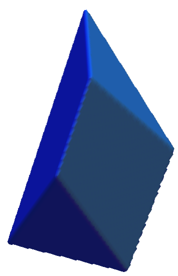
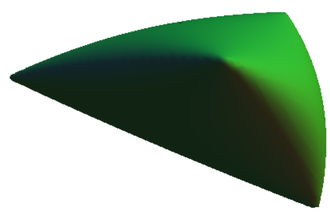
Numerical Algebraic Geometry and Optimization

by Jonathan D. Hauenstein[1]

Convex programming aims to minimize a convex objective function over a convex set, called the feasible set. For example, linear programming minimizes a linear function over a polytope (intersection of finitely many linear half-spaces as in Figure 1(a)) while semidefinite programming minimizes a linear function over a spectrahedron (intersection of the cone of positive semidefinite matrices with a linear space as in Figure 1(b)).



(a)



(b)

FIGURE 1. Example of (a) a polytope and (b) a spectrahedron.

When the feasible set has a nonempty interior, a standard approach for solving convex programs are interior point methods. Conversely, when the feasible set is empty, the program is said to be infeasible and the traditional Farkas' lemma is a standard approach for verifying infeasibility. For example, every infeasible linear program can be verified using the traditional Farkas' lemma. However, there are so-called weakly infeasible semidefinite programs where this is not the case. To illustrate, consider the following semidefinite program:

$$\text{(1)} \qquad \begin{array}{ll} \text{minimize} & x_{11} \\ \text{subject to} & \begin{bmatrix} x_{11} & 1 \\ 1 & 0 \end{bmatrix} \succeq 0 \end{array}$$

where $A \succeq 0$ means that A is a positive semidefinite matrix. Since the determinant of the matrix in (1) is -1, the program (1) is clearly infeasible. Moreover, (1) is weakly infeasible since the corresponding alternative via the traditional Farkas' lemma is also infeasible, i.e., there does not exist $y \in \mathbb{R}^2$ such that

$$\begin{array}{rcl} \begin{bmatrix} 0 & y_1 \\ y_1 & y_2 \end{bmatrix} & \succeq & 0 \\ 2 \cdot y_1 + 0 \cdot y_2 & = & -1. \end{array}$$

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One numerical challenge in identifying weakly infeasible semidefinite programs is that perturbations can be strongly infeasible or strictly feasible. For example,

$$\begin{array}{ll} \text{minimize} & x_{11} \\ \text{subject to} & \begin{bmatrix} x_{11} & 1 \\ 1 & \epsilon \end{bmatrix} \succeq 0 \end{array}$$

is strongly infeasible for $\epsilon < 0$ and strictly feasible for $\epsilon > 0$. Liu and Pataki [3] showed that many commonly-used software packages in semidefinite programming have difficulty identifying weakly infeasible semidefinite programs when the reason for infeasibility is not trivially obvious. Such *messy* instances were obtained by obscuring their structure via row operations and rotations. Thus, a change of perspective was needed for identifying weakly infeasible semidefinite programs.

Using the lens of numerical algebraic geometry [1, 4], the mathematical foundation of traditional interior point methods is to numerically track a solution path of a homotopy from a point in the interior of the feasible set to an optimizer. With this viewpoint, weakly infeasible semidefinite programs can be identified [2] using the following three techniques from numerical algebraic geometry: projective space for compactifying infinite length solution paths, adaptive precision path tracking for navigating through ill-conditioned areas, and endgames for accurately computing singular endpoints.

To illustrate, we consider the following convex program modified from (1):

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \begin{bmatrix} x_{11}+\lambda & 1 \\ 1 & \lambda \end{bmatrix} \succeq 0. \end{array} \tag{2}$$

The corresponding optimal value is easily observed to be $\lambda^* = 0$, but this is actually an infimum that is not attained as a minimum, a condition that is equivalent to (1) being weakly infeasible. Therefore, optimizers to (2) are "at infinity" meaning that a solution path defined by traditional interior point methods will have infinite length and approach an asymptote as represented in Figure 2(a). Compactification using projective space yields a finite length path that can be efficiently tracked as represented in Figure 2(b).

Complex analysis enters the scene to accurately compute the endpoint. The winding number (also called the cycle number) of the endpoint for the path displayed in Figure 2(b) is 2, meaning that the path over the complex numbers locally behaves like the complex square root function. Hence, the Cauchy integral theorem can be used to compute the endpoint of this path by integrating along a closed loop as shown in Figure 3. Due to periodicity, numerical integration by the trapezoid rule is exponentially convergent [5]. Such a procedure for computing the endpoint is called the Cauchy endgame. Since any endpoint with winding number larger than 1 is necessarily singular, ill-conditioning that necessarily arises near the endpoint can be controlled using adaptive precision path tracking methods.

This viewpoint for identifying weakly infeasible semidefinite programs using numerical algebraic geometry and the software package `Bertini` [1] along with

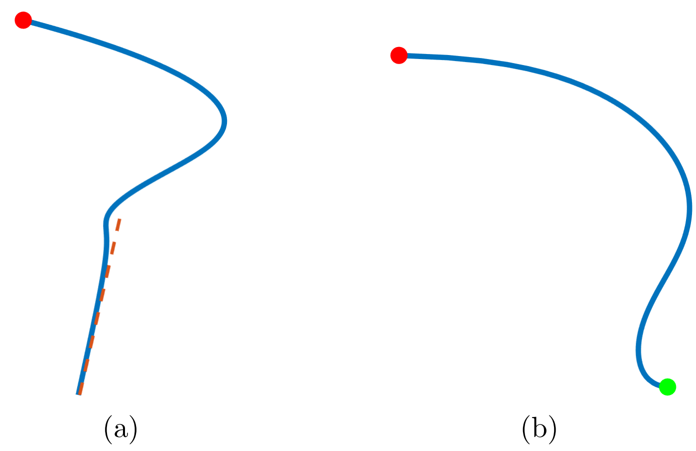

FIGURE 2. (a) A plot of paths at a given (red) point may have infinite length with limiting asymptote corresponding with $\lambda^* = 0$. (b) Compactification using projective space yields a finite-length path that can be efficiently tracked.

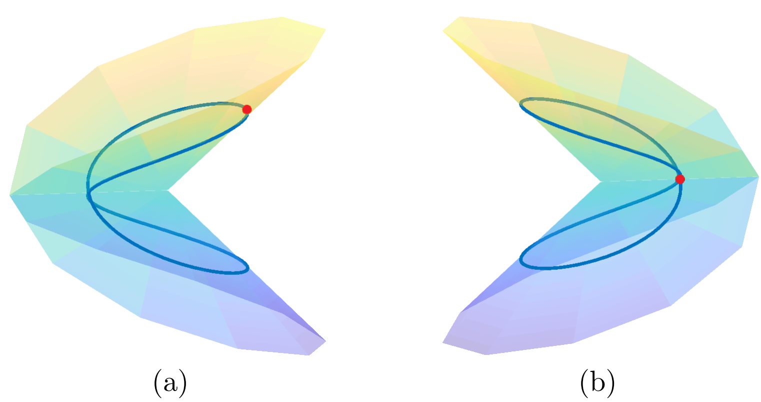

FIGURE 3. To compute the endpoint of a path as in Figure 2(b), one uses the Cauchy integral theorem and integrates along a closed loop like the one with winding number 2 with real (a) and imaginary (b) parts pictured here.

several other interactions of numerical algebraic geometry and optimization will be discussed in Arkansas.

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Jonathan D. Hauenstein has been honored with a DARPA Young Faculty Award, Sloan Research Fellowship, Army Young Investigator Award, and Office of Naval Research Young Investigator Award. Outside of mathematics, he enjoys spending time with his wife and four daughters.