Using numerical insights to improve symbolic computations

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***Abstract*—Numerical algebraic geometry provides a toolbox of numerical methods for performing computations involving systems of polynomial equations. Even though some of the computations which are performed on a computer using floating-point arithmetic are not certified, they can often be made very reliable using adaptive precision computations. Moreover, there is a wealth of information regarding the original problem which can be extracted from various numerical computation that can be used to improve subsequent symbolic computations to certify the result. This paper highlights two applications of such hybrid numeric-symbolic methods in algebraic geometry.**

***Keywords*-Numerical algebraic geometry; polynomial systems; homotopy continuation; witness sets; symbolic computations; hybrid numeric-symbolic computations.**

I. Introduction

Systems of nonlinear polynomial equations naturally arise in a variety of fields in mathematics, engineering, and science. Algebraic geometry is the mathematical field which studies the interplay of algebraic methods for manipulating systems of polynomial equations and geometric methods for computing and manipulating solution sets. Computational algebraic geometry has also followed along similar lines with methods based on both algebraic (e.g., Gröbner basis methods [1], [2]) and geometric manipulation (e.g., numerical algebraic geometry [3], [4]), each with their own advantages and disadvantages. Hybrid methods aim to combine computational approaches that accentuate advantages and minimize disadvantages.

Two applications of hybrid methods in computational algebraic geometry are described in Sections II and III. In particular, Section II describes witness sets which are used in numerical algebraic geometry to geometrically represent solution sets and applies them to analyzing some components arising from the cubic-centered 12-bar mechanism shown in Figure 2. Section III describes pseudowitness sets which are used in numerical elimination theory to geometrically represent projections of solution sets and applies them to a problem arising from 2×2 matrix multiplication.

II. Witness sets and the cubic-centered 12-bar mechanism

The solution set of a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$ is

$$\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\},$$

which is called the *variety* associated to f. Geometrically, each variety can be decomposed uniquely into inclusion maximal irreducible components. In particular, a variety V is *reducible* if there exists varieties $V_1, V_2 \subsetneq V$ with $V = V_1 \cup V_2$. An *irreducible* variety is a variety that is not reducible. Thus, the *irreducible decomposition* of V consists of irreducible varieties $V_1, \dots, V_k$ called the *irreducible components* of V which are unique up to relabeling with

$$V_i \not\subset \bigcup_{j \neq i} V_j \quad \text{and} \quad V = \bigcup_{i=1}^{k} V_i.$$

Each irreducible variety V_i naturally has a *dimension*, denoted $\dim V_i$, which is equal to the minimum of the dimension of the tangent space at each point $x \in V_i$. Then,

$$\dim V = \max_{i=1,\dots,k} \dim V_i.$$

Example II.1. The variety $V = \mathcal{V}(f)$ where

$$f(x, y, z) = \begin{bmatrix} xy - x - z \\ xz - 2x - 2z \end{bmatrix} \tag{1}$$

is reducible with two irreducible components:

$$V_1 = \mathcal{V}(x, y) \quad \text{and} \quad V_2 = \mathcal{V}(z - 2y, xz - 2x - 2z).$$

The real part is pictorial represented in Figure 1. Clearly, $\dim \mathcal{V}(f) = \dim V_1 = \dim V_2 = 1$.

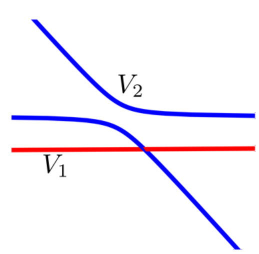

Figure 1. Real points in $\mathcal{V}(f)$ for f in (1) with two irreducible components

In numerical algebraic geometry (see [3], [4] for a general overview), each irreducible component V_i of $\mathcal{V}(f)$ is represented by a *witness set* $\{f, \mathcal{L}, V_i \cap \mathcal{L}\}$ where $\mathcal{L} \subset \mathbb{C}^N$ is a general linear space with $\operatorname{codim} \mathcal{L} = \dim V_i$. Therefore, V_i and $\mathcal{L}$ intersect in the maximum number of isolated points, called the *degree* of V_i, i.e., $\deg V_i = \#(V_i \cap \mathcal{L})$. In summary, a witness set for V_i consists of (1) a *witness system* f such that V_i is an irreducible component of $\mathcal{V}(f)$, (2) a *witness slice* $\mathcal{L}$ which is of complimentary dimension to V_i and intersects V_i transversely, and (3) a *witness point set* $V_i \cap \mathcal{L}$ consisting of $\deg V_i$ many points.

Example II.2. Continuing with Ex. II.1, for the linear space $\mathcal{L} = \mathcal{V}(2x+3y+z+1)$, the witness points sets for V_1 and V_2, respectively, are approximately

$$\{(0, -0.3333, 0)\} \quad \text{and}$$
$$\{(0.6180, -0.4472, -0.8944), (-1.6180, 0.4472, 0.8944)\}$$

with $\deg V_1 = 1$ and $\deg V_2 = 2$.

The small integer coefficients for $\mathcal{L}$ in Ex. II.2 were selected for illustrative purposes. In practice, the coefficients are randomly selected complex numbers to ensure genericity of the corresponding $\mathcal{L}$ with probability 1.

A witness set for each irreducible component V_i can be used to sample points on V_i by using a homotopy that deforms the linear space $\mathcal{L}$ along V_i. Let $\mathcal{M}$ be another general linear space with $\operatorname{codim} \mathcal{M} = \dim V_i$, and ℓ and m be systems of linear polynomials such that $\mathcal{L} = \mathcal{V}(\ell)$ and $\mathcal{M} = \mathcal{V}(m)$. Consider the homotopy

$$H(x,t) = \left[\begin{array}{c} f(x) \\ (1-t)\cdot m(x) + t \cdot \ell(x) \end{array} \right] = 0 \qquad (2)$$

with start point $w \in V_i \cap \mathcal{L}$ at $t = 1$ so that $H(w, 1) = 0$. Thus, there is a path $x(t) \subset V_i$ for $t \in [0, 1]$ such that $H(x(t), t) \equiv 0$ and $x(1) = w$. In particular, $x(0) \in V_i \cap \mathcal{M}$. Numerically, $x(t)$ can be tracked from $t = 1$ to $t = 0$ using adaptive precision path tracking [5] and the endpoint $x(0)$ can be computed to arbitrary accuracy (e.g., see [3, Ch. 7] to perform this computation using `Bertini` [6]).

Sample points can be used to gain insights into polynomials which identically vanish on an irreducible component. In fact, adding some new low degree polynomials to the other generators can provide significant improvements to the computational time used by Gröbner basis computations. With one sufficiently accurate sample point, an approach using exactness recovery algorithms such as LLL [7] or PSLQ [8] is described in [9] for computing vanishing polynomials with integer coefficients.

Example II.3. Continuing with Ex. II.1, consider the following sample point on V_2 approximated to 15 digits:

$$(x, y, z) =$$
$$(0.516108199672976, -0.347807164618899, -0.695614329237798).$$

The following computes a linear with integer coefficients, namely $z - 2y$, that vanishes on V_2 using `Maple`:

```
PSLQ([1,x,y,z])
                    [0, 0, -2, 1]
```

Since z is linearly dependent on y, the following computes an independent quadratic, namely $x + 2y - xy$, that also vanishes on V_2 using `Maple`:

```
PSLQ([1,x,y,x^2,x*y,y^2])
              [0, 1, 2, 0, -1, 0]
```

We note that another approach is to utilize interpolation (e.g., see [10]) at many sample points [11].

A. Cubic-centered 12-bar mechanism

The following applies numerical algebraic geometry to compute low degree polynomials for the cubic-centered 12-bar mechanism proposed in [12] and shown in Figure 2.

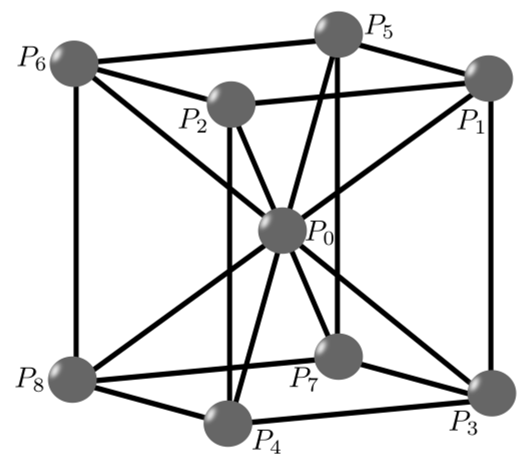

Figure 2. Cubic-centered 12-bar mechanism

To remove trivial rotations, a ground link is fixed via $P_0 = (0, 0, 0)$, $P_7 = (-1, 1, -1)$, and $P_8 = (-1, -1, -1)$. The remaining 6 vertices can move in space as long as they maintain their relative distance: length 2 along edges of the cube and length $\sqrt{3}$ to the center. Thus, there are $6 \cdot 3 = 18$ variables with the following 17 quadratic constraints:

$$\|P_i - P_j\|_2^2 - 4 = 0,$$
$$(i,j) \in \left\{ \begin{array}{l} (1,2), (1,3), (1,5), (2,4), (2,6), \\ (3,4), (3,7), (4,8), (5,6), (5,7), (6,8) \end{array} \right\},$$

$$\|P_i\|_2^2 - 3 = 0, \quad i = 1, \ldots, 6.$$

A full description of the irreducible components computed using `Bertini` is available at [13, Table 1]. This shows that there are many degenerate (dimension > 1) components that presumably are preventing symbolic computational software such as `Macaulay2` [14] from computing a primary decomposition. However, by using sample points, we can focus on only the components of interest which, in this case, are the two irreducible curves (i.e., one dimensional) of degree 6. Using the following sample point approximated to 15 digits:

$$\begin{aligned}
P_1 &= (1.3451601328564 + 0.787548261843099i,\\
&\quad 0.145383132109399 - 0.0948566874942974i,\\
&\quad 1.5094567350342 - 0.692691574348801i),\\
P_2 &= (1.81382227767393 - 4.88606735917172i,\\
&\quad -4.82454731880769 - 3.1478244462172i,\\
&\quad -3.63836959648162 + 1.73824291295452i),\\
P_3 &= (-1.26042307948286 + 4.77397655306057i,\\
&\quad 4.15712999740673 + 3.16122762841396i,\\
&\quad 4.41755307688959 - 1.61274892464661i),\\
P_4 &= (0.302526107876438 - 1.19098191498458i,\\
&\quad -2.14727885133525 + 0.215833989829908i,\\
&\quad 0.844752743458814 + 0.975147925154673i),\\
P_5 &= (0.446600801808932 + 0.11209080611115i,\\
&\quad 1.66741732140096 - 0.0134031821967617i,\\
&\quad 0.220816519592029 - 0.125493988307912i),\\
P_6 &= (-0.647686240732835 + 0.403433653141482i\\
&\quad 1.00189571922585 - 0.12097730233561i\\
&\quad -1.35420947849302 - 0.282456350805872i)
\end{aligned}$$

where $i = \sqrt{-1}$, exactness recovery [9] yields the following 10 vanishing linear polynomials where $P_j = (x_j, y_j, z_j)$:

$$\begin{array}{l}
z_2 + z_3 + z_5 - 1, x_6 + y_6 - z_1 + 2z_2 + 2z_3 - z_4 + 2z_5,\\
z_1 - z_2 - z_3 + z_4 - z_5 + z_6, y_1 + y_2 + y_3 + y_4 + y_5 + y_6,\\
x_4 + y_2 + y_3 + y_4 + y_5 + z_4, x_1 + x_4 + x_6 - y_2 - y_3 - y_5,\\
x_4 + x_6 - y_1 + 2y_2 + 2y_3 + 2y_5 - z_1, x_1 - x_2 - x_3 + x_4 - x_5 + x_6,\\
x_3 - x_4 + x_5 - x_6 + y_1 + y_2 + z_1 - z_2, x_3 + y_2 + y_5 + z_3.
\end{array}$$

Using the collection of these 10 linear and the original 17 quadratics, `Macaulay2` now trivially ($<$ 0.1 seconds) verifies that this collection defines a curve of degree 12. This curve is irreducible over $\mathbb{C}$ since, for $w = \sqrt{-3}$, the polynomial $g = g_1 \cdot g_2$ where

$$\begin{aligned}
g_1 &= 2(y_4 + y_6) + (1 + w)(z_4 + z_6) + 4\\
g_2 &= 2(y_4 + y_6) + (1 - w)(z_4 + z_6) + 4
\end{aligned}$$

has integer coefficients which vanishes on this curve of degree 12. Hence, it decomposes as two complex conjugate curves of degree 6. Moreover, the cubic-centered 12-bar mechanism shown in Figure 2, which has coordinates

$$\begin{aligned}
P_1 &= (1,1,1), & P_2 &= (1,-1,1),\\
P_3 &= (1,1,-1), & P_4 &= (1,-1,-1),\\
P_5 &= (-1,1,1), & P_6 &= (-1,-1,1),
\end{aligned}$$

lies at the intersection of these two complex conjugate curves showing that it corresponds with an isolated real solution. Hence, the mechanism shown in Figure 2 is rigid over $\mathbb{R}$ confirming the results of [15, § 9.4] based on using numerical computations to aid in exact symbolic verification.

III. Numerical elimination and 2×2 matrix multiplication

A witness system f for an irreducible variety V is an essential element of a witness set for V. The witness system permits additional computations on V, such as sampling summarized in Section II, due to the fact that V is an irreducible component of $\mathcal{V}(f)$. For problems in elimination theory (e.g., see [2, Ch. 3]), the main goal is to compute vanishing polynomials so a witness system is not readily available. To overcome this, numerical elimination theory utilizes *pseudowitness sets* [16] to geometrically represent projections of varieties.

Suppose that $V \subset \mathbb{C}^N$ is an irreducible variety with known witness system f. Let $\pi : \mathbb{C}^N \to \mathbb{C}^m$ be the projection onto the first m coordinates, i.e., $\pi(x) = (x_1, \ldots, x_m)$. Thus, $U = \overline{\pi(V)} \subset \mathbb{C}^m$, where the closure is taken in the usual Euclidean topology on $\mathbb{C}^m$, is also an irreducible variety. The underlying idea of numerical elimination theory is to replace computations on U with computations on V since a witness system for V is known. To that end, a pseudowitness set for U is $\{f, \pi, \mathcal{L}, V \cap \mathcal{L}\}$ where $\mathcal{L} = \mathcal{M} \times \mathcal{K}$ with $\operatorname{codim} \mathcal{L} = \dim V$ is constructed as follows. Let $\mathcal{M} \subset \mathbb{C}^m$ be a general linear space with $\operatorname{codim} \mathcal{M} = \dim U$ so that $U \cap \mathcal{M}$ is a witness point set for U with $\deg U = \#(U \cap \mathcal{M})$. Let $\mathcal{K} \subset \mathbb{C}^{N-m}$ be a general linear space with $\operatorname{codim} \mathcal{K} = \dim V - \dim U \geq 0$. Hence, $\mathcal{M}$ slices the "image variables" $x_1, \ldots, x_m$ while $\mathcal{K}$ slices the "fiber variables" $x_{m+1}, \ldots, x_N$. In particular, $\pi(V \cap \mathcal{L}) = U \cap \mathcal{M}$ and $\#(V \cap \mathcal{L}) = \deg U \cdot \deg_{gf}(V, \pi)$ with $\deg_{gf}(V, \pi) = \deg(\pi^{-1}(\pi(v)) \cap V)$ for a generic $v \in V$.

Example III.1. Let $V = \mathcal{V}(f) \subset \mathbb{C}^3$ be the twisted cubic curve where

$$f(x, y, z) = \begin{bmatrix} y - x^2 \\ z - xy \end{bmatrix}.$$

Let $\pi(x, y, z) = (x, y)$ be the projection onto the first two variables and $U = \overline{\pi(V)}$. The real part of U and V are pictorially shown in Figure 3. Since $\dim V = \dim U = 1$, we take $\mathcal{M} = \mathcal{V}(4x - 2y - 1) \subset \mathbb{C}^2$ and $\mathcal{K} = \mathbb{C}$ with $\mathcal{L} = \mathcal{M} \times \mathcal{K} \subset \mathbb{C}^3$. Hence, $V \cap \mathcal{L}$ consists of two points, approximately

$$(0.2929, 0.0858, 0.0251) \quad \text{and} \quad (1.7071, 2.9142, 4.9749).$$

Thus, $\deg U = \#\pi(V \cap \mathcal{L}) = 2$ and $\deg_{gf}(V, \pi) = 1$.

Obviously, one approach to sample points on U is to first sample points on V and then project via π. Additionally, one can also easily adapt the sampling procedure described in Section II to sample points on U along a given linear space by deforming $\mathcal{M}$ to the given linear space.

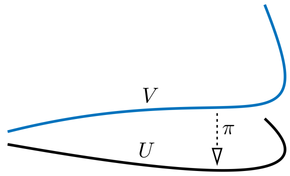

Figure 3. Illustrating the projection of the twisted cubic curve onto first two coordinates

A. Border rank of 2×2 matrix multiplication

Consider multiplying two 2×2 matrices

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix},$$

namely

$$X \cdot Y = \begin{bmatrix} x_1 \cdot y_1 + x_2 \cdot y_3 & x_1 \cdot y_2 + x_2 \cdot y_4 \\ x_3 \cdot y_1 + x_4 \cdot y_3 & x_3 \cdot y_2 + x_4 \cdot y_4 \end{bmatrix}.$$

As written, $X \cdot Y$ requires 8 multiplications to be performed:

$$x_1 \cdot y_1,\ x_2 \cdot y_3,\ x_1 \cdot y_2,\ x_2 \cdot y_4,\ x_3 \cdot y_1,\ x_4 \cdot y_3,\ x_3 \cdot y_2,\ x_4 \cdot y_4.$$

In 1969, Strassen [17] provided an alternative method to compute $X \cdot Y$ which requires only 7 multiplications:

$$X \cdot Y = \begin{bmatrix} I + IV - V + VII & III + V \\ II + IV & I - II + III + VI \end{bmatrix}$$

where

$$\begin{aligned} I &= (x_1 + x_4) \cdot (y_1 + y_4), \\ II &= (x_3 + x_4) \cdot y_1, \\ III &= x_1 \cdot (y_2 - y_4), \\ IV &= x_4 \cdot (y_3 - y_1), \\ V &= (x_1 + x_2) \cdot y_4, \\ VI &= (x_3 - x_1) \cdot (y_1 + y_2), \\ VII &= (x_2 - x_4) \cdot (y_3 + y_4). \end{aligned}$$

It was shown in [18] that seven multiplications was necessary. The *rank* of 2×2 matrix multiplication, which is the minimum number of scalar multiplications needed to compute the product, is 7 (e.g., see [19] for more information).

Rather than exactly compute the multiplication of two matrices, one could alternatively aim to approximate the result up to arbitrary accuracy. For example, the problem of multiplying the following matrices was studied in [20]:

$$P = \begin{bmatrix} p_1 & p_2 \\ 0 & p_4 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}. \tag{3}$$

Classically,

$$P \cdot Q = \begin{bmatrix} p_1 \cdot q_1 + p_2 \cdot q_3 & p_1 \cdot q_2 + p_2 \cdot q_4 \\ p_4 \cdot q_3 & p_4 \cdot q_4 \end{bmatrix}$$

requires 6 multiplications, which is necessary to exactly compute this product. However, for any $\epsilon \neq 0$, it is shown in [20] that 5 multiplications are sufficient to compute $P \cdot Q + O(\epsilon)$, namely

$$\begin{bmatrix} \epsilon^{-1}(I - II - III + IV) & \epsilon^{-1}(V - III) \\ IV & I - V \end{bmatrix}$$

where

$$\begin{aligned} I &= (p_1 + p_4) \cdot (q_4 + \epsilon q_1), \\ II &= p_4 \cdot (q_3 + q_4), \\ III &= p_1 \cdot q_4, \\ IV &= (p_4 + \epsilon p_2) \cdot (q_3 - \epsilon q_1), \\ V &= (p_1 + \epsilon p_2) \cdot (q_4 + \epsilon q_2). \end{aligned}$$

The *border rank* of a problem is the minimum number of scalar multiplications needed to approximate the result to arbitrary accuracy. In particular, the rank of multiplying P and Q in (3) is 6 while the border rank is 5. The difference between this rank and border rank suggested that one should investigate the border rank for 2×2 matrix multiplication. This remained an open problem until [21] provided an argument that both the rank and border rank for 2×2 matrix multiplication is 7. Due to a gap in this argument that was subsequently filled in the unpublished manuscript [22], an alternative approach guided by numerical insights was developed in [23] as follows.

To formulate in terms of polynomial systems, 2×2 matrix multiplication is written as a tensor in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \cong \mathbb{C}^{64}$. For example, the classical definition yields

$$\begin{aligned} M_2 \quad := \quad & (x_1 \otimes y_1 + x_2 \otimes y_3) \otimes z_1 + (x_1 \otimes y_2 + x_2 \otimes y_4) \otimes z_2 + \\ & (x_3 \otimes y_1 + x_4 \otimes y_3) \otimes z_3 + (x_3 \otimes y_2 + x_4 \otimes y_4) \otimes z_4 \end{aligned}$$

while Strassen's approach is equivalent to rewriting

$$\begin{aligned} M_2 \quad = \quad & (x_1 + x_4) \otimes (y_1 + y_4) \otimes (z_1 + z_4) + \\ & (x_3 + x_4) \otimes y_1 \otimes (z_3 - z_4) + \\ & x_1 \otimes (y_2 - y_4) \otimes (z_2 + z_4) + \\ & x_4 \otimes (y_3 - y_1) \otimes (z_1 + z_3) + \\ & (x_1 + x_2) \otimes y_4 \otimes (z_2 - z_1) + \\ & (x_3 - x_1) \otimes (y_1 + y_2) \otimes z_4 + \\ & (x_2 - x_4) \otimes (y_3 + y_4) \otimes z_1. \end{aligned}$$

A scalar multiplication corresponds with a term of the form

$$\ell_1(x) \otimes \ell_2(y) \otimes \ell_3(z)$$

where each ℓ_i is a linear form. Hence, the classical definition expresses M_2 using 8 terms while Strassen's approach uses 7 terms. Since the rank of 2×2 multiplication is 7, there does not exist linear forms ℓ_{ij} such that

$$M_2 = \sum_{j=1}^{6} \ell_{1j}(x) \otimes \ell_{2j}(y) \otimes \ell_{3j}(z).$$

The variety of all tensors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ of border rank at most 6 is

$$\sigma_6 = \overline{\left\{ \sum_{j=1}^{6} \ell_{1j}(x) \otimes \ell_{2j}(y) \otimes \ell_{3j}(z) \,\middle|\, \ell_{ij} \text{ is linear form} \right\}}.$$

Due to Strassen's approach, the border rank of M_2 is 7 if and only if $M_2 \notin \sigma_6$. Clearly, computing polynomials that vanish on σ_6 is a problem in elimination theory. With no previously known nonconstant polynomials vanishing on σ_6, numerical elimination theory was applied to σ_6 in [23] yielding a pseudowitness set for σ_6 showing $\operatorname{codim} \sigma_6 = 4$ and $\deg \sigma_6 = 15{,}456$. A homotopy membership test based on using pseudowitness sets [24] showed that $M_2 \notin \sigma_6$.

Numerical insight was then used to determine in which degree to search for polynomials vanishing on σ_6 with the aim of finding a polynomial vanishing on σ_6 that does not vanish at M_2. The difficulty of this is due to the number of variables. For example, the space of homogeneous polynomials of degree 18 in 64 variables has dimension

$$\binom{63+18}{18} = 456{,}703{,}981{,}505{,}085{,}600$$

which is too large to utilize the approaches [9], [11] suggested in Section II. Hence, one reduces the problem via necessary conditions by vanishing on a witness point set restricted to the witness slice [11], [23]. Since $\operatorname{codim} \sigma_6 = 4$, this reduces down to computing affine polynomials in 4 variables. Thus, to show that no nonzero polynomials of degree 18 vanish on σ_6, one only needs to consider a space of very manageable dimension

$$\binom{4+18}{18} = 7315.$$

In particular, restricting to the witness point set shows that the minimal degree of vanishing polynomials on σ_6 is 19 with at most 64 independent polynomials in degree 19.

This numerical insight guided where to apply representation theory to describe 64 polynomials of degree 19 that vanish on σ_6, the first known nonzero collection of vanishing polynomials for σ_6. An explicit polynomial g of degree 20 generated from these 64 vanishing polynomials was used in [23] to separate M_2 from σ_6 since $\sigma_6 \subset \mathcal{V}(g)$ and $g(M_2) \neq 0$. Therefore, this process created an explicit symbolic proof that the border rank of 2×2 matrix multiplication is 7 which was aided by pseudowitness sets and numerical elimination theory.

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References

[1] B. Buchberger, "Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal," *Journal of Symbolic Computation*, vol. 41, no. 3, pp. 475–511, 2006.

[2] D. A. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, 4th ed., ser. Undergraduate Texts in Mathematics. Springer, Cham, 2015.

[3] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler, *Numerically Solving Polynomial Systems with Bertini*. SIAM, 2013, vol. 25.

[4] A. J. Sommese and C. W. Wampler, *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific, 2005, vol. 99.

[5] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler, "Adaptive multiprecision path tracking," *SIAM J. Numer. Anal.*, vol. 46, no. 2, pp. 722–746, 2008.

[6] ——, "Bertini: Software for numerical algebraic geometry," Available at bertini.nd.edu.

[7] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász, "Factoring polynomials with rational coefficients," *Math. Ann.*, vol. 261, no. 4, pp. 515–534, 1982.

[8] H. Ferguson and D. Bailey, "A polynomial time, numerically stable integer relation algorithm," Technical report, 1991.

[9] D. J. Bates, J. D. Hauenstein, T. M. McCoy, C. Peterson, and A. J. Sommese, "Recovering exact results from inexact numerical data in algebraic geometry," *Exp. Math.*, vol. 22, no. 1, pp. 38–50, 2013.

[10] Z. A. Griffin, J. D. Hauenstein, C. Peterson, and A. J. Sommese, "Numerical computation of the Hilbert function and regularity of a zero dimensional scheme," in *Connections between algebra, combinatorics, and geometry*, ser. Springer Proc. Math. Stat. Springer, New York, 2014, vol. 76, pp. 235–250.

[11] N. S. Daleo and J. D. Hauenstein, "Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme," *J. Symbolic Comput.*, vol. 72, pp. 128–146, 2016.

[12] C. W. Wampler, B. T. Larson, and A. G. Erdman, "A new mobility formula for spatial mechanisms," in *Proceedings of the 2007 ASME International Design Engineering Technical Conferences/Computers and Information in Engineering Conference*, 2008, vol. 8, pp. 561–570.

[13] J. D. Hauenstein, "Numerically computing real points on algebraic sets," *Acta Appl. Math.*, vol. 125, pp. 105–119, 2013.

[14] D. R. Grayson and M. E. Stillman, "Macaulay2, a software system for research in algebraic geometry," Available at www.math.uiuc.edu/Macaulay2/.

[15] C. W. Wampler, J. D. Hauenstein, and A. J. Sommese, "Mechanism mobility and a local dimension test," *Mechanism and Machine Theory*, vol. 46, no. 9, pp. 1193–1206, 2011.

[16] J. D. Hauenstein and A. J. Sommese, "Witness sets of projections," *Appl. Math. Comput.*, vol. 217, no. 7, pp. 3349–3354, 2010.

[17] V. Strassen, "Gaussian elimination is not optimal," *Numer. Math.*, vol. 13, pp. 354–356, 1969.

[18] J. E. Hopcroft and L. R. Kerr, "On minimizing the number of multiplications necessary for matrix multiplication," *SIAM J. Appl. Math.*, vol. 20, pp. 30–36, 1971.

[19] J. M. Landsberg, *Geometry and complexity theory*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017, vol. 169.

[20] D. Bini, M. Capovani, F. Romani, and G. Lotti, "$O(n^{2.7799})$ complexity for $n \times n$ approximate matrix multiplication," *Inform. Process. Lett.*, vol. 8, no. 5, pp. 234–235, 1979.

[21] J. M. Landsberg, "The border rank of the multiplication of 2×2 matrices is seven," *J. Amer. Math. Soc.*, vol. 19, no. 2, pp. 447–459, 2006.

[22] ——, "The border rank of the multiplication of 2×2 matrices is seven," arXiv:math/0407224, 2006.

[23] J. D. Hauenstein, C. Ikenmeyer, and J. M. Landsberg, "Equations for lower bounds on border rank," *Exp. Math.*, vol. 22, no. 4, pp. 372–383, 2013.

[24] J. D. Hauenstein and A. J. Sommese, "Membership tests for images of algebraic sets by linear projections," *Appl. Math. Comput.*, vol. 219, no. 12, pp. 6809–6818, 2013.