(x, q) = 0$.

If the first part has succeeded, the second part attempts to produce new distinct real solutions and is outlined in the following summary.

1. Use points along the solution path tracked using H^c to compute a unit vector $v \in \mathbb{R}^N$ which is approximately tangent to the projection of path into the parameter space at $t = 0$ and points in the direction of the path as t decreases.

2. Compute Δp^* to be a nonzero vector in the direction of v.

3. Compute all solutions of $f(x, p^* + \Delta p^*) = 0$.

   (a) If the number of real solutions has increased, use the approach of Dietmaier [6] to attempt to increase the distance between the real solutions and then terminate the process. Otherwise, return to Item 2 and compute a shorter Δp^*.

If successful, the two-part process has passed through the discriminant locus in such a way to increase the number of real solutions.

3.3 Decrease the number of real solutions

The process for attempting to decrease the number of real solutions follows the same basic setup as the process for attempting to increase the number of real solutions in Section 3.2, with only a few minor changes.

The first part of the two-part process starts with two real solutions, y_1 and y_2, and uses the homotopy H^r defined in (8). The persistence of the real solutions in Items 1b and 2b is replaced with the persistence of nonreal solutions. Also, the local linearization in Item 2 is setup to minimize the distance between y_1 and y_2. Upon failure, this part is repeated using a new pair of real solutions.

For the second part, we replace H^c with H^r in Item 1. The only other change occurs in Item 3a where, if Δp^* has been computed such that the number of nonreal solutions has increased, we use the approach of Dietmaier [6] to attempt to increase the distance between the nonreal solutions and $\mathbb{R}^N$.

3.4 Illustrative example

As an illustration, reconsider the polynomial $f : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}$ from Example 2, namely

$$f(x, p) = x^2 + p_1 x + p_2.$$

We first consider increasing the number real solutions starting with $y = -1 + 2i$ and $q = (2, 5)$. For $\alpha = (2, -5, 7)$, the homotopy H^c defined by (7) yields $x^* = -2$ and $p^* = (4, 4)$. The first part was successful since x^* is a real singular solution of $f(x, p^*) = 0$.

In the second part, we took, to three decimal places, $v = (0.970, -0.243)$, which is the unit vector pointing in the direction of the vector between the parameter values at $t = 10^{-5}$ and p^*. Taking $\Delta p^* = 10^{-4}v$, the equation $f(x, p^* + \Delta p^*) = 0$ has two real solutions separated by a distance of 0.030.

We now consider decreasing the number of real solutions starting with $y_1 = -1$, $y_2 = 3$, and $q = (-2, -3)$. For $\alpha = (2, -5, 7)$, the homotopy H^r defined by (8) yields, to three decimal places, $x^* = -0.388$ and $p^* = (-0.777, 0.151)$. The first part was successful since x^* is a real singular solution of $f(x, p^*) = 0$.

In the second part, we took, to three decimal places, $v = (0.149, 0.989)$, which is the unit vector pointing in the direction of the vector between the parameter values at $t = 10^{-5}$ and p^*. Taking $\Delta p^* = 10^{-4}v$, the equation $f(x, p^* + \Delta p^*) = 0$ has two complex solutions separated by a distance of 0.021.

4 Real enumerative geometry: points, lines, and conics

The approach presented in Section 3 can be used to develop new results in real enumerative geometry. To demonstrate, consider the geometric problem of computing the plane conics in $\mathbb{C}^3$ which meet k given points and $8 - 2k$ given lines for $k = 0, 1, 2$. Table 1 shows the number of plane conics N_k when the k points and $8 - 2k$ lines are in general position.

Table 1: Number of plane conics

k	2	1	0
N_k	4	18	92

By solving random instances of these problems, data is presented in [13] showing that every possible number of real solutions can be achieved except

- 18 nonreal solutions for $k = 1$, and
- 92 real solutions for $k = 0$.

The following sections summarize using the approach presented in Section 3 to reproduce some of these results as well as complete the $k = 0$ case. That is, this approach has indeed computed eight lines in $\mathbb{R}^3$ with 92 real plane conics meeting them.

The following computations used `Bertini` [3] to perform the the path tracking and updating of solutions, MATLAB to solve the linear programs, and `alphaCertified` [14] to certify the number of real solutions using rational arithmetic. We note that all numbers provided in the following sections are exact. The website of the last author contains additional details regarding the computations described below.

4.1 Two points and four lines

We started with two randomly selected points $(-0.506, 1.57, -6.01)$ and $(0.725, 0.604, 2.84)$, and four randomly selected lines $L_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$ where

$$\begin{array}{ll} p_1 = (-0.297, 0.164, -0.846) & v_1 = (1.52, 1.69, -0.767) \\ p_2 = (-0.972, -1.32, 2.34) & v_2 = (-1.88, 69.9, 1.03) \\ p_3 = (0.475, -0.368, -0.863) & v_3 = (-0.696, 0.0421, 1.35) \\ p_4 = (1.1, 0.67, 0.902) & v_4 = (54.3, 0.0202, -3.24). \end{array}$$

There are four nonreal plane conics passing through these two points and meeting these four lines. The following summarizes using the approach of Section 3 to systematically increase the number of real solutions.

Since there are four nonreal plane conics, the first goal was to find two points and four lines which describe two real and two nonreal plane conics. This was accomplished using the approach of Section 3 which yielded the two points $(-0.469, 1.53, -6.05)$ and $(0.738, 0.594, 2.8)$, and the four lines L_i defined by

$$\begin{array}{ll} p_1 = (-0.334, 0.201, -0.809) & v_1 = (1.48, 1.73, -0.793) \\ p_2 = (-1.01, -1.36, 2.3) & v_2 = (-1.85, 69.9, 1.03) \\ p_3 = (0.482, -0.405, -0.853) & v_3 = (-0.722, 0.00478, 1.32) \\ p_4 = (1.14, 0.644, 0.939) & v_4 = (54.3, 0.0462, -3.28). \end{array}$$

The next goal was to force the two remaining nonreal solutions to merge and then become two distinct real solutions. The two points $(-0.416, 1.48, -6.1)$ and $(0.756, 0.54, 2.76)$, and the four lines L_i defined by

$$\begin{array}{ll} p_1 = (-0.362, 0.251, -0.756) & v_1 = (1.47, 1.74, -0.772) \\ p_2 = (-0.963, -1.31, 2.33) & v_2 = (-1.86, 69.9, 1.02) \\ p_3 = (0.429, -0.459, -0.906) & v_3 = (-0.776, -0.049, 1.27) \\ p_4 = (1.17, 0.641, 0.979) & v_4 = (54.2, 0.5006, -3.32) \end{array}$$

were computed by the approach of Section 3. It is easy to verify that there are four real plane conics passing through these two points and meeting these four lines.

4.2 One point and six lines

We started with the randomly selected point $(-1.01, -0.011, 0.01)$ and six randomly selected lines $L_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$ where

$$\begin{array}{ll} p_1 = (1.6, 0.136, -4.43) & v_1 = (2.99, -2.16, 2.05) \\ p_2 = (-2.59, 4.38, 4.43) & v_2 = (-3.63, -0.01, -4.24) \\ p_3 = (1.1, 2.67, 0.9) & v_3 = (-1.38, 2.47, -2.13) \\ p_4 = (-4.77, 1.52, -1.31) & v_4 = (1.51, 1.1, 1.42) \\ p_5 = (0.204, 0.85, 1.44) & v_5 = (-1.69, -1.37, 0.987) \\ p_6 = (-4.6, -1.01, 0.841) & v_6 = (-2.01, 0.755, -0.436). \end{array}$$

There are 4 real and 14 nonreal plane conics passing through this point and meeting these six lines. The approach of Section 3 was able to decrease the number of real solutions down to 2, but failed when trying to remove the remaining two real solutions. This computation is consistent with the results presented in Table 5 of [13]. In particular, for the point $(-1.01, 0.001, -0.001)$ and the six lines L_i defined by

$$\begin{array}{ll} p_1 = (1.59, 0.126, -4.44) & v_1 = (3.001, -2.16, 2.04) \\ p_2 = (-2.57, 4.39, 4.41) & v_2 = (-3.61, 0.0092, -4.26) \\ p_3 = (1.12, 2.65, 0.883) & v_3 = (-1.4, 2.49, -2.11) \\ p_4 = (-4.78, 1.51, -1.3) & v_4 = (1.49, 1.09, 1.44) \\ p_5 = (0.215, 0.839, 1.43) & v_5 = (-1.68, -1.36, 0.996) \\ p_6 = (-4.38, -1.01, 0.851) & v_6 = (-1.99, 0.745, -0.446), \end{array}$$

there are 2 real and 16 nonreal plane conics passing through this point and meeting these lines.

4.3 Eight lines

We started with the eight randomly selected lines $L_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$ where

$$\begin{array}{ll} p_1 = (0.4096, -3.903, -2.287) & v_1 = (3.222, 1.433, -0.3969) \\ p_2 = (3.027, 0.5909, 3.208) & v_2 = (0.3143, 1.228, 4.478) \\ p_3 = (0.2573, 0.9133, 0.6372) & v_3 = (-3.261, -1.43, 1.502) \\ p_4 = (-4.276, 3.802, -1.097) & v_4 = (-0.926, 3.681, -2.706) \\ p_5 = (1.505, -0.7109, -4.401) & v_5 = (-3.741, 4.067, -2.589) \\ p_6 = (0.6752, -4.367, -2.557) & v_6 = (1.009, 1.772, -1.654) \\ p_7 = (-3.625, 3.66, 1.698) & v_7 = (-4.53, 1.966, 1.868) \\ p_8 = (1.444, 3.607, 0.5243) & v_8 = (-3.045, -2.643, -0.7563) \end{array}$$

for which there are 82 real and 10 nonreal plane conics meeting these eight lines. The approach of Section 3 systematically increased the number of real solutions up to 92. The following theorem is a restatement of Theorem 1 which includes the eight lines.

Theorem 18. *There are 92 real plane conics meeting the* 8 *lines* $L_i = \{p_i + tv_i \mid t \in \mathbb{C}\}$ *defined by*

$$\begin{array}{ll} p_1 = (0.46978, -3.988, -2.3527) & v_1 = (2.9137, 1.546, -0.27448) \\ p_2 = (3.19, 0.5752, 3.0953) & v_2 = (0.56569, 1.108, 4.3629) \\ p_3 = (0.40308, 0.78659, 0.9053) & v_3 = (-3.0656, -1.4638, 1.4096) \\ p_4 = (-4.3743, 4.0046, -1.0243) & v_4 = (-0.9163, 3.6495, -2.6528) \\ p_5 = (1.5198, -0.86125, -4.5963) & v_5 = (-3.8418, 3.9541, -2.5494) \\ p_6 = (0.46801, -4.0308, -2.4411) & v_6 = (1.0225, 1.6422, -1.5925) \\ p_7 = (-3.3382, 3.8432, 1.693) & v_7 = (-4.4657, 1.9618, 1.6865) \\ p_8 = (1.3536, 3.6311, 0.42864) & v_8 = (-3.1442, -2.4915, -0.63586). \end{array}$$

Proof. For $i = 1, \dots, 8$, let $x_i(t)$, $y_i(t)$, and $z_i(t)$ be the first, second, and third coordinates of $p_i + tv_i$, respectively. Consider the polynomial system $F : \mathbb{C}^{16} \to \mathbb{C}^{16}$ defined by

$$F(t_1, \dots, t_8, c_1, \dots, c_5, a_1, a_2, a_3) = \begin{bmatrix} a_1 x_i(t_i) + a_2 y_i(t_i) + a_3 - z_i(t_i), & i = 1, \dots, 8 \\ C(x_i(t_i), y_i(t_i), c_1, \dots, c_5), & i = 1, \dots, 8 \end{bmatrix}$$

where $C(x, y, c_1, \dots, c_5) = c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x + c_5 y - 1$. In particular, if $(t, c, a) \in \mathcal{V}(F)$ where $t = (t_1, \dots, t_8)$, $c = (c_1, \dots, c_5)$, and $a = (a_1, a_2, a_3)$, then the plane conic

$$\mathcal{C}_{a,c} = \{(x, y, a_1 x + a_2 y + a_3) \in \mathbb{C}^3 \mid c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x + c_5 y = 1\}$$

passes through the point $p_i + t_i v_i \in L_i \subset \mathbb{C}^3$ for $i = 1, \dots, 8$.

Suppose that $(t, c, a), (\tilde{t}, \tilde{c}, \tilde{a}) \in \mathcal{V}(F)$ such that $\mathcal{C}_{a,c} = \mathcal{C}_{\tilde{a},\tilde{c}}$, and define $P_a = \mathcal{V}(a_1 x + a_2 y + a_3 - z)$ and $P_{\tilde{a}} = \mathcal{V}(\tilde{a}_1 x + \tilde{a}_2 y + \tilde{a}_3 - z)$. If $P_a \neq P_{\tilde{a}}$, then $\mathcal{C}_{a,c}$ is contained in the line $P_a \cap P_{\tilde{a}}$. However, since one can easily verify that no line meets the eight lines L_i, we must have $P_a = P_{\tilde{a}}$ which immediately implies that $a = \tilde{a}$. This shows that if $(t, c, a), (\tilde{t}, \tilde{c}, \tilde{a}) \in \mathcal{V}(F)$ such that $a \neq \tilde{a}$, then $\mathcal{C}_{a,c}$ and $\mathcal{C}_{\tilde{a},\tilde{c}}$ are distinct plane conics meeting the eight lines L_i.

We used `Bertini` [3] to compute a set of 92 points $X \subset \mathbb{C}^{16}$, represented using floating point, such that each point in X is heuristically within 10^{-50} of a point in $\mathcal{V}(F)$. Since F is a square system with rational coefficients, after approximating the coordinates of the points in X using rational numbers, the results of [13] based on α-theory [4, 20] implemented in `alphaCertified` [14] using exact rational arithmetic proved the following statements.

- Newton's method with respect to F starting at each point in X quadratically converges to a point in $\mathcal{V}(F)$.
- If $Z \subset \mathcal{V}(F)$ is the set of points which arise as the limit of Newton's method with respect to F starting at some point in X, then Z consists of 92 distinct real points.

This computation also proved that the maximum distance from each point in X to the corresponding point in $Z \subset \mathcal{V}(F)$ is bounded above by $3 \cdot 10^{-53}$. Since, for distinct $(t, c, a), (\tilde{t}, \tilde{c}, \tilde{a}) \in X$, we have $\|a - \tilde{a}\| \geq 0.01$, the triangle inequality with these two bounds show that the 92 points in Z indeed correspond to 92 distinct real conics passing through the eight given lines L_i. □

Theorem 18 together with Table 6 of [13] shows that, for $\ell = 0, 2, 4, \dots, 92$, there exists 8 real lines such that there are ℓ real and $92 - \ell$ nonreal plane conics meeting them.

Acknowledgments

Both authors would like to thank Frank Sottile for his help and support. The second author would also like to thank Chris Aholt and Rehka Thomas for related discussions.

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