Algebraic computations using Macaulay dual spaces

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Abstract

The algebraic operations of addition, intersection, elimination, and quotient are fundamental to computational algebraic geometry. This article describes how to perform these operations on homogeneous ideals using Macaulay dual spaces. If F is a polynomial system with finitely many solutions, these operations are used to compute the homogenization of the ideal generated by F which, in particular, yields the number of solutions, counting multiplicity, of F. These computations can be performed either using exact or floating point arithmetic and are naturally parallelizable.
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1 Introduction

If $I, J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ are homogeneous ideals and $1 \le \ell \le N$, four basic algebraic computations are $I+J$, $I\cap J$, $I\cap\mathbb{C}[x_0, x_1, \ldots, x_\ell]$, and $I : J$. When the coefficients of the generators of I and J are known exactly, standard methods based on Gröbner bases can be used to perform these computations. If the generators are only known approximately, which is common in many applications, additional steps are needed in order to maintain numerical integrity when performing these algebraic computations [11, 13].

This article uses Macaulay dual spaces, which are the modern form of inverse systems [7], to perform these algebraic computations. This approach is amenable to numerical approximations. In particular, this article computes $(I + J)_d$, $(I \cap J)_d$, $(I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_d$, and $(I : J)_d$ using dual

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spaces where K_d is the vector space of the homogeneous elements of degree d in K. Under mild assumptions, these algorithms are polynomial in N.

To compute the solutions to a polynomial system $F \subset \mathbb{C}[x_1, \ldots, x_N]$ using numerical algebraic geometry, one typically homogenizes each polynomial in F and solves the resulting homogenized system on a random coordinate patch. This is done to avoid tracking solution paths which have infinite length [8]. By saturating this resulting homogenized system with respect to the homogenizing variable, the number of solutions of F, counting multiplicity, can be computed. Saturation is performed by repeatedly computing quotient ideals. The ascending chain condition yields that this sequence of quotient ideals can only have finitely many strict containments. A dual space termination criterion is presented to determine when the saturation has been computed.

The rest of this section provides the necessary background information regarding algebra and Macaulay dual spaces. Section 2 describes the computation of $(I+J)_d$, $(I\cap J)_d$, $(I\cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_d$, and $(I : J)_d$ using dual spaces while Section 3 describes an algorithm which counts the number of solutions of zero-dimensional polynomial systems. Section 4 presents examples to demonstrate the algorithms developed in this article.

1.1 Algebra overview

This section, which follows [5], presents an overview of the required concepts from algebra. The following definition describes common operations performed on ideals.

Definition 1. *Let $I, J \subset \mathbb{C}[x_1, \ldots, x_N]$ be ideals and $f \in \mathbb{C}[x_1, \ldots, x_N]$. Define the following ideals*

1. *(sum)* $I + J = \{a + b \mid a \in I, b \in J\}$,
2. *(intersection)* $I \cap J = \{a \mid a \in I, a \in J\}$,
3. *(product)* $IJ = \left\{ \sum_{i=1}^k a_i b_i \;\middle|\; a_i \in I, b_i \in J, k > 0 \right\}$,
4. *(quotient)* $I : f = \{a \mid af \in I\}$,
5. *(quotient)* $I : J = \{a \mid aJ \subset I\}$,
6. *(saturation)* $I : f^\infty = \{a \mid af^m \in I \text{ for some } m > 0\}$, *and*
7. *(saturation)* $I : J^\infty = \{a \mid aJ^m \subset I \text{ for some } m > 0\}$.

The following proposition describes one way to compute quotients and saturations.

Proposition 2. *Let $I, J \subset \mathbb{C}[x_1, \ldots, x_N]$ be ideals.*

1. *If $J = \langle g_1, \ldots, g_k \rangle$, then $I : J = \bigcap_{i=1}^{k} (I : g_i)$.*
2. *For any $\ell > 0$, $I : J^{\ell+1} = (I : J^{\ell}) : J$.*
3. *There exists $p > 0$ such that $I : J^p = I : J^{\ell}$ for all $\ell \geq p$.*
4. *For any $p \geq 0$, $I : J^p = I : J^{p+1}$ if and only if $I : J^p = I : J^{\infty}$.*

This proposition is the basis of an algorithm for computing saturations. In particular, Items 3 and 4 show that such an algorithm must terminate and provide a stopping criterion.

Let $k \geq 0$ and $\mathbb{C}[x_1, \ldots, x_N]_{\leq k}$ be the vector space of polynomials of degree at most k. For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$, define

$$I_{\leq k} = I \cap \mathbb{C}[x_1, \ldots, x_N]_{\leq k}.$$

The *affine Hilbert function* of I is the function $H_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$H_I(k) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_N]_{\leq k} - \dim_{\mathbb{C}} I_{\leq k}.$$

Similarly, denote $\mathbb{C}[x_0, x_1, \ldots, x_N]_k$ as the space of homogeneous polynomials of degree k. For a homogeneous ideal $J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$, define

$$J_k = J \cap \mathbb{C}[x_0, x_1, \ldots, x_N]_k.$$

The *projective Hilbert function* of J is the function $H_J^p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$H_J^p(k) = \dim_{\mathbb{C}} \mathbb{C}[x_0, x_1, \ldots, x_N]_k - \dim_{\mathbb{C}} J_k.$$

The following proposition relates the Hilbert function of two ideals.

Proposition 3. *If $I, J \subset \mathbb{C}[x_1, \ldots, x_N]$ are ideals and $k \geq 0$,*

$$H_{I \cap J}(k) = H_I(k) + H_J(k) - H_{I+J}(k).$$

Similarly, if $I, J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ are homogeneous ideals and $k \geq 0$,

$$H_{I \cap J}^p(k) = H_I^p(k) + H_J^p(k) - H_{I+J}^p(k).$$

If $I \subset \mathbb{C}[x_1, \ldots, x_N]$ is an ideal and $J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a homogeneous ideal, the functions H_I and H_J^p eventually become a polynomial. That is, there exists polynomials HP_I and HP_J^p, called the *affine Hilbert polynomial* of I and the *projective Hilbert polynomial* of J, respectively, such that $H_I(k) = HP_I(k)$ and $H_J^p(k) = HP_J^p(k)$ for $k \gg 0$. The *index of regularity* is the minimum $k_0 \geq 0$ such that the Hilbert function and Hilbert polynomial agree for all $k \geq k_0$.

If I is a zero-dimensional ideal and k_0 is the index of regularity, then $H_I(k_0) = H_I(k) = HP_I(k)$ for all $k \geq k_0$. In particular, the Hilbert polynomial is constant and is equal to the number of solutions of I, counting multiplicity. Conversely, if there exists $k \geq 0$ such that $H_I(k) = H_I(k+1)$, then I is a zero-dimensional ideal with $H_I(k) = H_I(\ell) = HP_I(\ell)$ for all $\ell \geq k$. Similarly statements hold for homogeneous ideals as well. The following proposition relates the index of regularity to generators of the ideal.

Proposition 4.

1. *If $I \subset \mathbb{C}[x_1, \ldots, x_N]$ is a zero-dimensional ideal and r is the index of regularity of I, then $\langle I_{\leq r+1} \rangle = I$.*

2. *If $J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a zero-dimensional homogeneous ideal and r is the index of regularity of J, then $\langle J_{\leq r+1} \rangle = J$.*

There is a natural relationship between ideals on $\mathbb{C}[x_1, \ldots, x_N]$ and homogeneous ideals on $\mathbb{C}[x_0, x_1, \ldots, x_N]$. The following definition and proposition emphasize the relevant parts of this relationship.

Definition 5. *Let $f \in \mathbb{C}[x_1, \ldots, x_N]$ be a polynomial of degree d. The homogenization of f is the polynomial $f^h \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ where*

$$f^h(x_0, x_1, \ldots, x_N) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_N}{x_0}\right).$$

The homogenization of an ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$ is the homogeneous ideal $I^h \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ where

$$I^h = \langle f^h \mid f \in I \rangle.$$

Proposition 6. *For any $k \geq 0$, $H_I(k) = H_{I^h}^p(k)$.*

In general, the ideal generated by homogenizing the generators of I may be smaller than I^h. That is, if $I = \langle f_1, \ldots, f_n \rangle$, it is possible to have

$$\langle f_1^h, \ldots, f_n^h \rangle \subsetneq I^h.$$

The following proposition shows I^h can be computed via saturations.

Proposition 7. *Let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$ be an ideal. If $J = \langle f_1^h, \ldots, f_n^h \rangle \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$, then $I^h = J : x_0^\infty$.*

1.2 Macaulay dual space overview

Following the notation of [1, 6], for $\alpha \in (\mathbb{Z}_{\geq 0})^N$, define

$$|\alpha| = \alpha_1 + \cdots + \alpha_N, \quad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_N!, \quad \text{and} \quad \partial_\alpha = \tfrac{1}{\alpha!}\tfrac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

For $y \in \mathbb{C}^N$ and $g \in \mathbb{C}[x_1, \ldots, x_N]$, the differential functional $\partial_\alpha[y]$ is defined by

$$\partial_\alpha[y](g) = (\partial_\alpha g)(y).$$

When it is clear from the context, $\partial_\alpha[y]$ may be written as ∂_α.

For $y \in \mathbb{C}^N$, define $D_y = \operatorname{span}_{\mathbb{C}} \left\{\partial_\alpha[y] \mid \alpha \in (\mathbb{Z}_{\geq 0})^N\right\}$ as the vector space of differential functionals at y. If $I \subset \mathbb{C}[x_1, \ldots, x_N]$ is an ideal, the *Macaulay dual space*, or simply *dual space*, of I at y is the set of all differential functionals at y that vanish on I, namely

$$D_y(I) = \{\partial \in D_y \mid \partial(g) = 0 \text{ for all } g \in I\}. \tag{1}$$

A *dual basis* is a subset of a dual space that forms a $\mathbb{C}$-basis.

For $j = 1, \ldots, N$, Stetter and Thallinger [12, 15] construct $\Phi_j : D_y \to D_y$ to be the linear operator defined by

$$\Phi_j(\partial_\alpha) = \begin{cases} 0 & \text{if } \alpha_j = 0, \\ \partial_{\alpha - e_j} & \text{otherwise,} \end{cases} \tag{2}$$

where e_j is the j^{th} standard basis vector. The following proposition uses these linear operators to compute dual spaces.

Proposition 8. *Let $I = \langle f_1, \ldots, f_n\rangle$ be an ideal in $\mathbb{C}[x_1, \ldots, x_N]$, $y \in \mathbb{C}^N$, and $\partial \in D_y$. Then, $\partial \in D_y(I)$ if and only if $\partial(f_i) = 0$ for $i = 1, \ldots, n$ and $\Phi_j(\partial) \in D_y(I)$ for $j = 1, \ldots, N$.*

The dual space can be used for a basic ideal membership test. We state this basic membership test for dual spaces of homogeneous ideals at $0 \in \mathbb{C}^{N+1}$ which will be used to generate a *truncated* membership in Lemma 11.

Proposition 9. *If $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a homogeneous ideal and $f \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a polynomial, then $f \in I$ if and only if $\partial(f) = 0$ for all $\partial \in D_0(I)$.*

We can extend the natural grading of homogeneous polynomials to dual spaces. Let $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal and, for $j \geq 0$, define

$$D_0^j = \operatorname{span}_{\mathbb{C}}\{\partial_\alpha[0] \mid \alpha \in (\mathbb{Z}_{\geq 0})^{N+1}, |\alpha| = j\} \quad \text{and} \quad D_0^j(I) = D_0^j \cap D_0(I).$$

The vector space $D_0^j(I)$ is called the j^{th} order dual space of I. The following lemma shows that the dual space of homogeneous ideals is generated by the union of the j^{th} order dual spaces.

Lemma 10. *If $I \subset \mathbb{C}[x_0, x_1 \ldots, x_N]$ is a homogeneous ideal, then*

$$D_0(I) = \mathrm{span}_{\mathbb{C}} \left\{ \bigcup_{j=0}^{\infty} D_0^j(I) \right\}.$$

Proof. Let $\partial \in D_0(I)$ and suppose that $f \in I$ is a polynomial of degree d. We can write $\partial = \sum_{j=0}^{\infty} \partial_j$ where each $\partial_j \in D_0^j$ and $f = \sum_{j=0}^{d} f_j$ where each $f_j \in \mathbb{C}[x_0, x_1, \ldots, x_N]_j$. Since I is homogeneous, each $f_j \in I$. For any $k \geq 0$, homogeneity yields

$$\partial_k(f) = \partial_k(f_k) = \partial(f_k) = 0.$$

This shows that each $\partial_j \in D_0^j(I)$ and so $D_0(I) \subset \mathrm{span}_{\mathbb{C}} \left\{ \bigcup_{j=0}^{\infty} D_0^j(I) \right\}$. The other inclusion is trivial. $\square$

Dual spaces of homogeneous ideals yield a truncated membership test.

Lemma 11. *Let $I \subset \mathbb{C}[x_0, x_1 \ldots, x_N]$ be a homogeneous ideal and $f \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous polynomial of degree d. Then, $f \in I$ if and only if $\partial(f) = 0$ for all $\partial \in D_0^d(I)$.*

Proof. Assume that $\partial(f) = 0$ for all $\partial \in D_0^d(I)$ and let $\delta \in D_0(I)$. By Lemma 10, we can write $\delta = \sum_{j=0}^{\infty} \delta_j$ where $\delta_j \in D_0^j(I)$. Thus, $\delta_d(f) = 0$ and, for $j \neq d$, we trivially have $\delta_j(f) = 0$. Hence, $\delta(f) = 0$ and so $f \in I$ by Proposition 9. The other direction is trivial. $\square$

For a homogeneous ideal $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $k \geq 0$, define

$$Ann_k(D_0^k(I)) = \{f \in \mathbb{C}[x_0, x_1, \ldots, x_N]_k \mid \partial(f) = 0 \text{ for all } \partial \in D_0^k(I)\}.$$

Lemma 11 shows that $I_k = Ann_k(D_0^k(I))$ which immediately yields

$$H_I^p(k) = \dim_{\mathbb{C}} D_0^k(I). \tag{3}$$

Given $D \subset D_0^k$, the following algorithm computes a $\mathbb{C}$-basis for

$$Ann_k(\mathrm{span}_{\mathbb{C}} D) \subset \mathbb{C}[x_0, x_1, \ldots, x_N]_k.$$

Procedure $F = $ **Annihilator**(k, N, D)

Input Integers $k, N \geq 0$ and a set $D \subset D_0^k$.

Output A $\mathbb{C}$-basis for $Ann_k(\mathrm{span}_{\mathbb{C}} D) \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$.

Begin
1. Let $m := \binom{N+k}{k}$, and $\{\alpha_1, \ldots, \alpha_m\} = \{\beta \in (\mathbb{Z}_{\geq 0})^{N+1} \mid |\beta| = k\}$.
2. Let $d := \dim \mathrm{span}_{\mathbb{C}} D$, $\partial_1, \ldots, \partial_d$ be a basis for $\mathrm{span}_{\mathbb{C}} D$, and write $\partial_i = \sum_{j=1}^m a_{i,j} \partial_{\alpha_j}$.
3. Construct $A \in \mathbb{C}^{d \times m}$ where $A_{i,j} = a_{i,j}$.
4. Let $v_1, \ldots, v_{m-d}$ be a basis for null A.
5. For $i = 1, \ldots, m-d$, define $f_i := \sum_{j=1}^m v_{i,j} x^{\alpha_j}$.

Return $F := \{f_1, \ldots, f_{m-d}\}$.

Remark 12. *Since $\partial_1, \ldots, \partial_d$ are linearly independent and $d \leq m$, A is a rank d matrix. In particular, the null space of A has dimension $m - d$.*

Proposition 8 implicitly defines a vector space called the *closedness subspace* of I, namely

$$C_0(I) = \{\partial \in D_0 \mid \Phi_j(\partial) \in D_0(I) \text{ for } j = 0, \ldots, N\}.$$

For $j \geq 0$, the vector space $C_0^j(I) = C_0(I) \cap D_0^j$ is called the j^{th} order closedness subspace of I.

Let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal. Let $d_i = \deg f_i$, and upon reordering, we can assume that $d_1 \leq d_2 \leq \cdots \leq d_n$. For any $j \geq 0$, Proposition 8 yields

$$\begin{aligned} D_0^j(I) &= C_0^j(I) \cap \{\partial \in D_0^j \mid \partial(f_i) = 0 \text{ for } i = 1, \ldots, n\} \\ &= C_0^j(I) \cap \{\partial \in D_0^j \mid \partial(f_i) = 0 \text{ if } d_i = j\}. \end{aligned}$$

For $j < d_1$, we have $D_0^j(I) = D_0^j = C_0^j(I)$ and, for $j > d_n$, we have $D_0^j(I) = C_0^j(I)$. Under the assumption that $\min \emptyset = \max \emptyset = 0$, define

$$\begin{aligned} d^{\min}(I) &= \min\{j \geq 0 \mid D_0^j(I) \subsetneq C_0^j(I)\} \quad \text{and} \\ d^{\max}(I) &= \max\{j \geq 0 \mid D_0^j(I) \subsetneq C_0^j(I)\}. \end{aligned}$$

Then, I is minimally generated by homogeneous polynomials of degree at least $d^{\min}(I)$ and at most $d^{\max}(I)$. In particular, in the nondegenerate case, namely $n \geq 1$ and $d_1 > 0$, $d^{\min}(I) = d_1$ and $d^{\max}(I) \leq d_n$.

This yields the following representation for homogeneous ideals.

Definition 13. *Let I be a homogeneous ideal. For $d^{\min}(I) \leq k \leq d^{\max}(I)$, let D_k be a basis of $D_0^k(I)$. A* dual basis representation *for I is*

$$I_D = \left\{ d^{\min}(I), d^{\max}(I), D_{d^{\min}(I)}, \ldots, D_{d^{\max}(I)} \right\}. \tag{4}$$

If I and J are homogeneous ideals, it is easy to verify that $d^{\max}(I+J) = \max\{d^{\max}(I), d^{\max}(J)\}$. Unfortunately, the degree of the largest generator for $I \cap J$ and $I : J$ is not as well behaved.

2 Operations on dual spaces

2.1 Inclusion, sums, and intersections

The following lemma describes the relationships of the dual spaces under inclusion, sums, and intersections of homogeneous ideals. By Lemma 10, it is enough to describe the relationship using j^{th} order dual spaces.

Lemma 14. *Let $I, J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be homogeneous ideals.*

1. *$I \subset J$ if and only if, for every $j \geq 0$, $D_0^j(I) \supset D_0^j(J)$.*
2. *$D_0^j(I+J) = D_0^j(I) \cap D_0^j(J)$ for every $j \geq 0$.*
3. *$D_0^j(I \cap J) = D_0^j(I) + D_0^j(J)$ for every $j \geq 0$.*

Proof. Item 1 follows from Lemma 11 and Item 2 follows from Item 1.

For Item 3, since $I \cap J \subset I, J$, we have $D_0^j(I) + D_0^j(J) \subset D_0^j(I \cap J)$. Since

$$\begin{aligned}
\dim_{\mathbb{C}} D_0^j(I \cap J) &= H^p_{I \cap J}(j) \\
&= H^p_I(j) + H^p_J(j) - H^p_{I+J}(j) \\
&= \dim_{\mathbb{C}} D_0^j(I) + \dim_{\mathbb{C}} D_0^j(J) - \dim_{\mathbb{C}} \left(D_0^j(I) \cap D_0^j(J) \right) \\
&= \dim_{\mathbb{C}} \left(D_0^j(I) + D_0^j(J) \right),
\end{aligned}$$

we know that $D_0^j(I) + D_0^j(J) = D_0^j(I \cap J)$. $\square$

2.2 Elimination

Elimination is a basic operation in algebra which is used in many algorithms in computational algebra such as solving and implicitization algorithms.

Given a homogeneous ideal $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $1 \leq \ell \leq N$, we define the ℓ^{th} *eliminant ideal* of I as

$$J_\ell = I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell] \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell].$$

For each $j \geq 0$, one may consider $(J_\ell)_j$, that is, the vector space consisting of the homogeneous polynomials of degree j in J_ℓ, as either a vector subspace of $\mathbb{C}[x_0, x_1, \ldots, x_\ell]_j$ or $\mathbb{C}[x_0, x_1, \ldots, x_N]_j$. The following defines the dual basis elimination operator and then uses it to show the relationship between these vector spaces.

Definition 15. *For $1 \leq \ell \leq N$ and $\alpha = (\alpha_0, \ldots, \alpha_N)$, define $\Pi_\ell : (\mathbb{Z}_{\geq 0})^{N+1} \to (\mathbb{Z}_{\geq 0})^{\ell+1}$ by*

$$\Pi_\ell(\alpha) = \begin{cases} (\alpha_0, \ldots, \alpha_\ell) & \text{if } \alpha_i = 0 \text{ for } i > \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Define the dual basis elimination operator Π_ℓ *by*

$$\Pi_\ell\left(\sum_\alpha a_\alpha \partial_\alpha\right) = \sum_\alpha a_\alpha \partial_{\Pi_\ell(\alpha)}.$$

Lemma 16. *Let $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal, $1 \leq \ell \leq N$,*

$$J_\ell = I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell] \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell].$$

For each $j \geq 0$, $D_0^j(J_\ell)$ computed by considering $(J_\ell)_j$ as a vector subspace of $\mathbb{C}[x_0, x_1, \ldots, x_\ell]_j$ is equal to $\Pi_\ell(D_0^j(J_\ell))$ computed by considering $(J_\ell)_j$ as a vector subspace of $\mathbb{C}[x_0, x_1, \ldots, x_N]_j$.

Proof. Let $D^\ell = D_0^j(J_\ell)$ where $(J_\ell)_j \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell]_j$ and $D^N = D_0^j(J_\ell)$ where $(J_\ell)_j \subset \mathbb{C}[x_0, x_1, \ldots, x_N]_j$.

If $\partial = \sum_{|\alpha|=j} a_\alpha \partial_\alpha \in D^\ell$, clearly $\delta = \sum_{|\alpha|=j} a_\alpha \partial_{(\alpha,0)} \in D^N$ with $\Pi_\ell(\delta) = \partial$.

If $\partial = \sum_{|\beta|=j} a_\beta \partial_\beta \in D^N$, let

$$\partial_1 = \sum_{\substack{|\beta|=j \\ \Pi_\ell(\beta)\neq 0}} a_\beta \partial_\beta \quad \text{and} \quad \partial_2 = \sum_{\substack{|\beta|=j \\ \Pi_\ell(\beta)=0}} a_\beta \partial_\beta.$$

Clearly, $\partial = \partial_1 + \partial_2$ and $\Pi_\ell(\partial) = \Pi_\ell(\partial_1)$. If $f \in (J_\ell)_j \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell]_j$, define $g(x_0, \ldots, x_N) = f(x_0, \ldots, x_\ell)$. By construction, $\partial_2(g) = 0$ yielding

$$\Pi_\ell(\partial)(f) = \Pi_\ell(\partial_1)(f) = \partial_1(g) = \partial(g) = 0.$$

$\square$

The following theorem relates the dual spaces of J_ℓ and I.

Theorem 17. *If $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a homogeneous ideal, $1 \le \ell \le N$, and $J_\ell = I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell] \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell]$. Then, for all $j \ge 0$,*

$$D_0^j(J_\ell) = \Pi_\ell(D_0^j(I)).$$

Proof. Consider $(J_\ell)_j$ as a vector subspace of $\mathbb{C}[x_0, x_1, \ldots, x_N]_j$. Then,

$$(J_\ell)_j = (I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_j = I_j \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell]_j.$$

Since the proof of Lemma 14 only depends upon the vector space structure rather than the ideal structure, we know that

$$D_0^j(J_\ell) = D_0^j(I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell]) = D_0^j(I) + D_0^j(\mathbb{C}[x_0, x_1, \ldots, x_\ell]).$$

Clearly, $\Pi_\ell(D_0^j(\mathbb{C}[x_0, x_1, \ldots, x_\ell])) = \{0\}$ and thus

$$\Pi_\ell(D_0^j(J_\ell)) = \Pi_\ell(D_0^j(I)).$$

Lemma 16 completes the proof. $\square$

We note that if B is a basis for $D_0^j(I)$, then $\Pi_\ell(B)$ is a spanning set for $\Pi_\ell(D_0^j(I))$ which could be used to compute a basis if one is needed.

2.3 Quotients

In order to compute the dual space of a quotient ideal, we need to generalize the linear operator Φ_j defined by Equation 2. For any homogeneous polynomial $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $\partial \in D_0$, let $\Phi_g(\partial)$ be the differential functional in D_0 defined by

$$\Phi_g(\partial)(f) = \partial(gf) \quad \text{for every } f \in \mathbb{C}[x_0, x_1, \ldots, x_N]. \tag{5}$$

Clearly, if $a \in \mathbb{C}$, $\Phi_a(\partial) = a \cdot \partial$. Also, if $g = g_1 + g_2 = h_1 h_2$, then

$$\Phi_g = \Phi_{g_1+g_2} = \Phi_{g_1} + \Phi_{g_2} \quad \text{and} \quad \Phi_g = \Phi_{h_1 h_2} = \Phi_{h_1} \circ \Phi_{h_2} = \Phi_{h_2} \circ \Phi_{h_1}.$$

If g has degree d and $j \ge 0$, then $\Phi_g\left(D_0^{j+d}\right) \subset D_0^j$. Additionally, if $|\alpha| = j + d$, Leibniz rule yields

$$\Phi_g(\partial_\alpha) = \sum_{\substack{\gamma \le \alpha \\ |\gamma| = d}} \partial_\gamma(g) \partial_{\alpha-\gamma}. \tag{6}$$

Equation 6 immediately yields that Φ_{x_j} is the same map as Φ_j defined by Equation 2.

The following lemma relates the dual spaces of I and $I \cap \langle g \rangle$ under Φ_g.

Lemma 18. *Let $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal and $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous polynomial. Then, for every $j \geq 0$,*

$$\Phi_g(D_0^j(\langle g \rangle)) = \{0\} \quad \textit{and} \quad \Phi_g(D_0^j(I)) = \Phi_g(D_0^j(I \cap \langle g \rangle)).$$

Proof. Let $\partial \in D_0^j(\langle g \rangle)$. For every $f \in \mathbb{C}[x_0, x_1, \ldots, x_N]$, $gf \in \langle g \rangle$ so that $\Phi_g(\partial)(f) = \partial(gf) = 0$ Hence, $\Phi_g(\partial) = 0$.

Lemma 14 yields

$$\begin{aligned} \Phi_g(D_0^j(I \cap \langle g \rangle)) &= \Phi_g(D_0^j(I) + D_0^j(\langle g \rangle)) \\ &= \Phi_g(D_0^j(I)) + \Phi_g(D_0^j(\langle g \rangle)) \\ &= \Phi_g(D_0^j(I)). \end{aligned}$$

□

In order to prove the main theorem regarding dual spaces for quotient ideals, the existence of a one-sided inverse for Φ_g is needed when g is a nonzero homogeneous polynomial. For each j, a one-sided inverse of $\Phi_j = \Phi_{x_j}$ was constructed by Zeng [16] as the linear operator defined by $\Psi_j(\partial_\beta) = \partial_{\beta+e_j}$. In particular, $\Phi_j \circ \Psi_j$ is the identity operator. The following technical definition constructs such an operator with the remark after the proof of the lemma showing that this construction generalizes this Ψ_j operator.

Definition 19. *Let $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a nonzero homogeneous polynomial of degree d. Define the linear operator $\Psi_g : D_0 \to D_0$ as follows. Write $g = \sum_{|\alpha|=d} g_\alpha x^\alpha$ and let $\prec$ be the lexicographic ordering on $(\mathbb{Z}_{\geq 0})^{N+1}$. Let*

$$\alpha_0 = \min_{\prec}\{\alpha \mid |\alpha| = d \text{ and } g_\alpha \neq 0\}.$$

For any $\beta, \gamma \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\gamma| - |\beta| = d$, define

$$G(\beta, \gamma) = \begin{cases} g_{\gamma-\beta} & \text{if } \gamma \geq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

For any $n \geq 0$ and $\beta \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\beta| = n$, define

$$\Psi_g(\partial_\beta) = \sum_{|\alpha|=n+d} c_\alpha(\beta)\partial_\alpha$$

where

$$c_\alpha(\beta) = \begin{cases} \frac{1}{g_{\alpha_0}} \left(\delta(\alpha - \alpha_0, \beta) - \sum_{\substack{|\gamma|=|\alpha| \\ \gamma \succ \alpha}} G(\alpha - \alpha_0, \gamma) c_\gamma(\beta) \right) & \alpha \geq \alpha_0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\delta(\zeta, \epsilon)$ is Kronecker's delta.

Lemma 20. *If $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a nonzero homogeneous polynomial, then $\Phi_g \circ \Psi_g$ is the identity operator. In particular, for any $n \geq 0$ and $\partial \in D_0^n$, $\Psi_g(\partial) \in D_0^{n+d}$ with $\Phi_g(\Psi_g(\partial)) = \partial$.*

Proof. Fix $n \geq 0$ and $\beta \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\beta| = n$. Utilizing the notation from Definition 19, for any $\gamma \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\gamma| = n$, we claim

$$\delta(\gamma, \beta) = \sum_{|\alpha|=n+d} G(\gamma, \alpha) c_\alpha(\beta). \tag{7}$$

To prove this equation, split the summation as

$$\begin{aligned} \sum_{|\alpha|=n+d} G(\gamma, \alpha) c_\alpha(\beta) \quad &= \quad \sum_{\substack{|\alpha|=n+d \\ \alpha \prec \gamma+\alpha_0}} G(\gamma, \alpha) c_\alpha(\beta) \\ &\quad + G(\gamma, \gamma + \alpha_0) c_{\gamma+\alpha_0}(\beta) + \sum_{\substack{|\alpha|=n+d \\ \alpha \succ \gamma+\alpha_0}} G(\gamma, \alpha) c_\alpha(\beta). \end{aligned}$$

Let α be such that $|\alpha| = n + d$ and $\alpha \prec \gamma + \alpha_0$. If $\alpha \not\geq \alpha_0$, $c_\alpha(\beta) = 0$. Otherwise, we must have $G(\gamma, \alpha) = 0$ by construction of α_0. In particular,

$$\sum_{\substack{|\alpha|=n+d \\ \alpha \prec \gamma+\alpha_0}} G(\gamma, \alpha) c_\alpha(\beta) = 0.$$

Since $G(\gamma, \gamma + \alpha_0) = g_{\alpha_0}$, the definition of $c_{\gamma+\alpha_0}(\beta)$ yields

$$G(\gamma, \gamma + \alpha_0) c_{\gamma+\alpha_0}(\beta) = \delta(\gamma, \beta) - \sum_{\substack{|\alpha|=n+d \\ \alpha \succ \gamma+\alpha_0}} G(\gamma, \alpha) c_\alpha(\beta).$$

Equation 7 now follows immediately.

The following computation using Equations 6 and 7 completes the lemma:

$$\begin{aligned} \Phi_g(\Psi_g(\partial_\beta)) \quad &= \quad \sum_{|\alpha|=n+d} c_\alpha(\beta) \Phi_g(\partial_\alpha) = \sum_{|\alpha|=n+d} c_\alpha(\beta) \sum_{\gamma \leq \alpha} \partial_{\alpha-\gamma}(g) \partial_\gamma \\ &= \quad \sum_{|\alpha|=n+d} c_\alpha(\beta) \sum_{|\gamma|=n} G(\gamma, \alpha) \partial_\gamma \\ &= \quad \sum_{|\gamma|=n} \left(\sum_{|\alpha|=n+d} G(\gamma, \alpha) c_\alpha(\beta) \right) \partial_\gamma \\ &= \quad \sum_{|\gamma|=n} \delta(\gamma, \beta) \partial_\gamma = \partial_\beta. \end{aligned}$$

$\square$

Remark 21. *Consider the linear operator Ψ_{x_j}. We have $\alpha_0 = e_j$ and $g_\alpha = \delta(\alpha, e_j)$. Fix $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\alpha| = n+1$ and $|\beta| = n$. Since $\alpha \ngeq e_j$ implies $c_\alpha(\beta) = 0$, we assume that $\alpha \geq e_j$. For any $\gamma \in (\mathbb{Z}_{\geq 0})^{N+1}$ with $|\gamma| = n+1$, we have $G(\alpha - e_j, \gamma) = \delta(\alpha, \gamma)$. Hence,*

$$\sum_{\substack{|\gamma|=|\alpha| \\ \gamma \succ \alpha}} G(\alpha - e_j, \gamma) c_\gamma(\beta) = 0.$$

Thus, $c_\alpha(\beta) = \delta(\alpha - e_j, \beta)$ which yields $c_{\beta+e_j}(\beta) = 1$ and $c_\alpha(\beta) = 0$ otherwise. In particular, $\Psi_{x_j}(\partial_\beta) = \partial_{\beta+e_j}$.

The following theorem describes the dual space of the quotient of two homogeneous ideals.

Theorem 22. *Let $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal.*

1. *If $g \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ is a homogeneous polynomial of degree d, then, for each $j \geq 0$,*
$$D_0^j(I : g) = \Phi_g\left(D_0^{j+d}(I)\right) = \Phi_g\left(D_0^{j+d}(I \cap \langle g \rangle)\right). \tag{8}$$
2. *If $g_1, \ldots, g_k \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ are homogeneous polynomials with $d_i = \deg g_i$ and $J = \langle g_1, \ldots, g_k \rangle$, then, for each $j \geq 0$,*
$$D_0^j(I : J) = \sum_{i=1}^k D_0^j(I : g_i) = \sum_{i=1}^k \Phi_{g_i}\left(D_0^{j+d_i}(I)\right).$$

Proof. Since $I : 0 = \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $\Phi_0 = 0$, we know that

$$D_0(I : 0) = \Phi_g(D_0(I)) = \Phi_g(D_0(I \cap \langle g \rangle)) = \{0\}.$$

Thus, we can assume that $g \neq 0$.

Let $\partial \in D_0^{j+d}(I)$. For any $f \in I : g$, we know $\Phi_g(\partial)(f) = \partial(gf) = 0$ since $gf \in I$. Hence, $\Phi_g(\partial) \in D_0^j(I : g)$ yielding $\Phi_g(D_0^{j+d}(I)) \subset D_0^j(I : g)$.

Let $\partial \in D_0^j(I : g)$. Suppose that $f \in I \cap \langle g \rangle$. Then, $h = \frac{f}{g} \in I : g$ and

$$\Psi_g(\partial)(f) = \Psi_g(\partial)(gh) = \Phi_g(\Psi_g(\partial))(h) = \partial(h) = 0.$$

Thus, $\Psi_g(\partial) \in D_0^{j+d}(I \cap \langle g \rangle)$ and $\partial = \Phi_g(\Psi_g(\partial)) \in \Phi_g(D_0^{j+d}(I \cap \langle g \rangle))$. In particular, $D_0^j(I : g) \subset \Phi_g(D_0^{j+d}(I \cap \langle g \rangle))$.

Lemma 18 yields Equation 8 since

$$\Phi_g(D_0^{j+d}(I)) \subset D_0^j(I : g) \subset \Phi_g(D_0^{j+d}(I \cap \langle g \rangle)) = \Phi_g(D_0^{j+d}(I)).$$

The remaining statement follows from Lemma 14 and Proposition 2. $\square$

Theorem 22 together with an induction argument immediately yields the following corollary.

Corollary 23. *Let $I \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a homogeneous ideal. For any $j \geq 0$ and $m \geq 1$,*

$$D_0^j(I : x_0^m) = \Phi_{x_0^m}(D_0^{j+m}(I)) = \underbrace{\Phi_{x_0} \circ \cdots \circ \Phi_{x_0}}_{m \text{ times}}(D_0^{j+m}(I)).$$

If $S \subset D_0^j$ is a vector subspace, the support of S is

$$\text{supp } S = \left\{ \alpha \;\middle|\; \text{there exists } \sum_{|\beta|=j} a_\beta \partial_\beta \in S \text{ such that } a_\alpha \neq 0 \right\}.$$

The following corollary is the weak nullstellensatz.

Corollary 24. *Let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$ and $J = \langle f_1^h, \ldots, f_n^h \rangle \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ where f_i^h is the homogenization of f_i with respect to x_0. The following statements are equivalent.*

1. *$\mathcal{V}(I) = \mathcal{V}(f_1, \ldots, f_n) = \{z \in \mathbb{C}^N \mid f_i(z) = 0 \text{ for } i = 1, \ldots, n\} = \emptyset$.*
2. *There exists $m \geq 0$ such that $(m, 0) \notin$ supp $D_0^m(J)$.*

Proof. This immediately follows from the fact that, for $m \geq 0$, $(m, 0) \in$ supp $D_0^m(J)$ if and only if $D_0^0(J : x_0^m) = \Phi_{x_0}^m(D_0^m(J)) \neq \{0\}$ which occurs if and only if $1 \notin J : x_0^m$. □

2.4 Algorithms

For homogeneous ideals $I, J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $d, \ell \geq 0$, the following algorithms computes a $\mathbb{C}$-basis for $(I+J)_d$, $(I \cap J)_d$, $(I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_d$, and $(I : J)_d$. These algorithms first compute the d^{th} order dual space for the resulting ideal and then utilize **Annihilator**.

The first algorithm computes a basis for $(I + J)_d$ and $(I \cap J)_d$.

Procedure $(K_+, K_\cap) =$ **SumIntersection**(F_I, F_J, d)

Input Two finite sets $F_I, F_J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ of homogeneous polynomials and integer $d \geq 0$.

Output A $\mathbb{C}$-basis K_+ and $K_\cap$ for $(I+J)_d$ and $(I \cap J)_d$, respectively, where $I = \langle F_I \rangle$ and $J = \langle F_J \rangle$.

Return $(K_+, K_\cap) := (\textbf{Annihilator}(d, N, B_{I+J}), \textbf{Annihilator}(d, N, B_I \cup B_J))$ where B_{I+J}, B_I, and B_J are bases for $D_0^d(\langle F_I, F_J\rangle)$, $D_0^d(\langle F_I\rangle)$, and $D_0^d(\langle F_J\rangle)$, respectively.

The second algorithm computes a basis for $(I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_d$ which is considered as a subspace of $\mathbb{C}[x_0, x_1, \ldots, x_\ell]_d$.

Procedure $K = \textbf{Eliminate}(F, \ell, d)$

Input A finite set $F \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ of homogeneous polynomials and integers $\ell, d \geq 0$.

Output A $\mathbb{C}$-basis $K \subset \mathbb{C}[x_0, x_1, \ldots, x_\ell]$ for $(I \cap \mathbb{C}[x_0, x_1, \ldots, x_\ell])_d$ where $I = \langle F\rangle$.

Return $K := \textbf{Annihilator}(d, \ell, \Pi_\ell(B))$ where B is a basis for $D_0^d(\langle F\rangle)$.

The third algorithm computes a basis for $(I : J)_d$.

Procedure $K = \textbf{Quotient}(F_I, F_J, d)$

Input Two finite sets $F_I, F_J \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ of homogeneous polynomials and integer $d \geq 0$.

Output A $\mathbb{C}$-basis K for $(I : J)_d$ where $I = \langle F_I\rangle$ and $J = \langle F_J\rangle$.

Begin
1. Write $F_J = \{g_1, \ldots, g_\ell\}$.
2. For $i = 1, \ldots, \ell$
   (a) Compute a basis B_i for $D_0^{d+\deg g_i}(\langle F_I\rangle)$.
   (b) Compute $E_i := \Phi_{g_i}(B_i)$.

Return $K := \textbf{Annihilator}(d, N, E_1 \cup \cdots \cup E_\ell)$**.**

Since all of the linear algebra computations used in these algorithms are parallelizable, these algorithms are naturally parallelizable. If the coefficients of all the polynomials are exact, then these algorithms yield exact results. For numerically approximated coefficients, we utilize numerical linear algebra routines to yield approximate results.

Suppose that we fix d and consider $R_d = \mathbb{C}[x_0, x_1, \ldots, x_N]_{\leq d}$. Since the dimension of R_d is polynomial in N, namely $p(N) = \binom{N+d+1}{d}$, without loss of generality, we may assume that $|F_I \cap R_d|, |F_J \cap R_d| \leq p(N)$. Then, **SumIntersection** is polynomial in N by using the method of Dayton and

Zeng[6] for computing dual bases. In particular, since $D_0^d(F_I \cap R_d) = D_0^d(F_I)$, the method of [6] identifies the dual space $D_0^d(F_I \cap R_d)$ as the null space of the d^{th} order Macaulay matrix which has size $|F_I \cap R_d| \cdot \binom{N+d}{d-1} \times p(N)$. That is, the number of rows and columns is polynomial in N. The same statements holds for the matrix A setup in **Annihilator** which is of size $(\dim D_0^d(F_I)) \times \binom{N+d}{d}$.

Under the assumption that $|F \cap R_d| \leq p(N)$, a similar argument yields that **Eliminate** is polynomial in N. If we assume that each element in F_J has degree at most, say, e and that $|F_I \cap R_{d+e}|$ and $|F_J|$ are polynomial in N, then **Quotient** is also polynomial in N by a similar argument.

2.5 Illustrative examples

2.5.1 Sum, intersection, and quotient

Consider computing $I + J$, $I \cap J$, and $I : J$ for the homogeneous ideals $I = \langle y, x^2 \rangle$ and $J = \langle x, z^2 \rangle$ in $\mathbb{C}[x, y, z]$. It is easy to verify that

$$D_0(I) = \text{span}_{\mathbb{C}}\{\partial_{xz^n}, \partial_{z^n} \mid n \geq 0\} \quad \text{and} \quad D_0(J) = \text{span}_{\mathbb{C}}\{\partial_{y^n}, \partial_{y^n z} \mid n \geq 0\}.$$

Clearly, $D_0(I + J) = D_0(I) \cap D_0(J) = \text{span}_{\mathbb{C}}\{\partial_1, \partial_z\}$ and

$$D_0(I \cap J) = D_0(I) + D_0(J) = \text{span}_{\mathbb{C}}\{\partial_{xz^n}, \partial_{y^n}, \partial_{y^n z}, \partial_{z^n} \mid n \geq 0\}.$$

This yields $H^p_{I+J} = \{1, 1, 0, 0, \dots\}$ and $I + J = \langle x, y, z^2 \rangle$ as well as $H^p_{I \cap J} = \{1, 3, 4, 4, \dots\}$ and $I \cap J = \langle x^2, xy, yz^2 \rangle$.

For any $n \geq 0$,

$$\Phi_x(\partial_{xz^n}) = \partial_{z^n} \quad \text{and} \quad \Phi_x(\partial_{z^n}) = 0,$$

and, for any $n \geq 2$,

$$\Phi_{z^2}(\partial_{xz^n}) = \partial_{xz^{n-2}} \quad \text{and} \quad \Phi_{z^2}(\partial_{z^n}) = \partial_{z^{n-2}}.$$

This yields $D_0(I : x) = \text{span}_{\mathbb{C}}\{\partial_{z^n} \mid n \geq 0\}$ and $D_0(I : z^2) = D_0(I)$. In particular, $H^p_{I:x} = \{1, 1, \dots\}$, $H^p_{I:z^2} = \{1, 2, 2, \dots\}$,

$$I : x = \langle x, y \rangle, \quad \text{and} \quad I : z^2 = I = \langle y, x^2 \rangle.$$

Since $D_0(I : J) = D_0(I : x) + D_0(I : z^2) = D_0(I)$, we have $I : J = I$.

j	Basis for $D_0^j(I)$	Image under Π_2
0	∂_1	∂_1
1	∂_x ∂_y ∂_z	∂_x ∂_y 0
2	$\partial_{x^2} + \partial_{yz}$ ∂_{xy} ∂_{xz} ∂_{y^2}	∂_{x^2} ∂_{xy} 0 ∂_{y^2}
3	$\partial_{x^3} + \partial_{xyz}$ $\partial_{x^2y} + \partial_{y^2z}$ ∂_{xy^2} ∂_{y^3}	∂_{x^3} ∂_{x^2y} ∂_{xy^2} ∂_{y^3}
4	$\partial_{x^3y} + \partial_{xy^2z}$ $\partial_{x^2y^2} + \partial_{y^3z}$ ∂_{xy^3} ∂_{y^4}	∂_{x^3y} $\partial_{x^2y^2}$ ∂_{xy^3} ∂_{y^4}

Table 1: Applying Π_2 to a basis of $D_0^j(I)$

2.5.2 Elimination

Consider computing $J = I \cap \mathbb{C}[x, y] \subset \mathbb{C}[x, y]$ where $I = \langle x^2 - yz, z^2 \rangle \subset \mathbb{C}[x, y, z]$. Let Π_2 be the elimination operator defined by the projection map $(\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0, \alpha_1)$. For $0 \leq j \leq 4$, Table 1 presents a dual basis for $D_0^j(I)$ as well as Π_2 applied to each element in this dual basis.

We see that J is a zero-dimensional ideal since its projective Hilbert function is $\{1, 2, 3, 4, 4, \dots, \}$ with the index of regularity being 3. Since $Ann_j(D_0^j(J)) = \{0\}$ for $0 \leq j \leq 3$ and $Ann_4(D_0^4(J)) = \text{span}_{\mathbb{C}}\{x^4\}$, Proposition 4 yields

$$J = \langle x^4 \rangle.$$

3 Counting solutions

Let $I = \langle f_1, \dots, f_n \rangle \subset \mathbb{C}[x_1, \dots, x_N]$ be a zero-dimensional ideal. That is, $\mathcal{V}(I) = \mathcal{V}(f_1, \dots, f_n) = \{z \in \mathbb{C}^N \mid f_i(z) = 0 \text{ for } i = 1, \dots, n\}$ consists of finitely many points. A basis operation is to count the number of solutions of I with multiplicity, that is, to compute

$$Z(I) = \sum_{z \in \mathcal{V}(f)} \mathrm{mult}_I(z).$$

Since I is zero-dimensional, its affine Hilbert polynomial HP_I is constant with $HP_I \equiv Z(I)$.

Let $I^h \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$ and $f_i^h \in \mathbb{C}[x_0, x_1, \ldots, x_N]$ be the homogenization of I and f_i with respect to x_0, respectively, and $J = \langle f_1^h, \ldots, f_n^h \rangle$. We know that $I^h = J : x_0^\infty$ and the projective Hilbert polynomial of I^h is the same as the affine Hilbert polynomial of I. This provides the underlying approach for using dual bases to compute $Z(I)$.

Even though I is zero-dimensional, the homogeneous ideal J may be positive-dimensional. That is, J may have positive-dimensional components which must be contained in $\mathcal{V}(x_0)$ and are removed via saturation. The following theorem provides a stopping criterion for computing I^h from J.

Theorem 25. *Let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_1, \ldots, x_N]$ be a zero-dimensional ideal, $J = \langle f_1^h, \ldots, f_n^h \rangle \subset \mathbb{C}[x_0, x_1, \ldots, x_N]$, and $m \geq 0$. Then, $I^h = J : x_0^m$ if and only if there exists $k \geq 0$ such that*

1. $\dim D_0^k(J : x_0^m) = \dim D_0^{k+1}(J : x_0^m)$, *and*
2. *for* $0 \leq \ell \leq k+1$, $\dim D_0^\ell(J : x_0^m) = \dim D_0^\ell(J : x_0^{m+1})$.

In this case, $Z(I) = \dim D_0^k(J : x_0^m) = \dim \Phi_{x_0^m}(D_0^{k+m}(J))$.

Proof. Suppose that $I^h = J : x_0^m$ and k is the index of regularity of I^h. We know that

$$\begin{aligned} \dim D_0^k(J : x_0^m) &= \dim D_0^k(I^h) = H_{I^h}^p(k) \\ &= H_{I^h}^p(k+1) = \dim D_0^{k+1}(I^h) \\ &= \dim D_0^{k+1}(J : x_0^m). \end{aligned}$$

Item 2 holds since $I^h = J : x_0^m = J : x_0^{m+1}$.

Suppose that Items 1 and 2 hold. Since $J : x_0^m \subset J : x_0^{m+1}$, Item 2 yields

$$(J : x_0^m)_\ell = (J : x_0^{m+1})_\ell \ \text{ for } \ 0 \leq \ell \leq k+1.$$

Item 1 yields that $J : x_0^m$ is zero dimensional with index of regularity at most k. In particular, we know that $J : x_0^m$ is generated in degree at most $k+1$ which yields $J : x_0^m = J : x_0^{m+1}$. Therefore, $J : x_0^m = J : x_0^\infty = I^h$. $\square$

3.1 Algorithm

The following algorithm computes the number of solutions, counting multiplicity, for a given polynomial system under the assumption that the polynomial system has finitely many solutions.

Procedure $Z = \textbf{CountSolutions}(F)$

Input A finite set $F \subset \mathbb{C}[x_1, \ldots, x_N]$ of polynomials such that $|\mathcal{V}(F)| < \infty$.

Output The number of solutions, counting multiplicity, of F, namely $Z(\langle F\rangle)$.

Begin

1. Write $F = \{f_1, \ldots, f_n\}$ and compute f_i^h. Define $J := \langle f_1^h, \ldots, f_n^h\rangle$.
2. For $j = 0, 1, 2, \ldots$
   (a) Compute a basis B_j for $D_0^j(J)$.
   (b) For $0 \leq m \leq j$, compute $H(m, j-m) := \dim \operatorname{span}_{\mathbb{C}} \Phi_{x_0^m}(B_j)$.
   (c) For $m = 0, 1, \ldots, j-2$,
      i. If $H(m, \ell) = H(m+1, \ell)$ for $0 \leq \ell \leq j-m-1$ and $H(m, j-m-2) = H(m, j-m-1)$, then **Return** $Z := H(m, j-m-2)$.

3.2 Illustrative example for counting solutions

Consider the ideals $I = \langle F\rangle \subset \mathbb{C}[x, y]$ and $J = \langle F^h\rangle \subset \mathbb{C}[x, y, z]$ where

$$F(x,y) = \begin{bmatrix} x \\ xy - 1 \end{bmatrix} \quad \text{and} \quad F^h(x,y,z) = \begin{bmatrix} x \\ xy - z^2 \end{bmatrix}.$$

It is clear that $Z(I) = 0$ and we will show this using dual bases.

Table 2 lists a dual basis for $D_0^j(J)$ and its image under Φ_{z^m}. This computation shows that $H_{I^h}^p = \{0, 0, \ldots\}$ which yields $Z(I) = 0$. In particular, we know that $J : z = \langle x, z\rangle$ and $I^h = J : z^2 = \langle 1\rangle$.

4 Examples

4.1 A dense polynomial system

Consider the ideal $I = \langle F\rangle \subset \mathbb{C}[x, y, z]$ where

$$F(x,y,z) = \begin{bmatrix} x^2 + 6xy + 4xz + 9y^2 + 12yz + 4z^2 - 9 \\ 4x^2 + 10xy + 9xz - 6y^2 - yz + 2z^2 - 1 \\ 5x^2 + 12xy + 9xz - 15x - 9y^2 - 9yz + 9y - 2z^2 + 3z - 1 \end{bmatrix}$$

j	Basis for $D_0^j(J)$	Φ_z	Φ_{z^2}	Φ_{z^3}	Φ_{z^4}
0	∂_1				
1	∂_y	0			
	∂_z	∂_1			
2	∂_{y^2}	0	0		
	∂_{yz}	∂_y	0		
3	∂_{y^3}	0	0	0	
	$\partial_{y^2 z}$	∂_{y^2}	0	0	
4	∂_{y^4}	0	0	0	0
	$\partial_{y^3 z}$	∂_{y^3}	0	0	0

Table 2: Applying Φ_{z^m} to a basis of $D_0^j(J)$

and the homogeneous ideal $J \subset \mathbb{C}[w, x, y, z]$ generated by homogenizing F with respect to w. If $Z(I) < \infty$, we know that $Z(I) \leq 2^3 = 8$ since each polynomial in F is quadratic. Based on the dense nature of F in terms of the monomials which appear, the Bézout count is equal to the mixed volume. Due to the relationships between the polynomials, the following computation shows that $Z(I) = 1$.

Table 3 shows the projective Hilbert functions for $J : w^m$ until the stopping criterion described in Theorem 25 is satisfied. One result of this computation is that J is a positive-dimensional homogeneous ideal since $H_J^p(4) = 9 > 8$. This computation also shows that $I^h = J : w^3$ with $H_{I^h}^p = \{1, 1, \ldots\}$ and $Z(I) = 1$.

k	0	1	2	3	4	5
J	1	4	7	8	9	10
$J : w$	1	4	4	4	4	
$J : w^2$	1	2	2	2		
$J : w^3$	1	1	1			
$J : w^4$	1	1				
$J : w^5$	1					

Table 3: Projective Hilbert functions for $J : w^m$

One can verify that $D_0^1(J : w^3)$ is spanned by

$$13\partial_x + 21\partial_y - 2\partial_z - 24\partial_w.$$

which yields

$$I^h = J : w^3 = \langle (J : w^3)_1 \rangle = \langle 24x + 13w, 24y + 21w, 24z - 2w \rangle.$$

In particular, the unique point in $\mathcal{V}(I)$ is

$$(x, y, z) = \left(\frac{-13}{24}, \frac{-21}{24}, \frac{2}{24}\right) = \left(\frac{-13}{24}, \frac{-7}{8}, \frac{1}{12}\right).$$

4.2 Inverse kinematics of an RR dyad

Consider the inverse kinematic problem of an RR dyad. As shown in Figure 1, the RR dyad consists of two legs of length ℓ_1 and ℓ_2 together with two pin joints. The mechanism is anchored at point O which, without loss of generality, we may assume is the origin. Given a point $P = (p_x, p_y)$, the problem is to find the angles θ_1 and θ_2 so that the end of the second leg is located at P. This problem is described by the equations

$$\ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) - p_x = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) - p_y = 0.$$

In [14], these equations are transformed into a polynomial system by substituting in the above equations

$$\sin(\theta_j) = \frac{2t_j}{1+t_j^2} \quad \text{and} \quad \cos(\theta_j) = \frac{1-t_j^2}{1+t_j^2}$$

and clearing denominators. The approach we will consider is to treat $\sin(\theta_j)$ and $\cos(\theta_j)$ as indeterminants, namely, s_j and c_j, together with $s_j^2 + c_j^2 = 1$. This yields the polynomial system

$$f(c_1, c_2, s_1, s_2) = \begin{bmatrix} \ell_1 c_1 + \ell_2 c_2 - p_x \\ \ell_1 s_1 + \ell_2 s_2 - p_y \\ s_1^2 + c_1^2 - 1 \\ s_2^2 + c_2^2 - 1 \end{bmatrix}.$$

Consider the homogenization of f with respect to z, namely

$$f^h(c_1, c_2, s_1, s_2, z) = \begin{bmatrix} \ell_1 c_1 + \ell_2 c_2 - p_x z \\ \ell_1 s_1 + \ell_2 s_2 - p_y z \\ s_1^2 + c_1^2 - z^2 \\ s_2^2 + c_2^2 - z^2 \end{bmatrix}.$$

The first objective is count the number of solutions of f for a general set of parameters. Let $I = \langle f \rangle$ and $J = \langle f^h \rangle$. Table 4 shows the projective Hilbert functions for $J : z^m$ until the stopping criterion described in

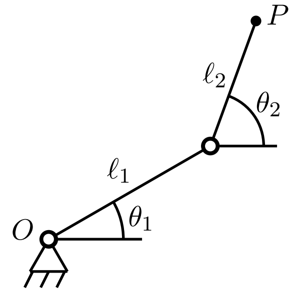

Figure 1: RR dyad

k	0	1	2	3	4
J	1	3	4	4	4
$J : z$	1	2	2	2	
$J : z^2$	1	2	2		
$J : z^3$	1	2			
$J : z^4$	1				

Table 4: Projective Hilbert functions for $J : z^m$

Theorem 25 is satisfied. This computation also shows that $I^h = J : z$ with $H^p_{I^h} = \{1, 2, 2, \dots\}$ and $Z(I) = 2$. In particular, we know

$$I^h = \left\langle \begin{array}{l} \ell_1 c_1 + \ell_2 c_2 - p_x z, \ell_1 s_1 + \ell_2 s_2 - p_y z, s_2^2 + c_2^2 - z^2, \\ 2\ell_2 p_x c_2 + 2\ell_2 p_y s_2 + (\ell_1^2 - \ell_2^2 - p_x^2 - p_y^2) z \end{array} \right\rangle.$$

The second objective is to describe the reality of the solutions of I for generic mechanically meaningful values of the parameters. In particular, we will assume that the parameters $\ell_1, \ell_2, p_x, p_y \in \mathbb{R}$ are general such that $\ell_1, \ell_2 > 0$. Consider $K = I \cap \mathbb{C}[c_1]$ and $K^h = I^h \cap \mathbb{C}[c_1, z]$. Using the dual space for $I^h = J : z$, we computed $K^h = \langle Ac_1^2 + Bc_1 z + Cz^2 \rangle$ and hence $K = \langle Ac_1^2 + Bc_1 + C \rangle$ where

$$\begin{array}{rcl} A & = & 4\ell_1^2(p_x^2 + p_y^2), \\ B & = & -4\ell_1 p_x(\ell_1^2 - \ell_2^2 + p_x^2 + p_y^2), \\ C & = & (\ell_1^2 + 2\ell_1 p_y - \ell_2^2 + p_x^2 + p_y^2)(\ell_1^2 - 2\ell_1 p_y - \ell_2^2 + p_x^2 + p_y^2). \end{array}$$

Clearly, $A \neq 0$. By the Shape Lemma [4], the solutions of I and K are real if and only if

$$0 \leq B^2 - 4AC = -16\ell_1^2 p_y^2((\ell_1 - \ell_2)^2 - p_x^2 - p_y^2)((\ell_1 + \ell_2)^2 - p_x^2 - p_y^2).$$

The assumptions on the parameters yield

$$((\ell_1 - \ell_2)^2 - p_x^2 - p_y^2)((\ell_1 + \ell_2)^2 - p_x^2 - p_y^2) \leq 0.$$

Since $\ell_1, \ell_2 > 0$, we know $(\ell_1 - \ell_2)^2 < (\ell_1 + \ell_2)^2$. Therefore, the solutions of I are real if and only if

$$(\ell_1 - \ell_2)^2 \leq p_x^2 + p_y^2 \leq (\ell_1 + \ell_2)^2.$$

Since $\|P\| = \sqrt{p_x^2 + p_y^2}$, this simplifies to

$$|\ell_1 - \ell_2| \leq \|P\| \leq \ell_1 + \ell_2.$$

The third objective is to describe the parameter values for which $\mathcal{V}(I)$ is positive dimensional. We could utilize the fiber product method of [10] to compute such values. However, the approach we will utilize is based on Hilbert functions.

Suppose that the parameter values are chosen so that $(I^h)_1$ is three dimensional. If $c_2^2 + s_2^2 - z^2 \notin \langle (I^h)_1 \rangle$, then $H_{I^h}^p = \{1, 2, 2, \dots\}$. It is easy to verify that $c_2^2 + s_2^2 - z^2 \in \langle (I^h)_1 \rangle$ implies $l_1 = p_x = p_y = 0$. In order for $\mathcal{V}(I)$ to be nonempty, we must also have $l_2 = 0$ yielding that $(I^h)_1 = \langle 0 \rangle$. Therefore, the only case we need to consider is when $\dim(I^h)_1 < 3$ meaning that $H_{I^h}^p(1) > 2$.

We know that $H_{I^h}^p(1)$ is equal to the null space of the Jacobian matrix of the generators of I^h evaluated at 0. Thus, $\mathcal{V}(I)$ is positive dimensional if and only if $\dim \operatorname{null} M > 2$ where

$$M = \begin{bmatrix} \ell_1 & \ell_2 & 0 & 0 & -p_x \\ 0 & 0 & \ell_1 & \ell_2 & -p_y \\ 0 & 2\ell_2 p_x & 0 & 2\ell_2 p_y & \ell_1^2 - \ell_2^2 - p_x^2 - p_y^2 \end{bmatrix}$$

which occurs if and only $\operatorname{rank} M < 3$. We could utilize the method of [2] to determine the parameter values where $\dim \operatorname{null} M > 2$. However, based on the simplicity of the problem, we will use the classical determinantal approach. That is, $\operatorname{rank} M < 3$ if and only if the ten 3×3 minors of M vanish. Solving this polynomial system using Bertini [3] yields five irreducible components which correspond to

$$\begin{aligned} C_1 &= \{\ell_1 = 0, \ell_2^2 - p_x^2 - p_y^2 = 0\}, \\ C_2 &= \{\ell_2 = 0, \ell_1^2 - p_x^2 - p_y^2 = 0\}, \\ C_3 &= \{\ell_1 = \ell_2, p_x = p_y = 0\}, \\ C_4 &= \{\ell_1 = -\ell_2, p_x = p_y = 0\}, \text{ and} \\ C_5 &= \{\ell_1 = 0, \ell_2 = 0\}. \end{aligned}$$

If $\ell_1 = \ell_2 = 0$, then $\mathcal{V}(I)$ is nonempty if and only if $p_x = p_y = 0$. Since $\ell_1 = \ell_2 = p_x = p_y = 0$ is an point in each C_j, this computation yields that $\mathcal{V}(I)$ is positive dimensional if and only if the parameters lie in $\bigcup_{j=1}^4 C_j$.

4.3 Counting solutions for a robotics problem

Consider the polynomial system, denoted F, for the inverse kinematics problem of a general six-revolute serial-link robot presented in [9]. The polynomial system is available at [3] where the coefficients are computed to 300 digits. This polynomial system consists of 8 quadratic polynomials in the 8 variables $x_1, \ldots, x_8$. Let $I = \langle F \rangle$ and $J = \langle F^h \rangle$ where F^h is the homogenization of F with respect to x_0. Assuming $Z(I) < \infty$, the Bézout count yields $Z(I) \le 2^8 = 256$. In fact, the 2-homogeneous Bézout count is 96 while the mixed volume is 64. We will utilize **CountSolutions** to show the well-known result that $Z(I) = 32$ for a general set of parameters.

Table 5 shows the projective Hilbert functions for $J : w^m$ until the stopping criterion described in Theorem 25 is satisfied. One result of this computation is that J is a positive-dimensional homogeneous ideal since $H_J^p(7) = 296 > 256$. This computation also shows that $I^h = J : w^4$ with $H_{I^h}^p = \{1, 9, 31, 32, 32, \ldots\}$ and $Z(I) = 32$.

k	0	1	2	3	4	5	6	7	8	9
J	1	9	37	93	163	219	256	296	336	376
$J : w$	1	9	37	82	115	128	144	160	176	
$J : w^2$	1	9	36	57	56	56	56	56		
$J : w^3$	1	9	34	32	32	32	32			
$J : w^4$	1	9	31	32	32	32				
$J : w^5$	1	9	31	32	32					
$J : w^6$	1	9	31	32						
$J : w^7$	1	9	31							
$J : w^8$	1	9								
$J : w^9$	1									

Table 5: Projective Hilbert functions for $J : w^m$

5 Conclusion

We have shown that many basic algebraic operations performed on homogeneous ideals can be translated into operations performed on Macaulay dual spaces. These new algorithms can either utilize exact or floating point arithmetic and can be parallelized by utilizing parallel linear algebra routines. With an efficient method for computing the closedness subspace $C_0^{j+1}(I)$ from the dual space $D_0^j(I)$, these parallelizable algorithms may be competitive against the inherently serial Gröbner based algorithms for zero-dimensional ideals and for positive-dimensional ideals when one only wants to compute the resulting ideal in certain degrees.

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